Covariant Functional Calculi from the Affine Groups

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Abstract

Invoking the Clifford-Hermite Wavelets from Clifford analysis, we use the covariances of affine groups to construct a kind of functional calculi for several non-commuting bounded operators. Functional calculi are the intertwining transforms between the representations of affine groups in the space $L^2(\mathbb{R}^m)$ and in the space of bounded operators. It turns out that the Weyl calculus is the value of this new functional calculus at the identity of affine groups. Our approach is inspired by the mathematical ideas contained in the paper "V. V. Kisil. Wavelets in Banach spaces. Acta Appl. Math. 1999, **59**(1): 79-109".

1 Introduction

An abstract theory of functional calculus has been proposed in [8]. The main idea is to use group covariance to construct a functional calculus, which is the intertwining transform between two representations of a group. In this way the Weyl calculus (see [1]) is reobtained from the Heisenberg group, like the Riesz-Dunford type calculus and the monogenic functional calculus are connected to the fractional linear transform and the Möbius group respectively (see [7-11]).

Recently Clifford-Hermite Continuous Wavelet Transforms (CHCWT) in Clifford analysis have been fully developed by the Ghent Clifford research group (see [3, 4, 5]). We find that they are suitable tools to realize the abstract theory

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from [8] for two affine groups, which is the main purpose of this paper. Although the results are obtained using the ideas contained in [8], they have a value in their own for at least two reasons. First, they can be used to verify the general theory developed in [8], and at the same time they provide a new kind of functional calculus, in particular they provide the possibility to study functional calculi for non-commuting unbounded operators. Secondly, they could not have been realized without the latest developments of CHCWT in Clifford analysis. So we find new applications of CHCWT besides their traditional applications in signal analysis and data compression.

The structure of this paper is as follows. In Section 2 we recall some results about two kinds of Clifford-Hermite wavelets and some properties of two affine groups. In Section 3 functional calculi are constructed as intertwining transforms between the representations of each affine group in $L^2(\mathbb{R}^m)$ and in the Banach space of bounded operators respectively. In each case the Weyl calculus is reobtained, more specifically the Weyl calculus is the value of functional calculus at the identity of each affine group.

2 Prerequisites

2.1 Basic notions on the Clifford-Hermite wavelets

Denote by $\mathbf{R}_{(m)}$ be the 2^m-dimensional real Clifford algebra with basis (see e.g. [2,5])

$$\{e_0, e_1, ..., e_m, ..., e_{j_1...j_r} = e_{j_1} \cdots e_{j_r}, ..., e_{1...m} : 1 \leq j_1 < ... < j_r \leq m\},$$

where $e_0 = 1$ is the unit element, and $e_1, ..., e_m$ are orthonormal vectors satisfying the relations

$$e_i e_j + e_j e_i = -2\delta_{ij}, \quad i, j = 1, \cdots, m. \tag{2.1.1}$$

The Clifford involution of $\mathbf{R}_{(m)}$ is the anti-involution defined by

$$\overline{e_{j_1...j_r}} = \overline{e_{j_r}}...\overline{e_{j_1}}, \quad (1 \leq j_1 < ... < j_r \leq m), \tag{2.1.2}$$

where $\overline{e}_j = -e_j, j = 1, ..., m$.

Identifying a vector $\underline{x}$ in $\mathbb{R}^m$ with the Clifford vector $\underline{x} = x_1 e_1 + ... + x_m e_m$, then $\overline{\underline{x}} = x_1 \overline{e_1} + ... + x_m \overline{e_m}$, $|\underline{x}| = \sqrt{\underline{x}\,\overline{\underline{x}}} = (\sum_{j=1}^m x_j^2)^{\frac{1}{2}}$ is the norm of $\underline{x} \in \mathbb{R}^m$. So the inverse of $\underline{x} \in \mathbb{R}^m$ is $\underline{x}^{-1} = \overline{\underline{x}}|\underline{x}|^{-2} \in \mathbb{R}^m$(see e.g. [2, 5]).

Denote by $\partial_{\underline{x}} = \sum_{j=1}^m e_j \frac{\partial}{\partial x_j}$ the Dirac operator in $\mathbb{R}^m$. A function f is called *left monogenic* if $\partial_{\underline{x}} f = \sum_{j=1}^m e_j \frac{\partial f}{\partial_{x_j}} = 0$. Similarly a function f is called *right monogenic* if $f\partial_{\underline{x}} = \sum_{j=1}^m \frac{\partial f}{\partial_{x_j}} e_j = 0$. The unit sphere in $\mathbb{R}^m$ is $S^{m-1} = \{\underline{x} \in \mathbb{R}^m : |\underline{x}| = 1\}$. Then

$$\text{Spin}(m) = \{\omega_1...\omega_{2l} : \omega_j \in S^{m-1}, j = 1, ..., 2l, l \in N\}, \tag{2.1.3}$$

is called the *Spin group* with respect to the usual Clifford multiplication. A spin element $s = \omega_1...\omega_{2l} \in \text{Spin}(m)$ has a unique inverse $s^{-1} = \overline{\omega_1...\omega_{2l}} = \overline{\omega_{2l}}...\overline{\omega_1} \in \text{Spin}(m)$ and e_0 is the identity element (see e.g. [5]).

Assume $n > 0$. Put

$$\psi_n(\underline{x}) = (-1)^n \partial_{\underline{x}}^n \exp\left(-\frac{|\underline{x}|^2}{2}\right), \tag{2.1.4}$$

$$\psi_n^{a,\underline{b}}(\underline{x}) = \frac{1}{a^{m/2}}\psi_n\left(\frac{\underline{x}-\underline{b}}{a}\right), \quad a \in \mathbb{R}_+,\ \underline{b} \in \mathbb{R}^m, \tag{2.1.5}$$

$$\psi_{n,k}(\underline{x}) = (-1)^n \partial_{\underline{x}}^n \left(\exp\left(-\frac{|\underline{x}|^2}{2}\right) P_k(\underline{x})\right), \tag{2.1.6}$$

$$\psi_{n,k}^{a,\underline{b},s}(\underline{x}) = \overline{\frac{1}{a^{m/2}} s\psi_{n,k}\left(\frac{\overline{s}(\underline{x}-\underline{b})s}{a}\right)}, \tag{2.1.7}$$

where $P_k(\underline{x})$ is a homogeneous monogenic polynomial of degree k and $s \in \mathrm{Spin}(m)$. The functions $\psi_n(\underline{x})$ and $\psi_{n,k}(\underline{x})$ are the so-called Clifford-Hermite wavelets and generalized Clifford-Hermite wavelets respectively; $\psi_n^{a,\underline{b}}(\underline{x})$ and $\psi_{n,k}^{a,\underline{b},s}(\underline{x})$ are the corresponding respective CHCTW (see [3, 4]).

Denote the inner product of $L^2(\mathbb{R}^m)$ by

$$\langle f, g\rangle = \int_{\mathbb{R}^m} \overline{f(\underline{x})} g(\underline{x}) d\underline{x}. \tag{2.1.8}$$

Here $\overline{f(\underline{x})}$ stands for the complex-conjugate, or the Clifford anti-involution or both of them, depending on whether $f(\underline{x})$ is complex-valued, real Clifford algebra-valued or complex Clifford-algebra valued.

Denote the Fourier transform of $f \in L^1 \cap L^2(\mathbb{R}^m)$ by

$$\hat{f}(\underline{u}) = \int_{\mathbb{R}^m} e^{-i\langle \underline{x},\underline{u}\rangle} f(\underline{x}) d\underline{x} = \langle e^{i\langle \underline{x},\underline{u}\rangle}, f\rangle, \tag{2.1.9}$$

where $\underline{x} = \sum_{j=1}^m x_j e_j$, $\underline{u} = \sum_{j=1}^m u_j e_j$, $\langle \underline{x}, \underline{u}\rangle = \sum_{j=1}^m x_j u_j$. Then (see [3,4])

$$\widehat{\psi_n}(\underline{u}) = (2\pi)^{m/2}(-i)^n \underline{u}^n \exp\left(-\frac{|\underline{u}|^2}{2}\right), \tag{2.1.10}$$

$$\widehat{\psi_n^{a,\underline{b}}}(\underline{u}) = a^{m/2} e^{-i\langle \underline{b},\underline{u}\rangle} \widehat{\psi_n}(a\underline{u}), \tag{2.1.11}$$

$$\widehat{\psi_{n,k}}(\underline{u}) = (2\pi)^{m/2}(-i)^n \underline{u}^n P_k(i\partial_{\underline{u}}) \exp\left(-\frac{|\underline{u}|^2}{2}\right), \tag{2.1.12}$$

$$\widehat{\psi_{n,k}^{a,\underline{b},s}}(\underline{u}) = a^{m/2} e^{-i\langle \underline{b},\underline{u}\rangle} \widehat{\psi_{n,k}}(a\overline{s}\underline{u}s)\overline{s}. \tag{2.1.13}$$

2.2 Some results on the $ax + b$ group

Let U denote the half space

$$U = \{(a, \underline{b}) : a \in \mathbb{R}^+, \underline{b} \in \mathbb{R}^m\}. \tag{2.2.1}$$

For any $(a_j, \underline{b}_j) \in U$, $j = 1, 2$, define the binary multiplication $\cdot$ by

$$(a_1, \underline{b}_1) \cdot (a_2, \underline{b}_2) = (a_1 a_2, \underline{b}_1 + a_1 \underline{b}_2). \tag{2.2.2}$$

U is the usual $ax+b$ group. In the case where $m=1$, some properties of U can be found in [12], §18. The following considerations are similar to those in [12].

Theorem 2.2.1 *U is a non-abelian group; $(1,\underline{0})$ is its identity element and $(a,\underline{b})^{-1}=(\frac{1}{a},-\frac{\underline{b}}{a})$ is the inverse of $(a,\underline{b})$.*

Proof. Assume $(a_j,\underline{b}_j)\in U$, $j=1,2,3$, then

$$\begin{aligned}((a_1,\underline{b}_1)\cdot(a_2,\underline{b}_2))\cdot(a_3,\underline{b}_3) &= (a_1a_2,\underline{b}_1+a_1\underline{b}_2)\cdot(a_3,\underline{b}_3)\\ &= (a_1a_2a_3,\underline{b}_1+a_1\underline{b}_2+a_1a_2\underline{b}_3),\\ (a_1,\underline{b}_1)\cdot((a_2,\underline{b}_2)\cdot(a_3,\underline{b}_3)) &= (a_1,\underline{b}_1)\cdot(a_2a_3,\underline{b}_2+a_2\underline{b}_3)\\ &= (a_1a_2a_3,\underline{b}_1+a_1(\underline{b}_2+a_2\underline{b}_3)),\end{aligned}$$

thus $((a_1,\underline{b}_1)\cdot(a_2,\underline{b}_2))\cdot(a_3,\underline{b}_3)=(a_1,\underline{b}_1)\cdot((a_2,\underline{b}_2)\cdot(a_3,\underline{b}_3))$, the associative law is verified.

Next, as

$$\begin{aligned}(a,\underline{b})\cdot(1,\underline{0}) &= (a,\underline{b}), & (1,\underline{0})\cdot(a,\underline{b}) &= (a,\underline{b}),\\ (a,\underline{b})\cdot(\frac{1}{a},-\frac{\underline{b}}{a}) &= (1,\underline{0}), & (\frac{1}{a},-\frac{\underline{b}}{a})\cdot(a,\underline{b}) &= (1,\underline{0}),\end{aligned}$$

$(1,\underline{0})$ is the identity of U and $(\frac{1}{a},-\frac{\underline{b}}{a})$ is the inverse of $(a,\underline{b})$.

Finally, obviously $(a_1,\underline{b}_1)\cdot(a_2,\underline{b}_2)\neq(a_2,\underline{b}_2)\cdot(a_1,\underline{b}_2)$, so U is a non-abelian group. ■

Theorem 2.2.2 *The left and right Haar measures of U are*

$$\frac{d\underline{b}da}{a^{m+1}} \quad\text{and}\quad \frac{d\underline{b}da}{a} \tag{2.2.3}$$

respectively, which shows that U is a non-unimodular group.

Proof. Let g be integrable on U with respect to $\frac{d\underline{b}da}{a^{m+1}}$. For any $(a',\underline{b}')\in U$, we have

$$\begin{aligned}\int_U g((a',\underline{b}')\cdot(a,\underline{b}))\frac{d\underline{b}da}{a^{m+1}} &= \int_0^{+\infty}\int_{\mathbb{R}^m} g(a'a,\underline{b}'+a'\underline{b})\frac{d\underline{b}da}{a^{m+1}}\\ &= \int_0^{+\infty}\int_{\mathbb{R}^m} g(a'a,a'\underline{b})\frac{d\underline{b}da}{a^{m+1}}\\ &= \int_0^{+\infty}\int_{\mathbb{R}^m} g(\alpha,\underline{\beta})\frac{d\underline{\beta}d\alpha}{\alpha^{m+1}},\end{aligned} \tag{2.2.4}$$

where $\alpha=a'a,\underline{\beta}=a'\underline{b}$ are used in the above. Thus $\frac{d\underline{b}da}{a^{m+1}}$ is the left Haar measure of U.

Similarly, let g be integrable on U with respect to $\frac{d\underline{b}da}{a}$, then

$$\int_U g((a,\underline{b})\cdot(a',\underline{b}'))\frac{d\underline{b}da}{a}=\int_0^{\infty}\int_{\mathbb{R}^m} g(aa',\underline{b}+a\underline{b}')\frac{d\underline{b}da}{a}.$$

Denote $\alpha=a'a$, $\underline{\beta}=\underline{b}+a\underline{b}'$, then $d\alpha d\underline{\beta}=a'dad\underline{b}$, thus

$$\int_U g((a,\underline{b})\cdot(a',\underline{b}'))\frac{d\underline{b}da}{a}=\int_0^{\infty}\int_{\mathbb{R}^m} g(\alpha,\underline{\beta})\frac{d\underline{\beta}d\alpha}{a'a}=\int_0^{\infty}\int_{\mathbb{R}^m} g(\alpha,\underline{\beta})\frac{d\underline{\beta}d\alpha}{\alpha},$$

which shows that $\frac{d\underline{b}da}{a}$ is the right Haar measure of U.

Obviously, $\frac{d\underline{b}da}{a^{m+1}}$ is different from $\frac{d\underline{b}da}{a}$, so U is a non-unimodular group. ■

Define $\chi_{\pm}(\underline{u}) = \frac{1}{2}\left(e_0 \pm \frac{i\underline{u}}{|\underline{u}|}\right)$ and

$$H^2_{\pm}(\mathbb{R}^m) = \left\{ f(\underline{x}) : \hat{f}(\underline{u}) = \chi_{\pm}(\underline{u})\hat{f}_0(\underline{u}),\ where\ \ f_0(\underline{x}) \in L^2(\mathbb{R}^m) \right\},$$

then $H^2_+(\mathbb{R}^m)$ and $H^2_-(\mathbb{R}^m)$ are the positive and negative boundary values of Cauchy-type integral defined on the hyperplane $\mathbb{R}^m$ respectively (e.g., [6,12]). As $(i\underline{u})^2 = |\underline{u}|^2$, then $\chi_+(\underline{u})\chi_-(\underline{u}) = \chi_-(\underline{u})\chi_+(\underline{u}) = 0$ and $(\chi_+(\underline{u}))^2 = \chi_+(\underline{u})$, $(\chi_-(\underline{u}))^2 = \chi_-(\underline{u})$. So both $H^2_+(\mathbb{R}^m)$ and $H^2_-(\mathbb{R}^m)$ are closed subspaces of $L^2(\mathbb{R}^m)$. Moreover $L^2(\mathbb{R}^m)$ has the orthogonal decomposition

$$L^2(\mathbb{R}^m) = H^2_+(\mathbb{R}^m) \oplus H^2_-(\mathbb{R}^m).$$

Theorem 2.2.3 *Let $f \in H^2_+(\mathbb{R}^m)$ (or $H^2_-(\mathbb{R}^m)$), and for any $(a, \underline{b}) \in U$, define*

$$(\rho_{(a,\underline{b})} f)(\underline{x}) = \frac{1}{a^{m/2}} f\left(\frac{\underline{x} - \underline{b}}{a}\right). \tag{2.2.5}$$

Then ρ is an irreducible and unitary representation of U in $H^2_{\pm}(\mathbb{R}^m)$ respectively.

Proof. Take $(a_j, \underline{b}_j) \in U$, $j = 1, 2$, then by (2.2.2) and (2.2.5), we have

$$\rho_{(a_1,\underline{b}_1)}\rho_{(a_2,\underline{b}_2)} f(\underline{x}) = \frac{1}{a_1^{m/2}} \rho_{(a_2,\underline{b}_2)} f\left(\frac{\underline{x} - \underline{b}_1}{a_1}\right) = \frac{1}{a_2^{m/2}} \frac{1}{a_1^{m/2}} f\left(\frac{\frac{\underline{x}-\underline{b}_1}{a_1} - \underline{b}_2}{a_2}\right), \tag{2.2.6}$$

$$\rho_{((a_1,\underline{b}_1)\cdot(a_2,\underline{b}_2))} f(\underline{x}) = \rho_{(a_1 a_2, \underline{b}_1 + a_1 \underline{b}_2)} f(\underline{x}) = \frac{1}{(a_1 a_2)^{m/2}} f\left(\frac{\underline{x} - \underline{b}_1 - a_1 \underline{b}_2}{a_1 a_2}\right), \tag{2.2.7}$$

which means that $\rho_{(a_1,\underline{b}_1)}\rho_{(a_2,\underline{b}_2)} = \rho_{((a_1,\underline{b}_1)\cdot(a_2,\underline{b}_2))}$.

Let $f \in H^2_{\pm}(\mathbb{R}^m)$, by assumption that there exists a $f_0(\underline{x}) \in L^2(\mathbb{R}^m)$ such that $\hat{f}(u) = \chi_{\pm}(\underline{u})\hat{f}_0(u)$. Since

$$\widehat{\rho_{(a,\underline{b})} f} = e^{-i\langle \underline{b}, \underline{u} \rangle} \hat{f}(a\underline{u}) = e^{-i\langle \underline{b}, \underline{u} \rangle} \chi_{\pm}(a\underline{u}) \hat{f}_0(a\underline{u}) = e^{-i\langle \underline{b}, \underline{u} \rangle} \chi_{\pm}(\underline{u}) \hat{f}_0(a\underline{u}).$$

Obviously, $\rho_{(a,\underline{b})} f \in H^2_{\pm}(\mathbb{R}^m)$.

It is easily seen that $\|\rho_{(a,\underline{b})} f\|_2 = \|f\|_2$, where $\|\cdot\|_2$ is the norm of $L^2(\mathbb{R}^m)$ induced by (2.1.8). So ρ is a unitary representation of U in the space $H^2_{\pm}(\mathbb{R}^m)$ respectively.

Let $f \in H^2_+(\mathbb{R}^m)$ be any fixed nonzero function, put $S = \{\rho_{(a,\underline{b})} f : \forall (a, \underline{b}) \in U\}$, then S is a closed subspace of $H^2_+(\mathbb{R}^m)$.

To prove the irreducibility of ρ in the space $H^2_+(\mathbb{R}^m)$, we only need to prove that $S = H^2_+(\mathbb{R}^m)$. If it were not, assume $g(\underline{x}) \in H^2_+(\mathbb{R}^m) \backslash S$ to be orthogonal to S; then by Plancherel's Theorem, we have

$$\begin{aligned}\int_{\mathbb{R}^m} \overline{g(\underline{x})} \rho_{(a,\underline{b})} f(\underline{x}) = 0 = \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} \overline{\widehat{g}(\underline{u})} \widehat{\rho_{(a,\underline{b})} f}(\underline{u}) d\underline{u} \\ = \frac{a^{m/2}}{(2\pi)^m} \int_{\mathbb{R}^m} e^{-i\langle \underline{b}, \underline{u} \rangle} \overline{\hat{g}(\underline{u})} \widehat{f}(a\underline{u}) d\underline{u},\end{aligned}$$

then $\overline{\hat{g}(\underline{u})}\hat{f}(a\underline{u}) = 0$ for almost all $\underline{u} \in \mathbb{R}^m$. As $\hat{g}(\underline{u}) \neq 0$ with a positive measure and both $f, g \in H^2_+(\mathbb{R}^m)$, then $\hat{f}(a\underline{u}) = 0 \Leftrightarrow \hat{f}(\underline{u}) = 0$, and hence $f(\underline{x}) \equiv 0$, which contradicts the assumption that f is a nonzero function.

The irreducibility of ρ in the space $H^2_-(\mathbb{R}^m)$ can be proved similarly. ■

2.3 Some results on the affine group V

Put $V = U \times \mathrm{Spin}(m)$, i.e.

$$V = \{(a, \underline{b}, s) : a \in \mathbb{R}^+, \underline{b} \in \mathbb{R}^m, s \in \mathrm{Spin}(m)\}, \tag{2.3.1}$$

its multiplication law being defined by

$$(a_1, \underline{b}_1, s_1) \cdot (a_2, \underline{b}_2, s_2) = (a_1 a_2, \underline{b}_1 + a_1 s_1 \underline{b}_2 \overline{s_1}, s_1 s_2). \tag{2.3.2}$$

Theorem 2.3.1 *V is a non-abelian group.*

Proof. Take $(a_j, \underline{b}_j, s_j) \in V, j = 1, 2, 3$, then by (2.3.2), we have

$$\begin{aligned}((a_1, \underline{b}_1, s_1) \cdot (a_2, \underline{b}_2, s_2)) \cdot (a_3, \underline{b}_3, s_3) &= (a_1 a_2, \underline{b}_1 + a_1 s_1 \underline{b}_2 \overline{s_1}, s_1 s_2) \cdot (a_3, \underline{b}_3, s_3)\\ &= (a_1 a_2 a_3, \underline{b}_1 + a_1 s_1 \underline{b}_2 \overline{s_1} + a_1 a_2 s_1 s_2 \underline{b}_3 \overline{s_1 s_2}, s_1 s_2 s_3)\end{aligned}$$

and

$$\begin{aligned}(a_1, \underline{b}_1, s_1) \cdot ((a_2, \underline{b}_2, s_2) \cdot (a_3, \underline{b}_3, s_3)) &= (a_1, \underline{b}_1, s_1) \cdot (a_2 a_3, \underline{b}_2 + a_2 s_2 \underline{b}_3 \overline{s_2}, s_2 s_3)\\ &= (a_1 a_2 a_3, \underline{b}_1 + a_1 s_1 (\underline{b}_2 + a_2 s_2 \underline{b}_3 \overline{s_2}) \overline{s_1}, s_1 s_2 s_3)\\ &= (a_1 a_2 a_3, \underline{b}_1 + a_1 s_1 \underline{b}_2 \overline{s_1} + a_1 a_2 s_1 s_2 \underline{b}_3 \overline{s_2}\, \overline{s_1}, s_1 s_2 s_3),\end{aligned}$$

thus $((a_1, \underline{b}_1, s_1) \cdot (a_2, \underline{b}_2, s_2)) \cdot (a_3, \underline{b}_3, s_3) = (a_1, \underline{b}_1, s_1) \cdot ((a_2, \underline{b}_2, s_2) \cdot (a_3, \underline{b}_3, s_3))$.

Next, as

$$\begin{aligned}&(a, \underline{b}, s) \cdot (1, \underline{0}, e_0) = (a, \underline{b}, s), && (1, \underline{0}, e_0) \cdot (a, \underline{b}, s) = (a, \underline{b}, s),\\ &(a, \underline{b}, s) \cdot (\frac{1}{a}, -\frac{\overline{s}\underline{b}s}{a}, \overline{s}) = (1, \underline{0}, e_0), && (\frac{1}{a}, -\frac{\overline{s}\underline{b}s}{a}, \overline{s}) \cdot (a, \underline{b}, s) = (1, \underline{0}, e_0),\end{aligned}$$

$(1, \underline{0}, e_0)$ is the identity of V, and $(\frac{1}{a}, -\frac{\overline{s}\underline{b}s}{a}, \overline{s})$ is the unique inverse of $(a, \underline{b}, s)$.

So V is a non-abelian group with respect to the multiplication law (2.3.2). ■

V is called a generalized affine group.

Theorem 2.3.2 *V is a non-unimodular group; its left and right Haar measures are given by $\frac{da}{a^{m+1}} d\underline{b} ds$ and $\frac{da}{a} d\underline{b} ds$ respectively, ds being the Haar measure of* Spin(m).

Proof. Let g be integrable on V with respect to $\frac{d\underline{b}dads}{a^{m+1}}$, then for any $(a', \underline{b}', s') \in V$,

$$\begin{aligned}\int_V g((a', \underline{b}', s') \cdot (a, \underline{b}, s)) \frac{d\underline{b}dads}{a^{m+1}} &= \int_0^\infty \int_{\mathbb{R}^m} \int_{S^{m-1}} g(a'a, \underline{b}' + a's'\underline{b}\overline{s'}, s's) \frac{d\underline{b}dads}{a^{m+1}}\\ &= \int_0^\infty \int_{\mathbb{R}^m} \int_{S^{m-1}} g(\alpha, \underline{\beta}, \gamma) \frac{d\underline{\beta}d\alpha d\gamma}{\alpha^{m+1}},\end{aligned}$$

where we have put $\alpha = a'a$, $\underline{\beta} = \underline{b}' + a's'\underline{b}\overline{s'}$, $\gamma = s's$ and $d\underline{\beta}d\alpha d\gamma = a'^{m+1} d\underline{b}dads$. Thus $\frac{d\underline{b}dads}{a^{m+1}}$ is the left Haar measure of V.

Similarly, let g be integrable on V with respect to $\frac{d\underline{b}dads}{a}$, then for any $(a', \underline{b}', s') \in V$,

$$\int_U g((a,\underline{b},s)\cdot(a',\underline{b}',s'))\frac{d\underline{b}dads}{a} = \int_0^\infty \int_{\mathbb{R}^m} \int_{S^{m-1}} g(aa', \underline{b} + as\underline{b}'\overline{s}, ss')\frac{d\underline{b}dads}{a}.$$

Put $\alpha = aa'$, $\underline{\beta} = \underline{b} + as\underline{b}'\overline{s}$, $\gamma = s's$, then $d\alpha d\underline{\beta} d\gamma = a'dad\underline{b}ds$. As

$$\int_0^\infty \int_{\mathbb{R}^m} \int_{S^{m-1}} g(aa', \underline{b} + as\underline{b}'\overline{s}, ss')\frac{d\underline{b}dads}{a} = \int_0^\infty \int_{\mathbb{R}^m} \int_{S^{m-1}} g(\alpha, \underline{\beta}, \gamma)\frac{d\underline{\beta}d\alpha d\gamma}{\alpha},$$

it is seen that $\frac{d\underline{b}dads}{a}$ is the right Haar measure of V.

As $\frac{d\underline{b}dads}{a^{m+1}}$ is different from $\frac{d\underline{b}dads}{a}$, V is a non-unimodular group. ■

Theorem 2.3.3 *Let $f \in H^2_+(\mathbb{R}^m)$ (or $H^2_-(\mathbb{R}^m)$), $(a, \underline{b}, s) \in V$. Define*

$$\beta_{(a,\underline{b},s)} f(\underline{x}) = \frac{s}{a^{m/2}} f(\frac{\overline{s}(\underline{x} - \underline{b})s}{a}), \tag{2.3.3}$$

then β is an irreducible and unitary representation of V in $H^2_\pm(\mathbb{R}^m)$ respectively.

Proof. Take $(a_j, \underline{b}_j, s_j) \in V$, $j = 1, 2$, and $f \in L^2(\mathbb{R}^m)$. Then we have

$$\begin{aligned}\beta_{(a_1,\underline{b}_1,s_1)\cdot(a_2,\underline{b}_2,s_2)} f(\underline{x}) &= \beta_{(a_1a_2,\underline{b}_1+a_1s_1\underline{b}_2\overline{s_1},s_1s_2)} f(\underline{x})\\ &= \frac{s_1s_2}{(a_1a_2)^{m/2}} f\left(\frac{\overline{s_1s_2}(\underline{x} - \underline{b}_1 - a_1s_1\underline{b}_2\overline{s_1})s_1s_2}{a_1a_2}\right)\\ &= \frac{s_1s_2}{(a_1a_2)^{m/2}} f\left(\frac{\overline{s_1s_2}(\underline{x} - \underline{b}_1)s_1s_2 - a_1\overline{s_2}\underline{b}_2s_2}{a_1a_2}\right)\end{aligned}$$

and

$$\begin{aligned}\beta_{(a_1,\underline{b}_1,s_1)}\beta_{(a_2,\underline{b}_2,s_2)} f(\underline{x}) &= \frac{s_1}{a_1^{m/2}}\beta_{(a_2,\underline{b}_2,s_2)} f(\frac{\overline{s_1}(\underline{x} - \underline{b}_1)s_1}{a_1})\\ &= \frac{s_1}{a_1^{m/2}}\frac{s_2}{a_2^{m/2}} f\left(\frac{\overline{s_2}(\frac{\overline{s_1}(\underline{x}-\underline{b}_1)s_1}{a_1} - \underline{b}_2)s_2}{a_2}\right)\\ &= \frac{s_1s_2}{(a_1a_2)^{m/2}} f\left(\frac{\overline{s_2}\,\overline{s_1}(\underline{x} - \underline{b}_1)s_1s_2 - a_1\overline{s_2}\underline{b}_2s_2}{a_1a_2}\right),\end{aligned}$$

and also $\beta_{(a_1,\underline{b}_1,s_1)}\beta_{(a_2,\underline{b}_2,s_2)} = \beta_{(a_1,\underline{b}_1,s_1)\cdot(a_2,\underline{b}_2,s_2)}$.

By a straightforward calculation it is seen that $\|\beta_{(a,\underline{b},s)} f\|_2 = \|f\|_2$.

Let $f \in H^2_+(\mathbb{R}^m)$ be any fixed nonzero function, i.e., there exists $f_0(\underline{x}) \in L^2(\mathbb{R}^m)$ such that $\hat{f}(\underline{u}) = \chi_+(\underline{u})\hat{f}_0(\underline{u})$. Then

$$\widehat{\beta_{(a,\underline{b},s)}} f(\underline{u}) = a^{m/2}e^{-i\langle\underline{b},\underline{u}\rangle} s\hat{f}(a\overline{s}\underline{u}s) = a^{m/2}e^{-i\langle\underline{b},\underline{u}\rangle} s\chi_+(a\overline{s}\underline{u}s)\hat{f}_0(a\overline{s}\underline{u}s),$$

which means $\beta_{(a,\underline{b},s)} f(\underline{x}) \in H^2_+(\mathbb{R}^m)$.

Let $f(\underline{x})$ be a non-zero function in $H^2_+(\mathbb{R}^m)$, and put $S = \{\beta_{(a,\underline{b},s)}f(\underline{x}) : (a,\underline{b},s) \in V\}$. Assume that $g(\underline{x}) \in H^2_+(\mathbb{R}^m)\backslash S$ is orthogonal to S; then by Plancherel's Theorem,

$$\int_{\mathbb{R}^m} \overline{g(\underline{x})}\beta_{(a,\underline{b},s)}f(\underline{x}) = 0 = \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m} \overline{\widehat{g}(\underline{u})}\widehat{\beta_{(a,\underline{b},s)}f}(\underline{u})d\underline{u} =$$

$$a^{m/2}\int_{\mathbb{R}^m} e^{-i\langle \underline{b},\underline{u}\rangle}\overline{\widehat{g}(\underline{u})}s\widehat{f}(a\bar{s}\underline{u}s)d\underline{u},$$

thus $a^{m/2}\overline{\hat{g}(\underline{u})}s\widehat{f}(a\bar{s}\underline{u}s) = 0$ for almost every $\underline{u} \in \mathbb{R}^m$. As f, g belong to $H^2_+(\mathbb{R}^m)$, and $\hat{g}(\underline{u}) \neq 0$ with a positive measure in $\mathbb{R}^m$, then $a^{m/2}s\hat{f}(a\bar{s}\underline{u}s) = 0 \Leftrightarrow \hat{f}(\underline{u}) = 0$, and hence that $f(\underline{x}) \equiv 0$, which contradicts the assumption that f is a non-zero function. So β is an irreducible representation of V in $H^2_+(\mathbb{R}^m)$.

Similarly β is proved to be irreducible and unitary in the space $H^2_-(\mathbb{R}^m)$. ■

3 Covariant Functional Calculi

Let X be a Banach space, let $L(X)$ be the space of linear bounded operators from X to X. Let $X_{(m)} = X \otimes_{\mathbb{R}} \mathbf{R}_{(m)}$, and let $L_{(m)}(X_{(m)})$ be the space of linear bounded operators from $X_{(m)}$ to $X_{(m)}$. By natural embeddings $X \subset X_{(m)}$, $L(X) \subset L_{(m)}(X_{(m)})$.

Denote by $A = A_1e_1 + ... + A_me_m$ an m-tuple operator, where the $A_j \in L(X)$ do not mutually commute, but however commute with the vectors in $\mathbf{R}_{(m)}$. For $\langle A,\underline{u}\rangle = \sum_{j=1}^m u_jA_j$, let $\sigma(\langle A,\underline{u}\rangle)$ be the spectrum of $\langle A,\underline{u}\rangle$. Assume that A satisfies the condition $\sigma(\langle A,\underline{u}\rangle) \subset \mathbb{R}$ for all $\underline{u} \in \mathbb{R}^m$; then the joint spectrum $\gamma(A)$ of A is a bounded subset of $\mathbb{R}^m$ (see [7]). Obviously, $A \in L_{(m)}(X_{(m)})$.

Next we will use the abstract theory of [8] to construct functional calculi for the affine groups U and V.

3.1 Covariant Functional Calculus derived from U

It is clear that $\psi_n^{a,\underline{b}}(\underline{x}) = \rho_{(a,\underline{b})}\psi_n(\underline{x})$. Denote the CHCWT of $f \in L^2(\mathbb{R}^m)$ and $g \in L^2(\mathbb{R}^m)$ by

$$F_n(a,\underline{b}) = \langle \rho_{(a,\underline{b})}\psi_n, f\rangle = \frac{1}{(2\pi)^m}\langle \widehat{\rho_{(a,\underline{b})}\psi_n}, \hat{f}\rangle \tag{3.1.1}$$

and

$$G_n(a,\underline{b}) = \langle \rho_{(a,\underline{b})}\psi_n, g\rangle = \frac{1}{(2\pi)^m}\langle \widehat{\rho_{(a,\underline{b})}\psi_n}, \hat{g}\rangle \tag{3.1.2}$$

respectively. The following Parseval formula holds (see [3]):

$$[G_n, F_n] = \langle g, f\rangle = \frac{1}{(2\pi)^m}\langle \hat{g}, \hat{f}\rangle, \tag{3.1.3}$$

where the inner product of $L_2(U, C_n^{-1}a^{-(m+1)}dad\underline{b})$ is defined by

$$[G_n, F_n] = \frac{1}{C_n}\int_U \overline{G_n(a,\underline{b})}F_n(a,\underline{b})\frac{da}{a^{m+1}}d\underline{b}. \tag{3.1.4}$$

Here C_n is the so-called admissibility constant given by

$$C_n = \int_0^{+\infty} |\widehat{\psi_n}(a\underline{u})|^2 \frac{da}{a} = \frac{1}{A_m} \int_{\mathbb{R}^m} \frac{|\widehat{\psi_n}(\underline{u})|^2}{|\underline{u}|^m} d\underline{u} = (2\pi)^m \frac{(n-1)!}{2}. \tag{3.1.5}$$

The original function f is recovered from (3.1.3) in the $L^2(\mathbb{R}^m)$-norm by

$$f(\underline{x}) = \frac{1}{C_n} \int_U (\rho_{(a,\underline{b})}\psi_n)(\underline{x}) \langle (\rho_{(a,\underline{b})}\psi_n)(\underline{y}), f(\underline{y}) \rangle \frac{da}{a^{m+1}} d\underline{b}. \tag{3.1.6}$$

Define $\mathbf{\Psi}_n : L^2(\mathbb{R}^m) \to L^2(\mathbb{R}^m)$ by

$$\mathbf{\Psi}_n(f)(\underline{x}) = \int_{\mathbb{R}^m} \Psi_n(\underline{x}, \underline{y}) f(\underline{y}) d\underline{y}, \tag{3.1.7}$$

where

$$\Psi_n(\underline{x}, \underline{y}) = \frac{1}{C_n} \int_U \rho_{(a,\underline{b})}\psi_n(\underline{x}) \overline{(\rho_{(a,\underline{b})}\psi_n)(\underline{y})} \frac{da}{a^{m+1}} d\underline{b}. \tag{3.1.8}$$

It follows from (3.1.6) that $\Psi_n(\underline{x}, \underline{y})$ is the reproducing kernel of $\mathbf{\Psi}_n$. Because the Clifford-Hermite wavelet $\psi_n(\underline{x})$ has a component $e^{-\frac{|\underline{x}|^2}{2}}$, by Fubini's Theorem the order of integration may be interchanged from (3.1.6) to (3.1.8).

Let $e^{i\langle A, \underline{u}\rangle}$ be defined by the Trotter-Daletskii formula. The Weyl calculus is defined by (see [1])

$$\mathcal{W}_A(f) = \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} e^{i\langle A, \underline{u}\rangle} \hat{f}(\underline{u}) d\underline{u}. \tag{3.1.9}$$

Assume $\overline{e^{i\langle A, \underline{u}\rangle}} = e^{-i\langle A, \underline{u}\rangle}$. Substituting $\hat{g}(\underline{u}) = e^{-i\langle A, \underline{u}\rangle}$ in (3.1.2) and (3.1.3) yields

$$\mathcal{W}_A(f) = \frac{1}{C_n} \int_U \overline{G_{n,A}(a, \underline{b})} F_n(a, \underline{b}) \frac{da}{a^{m+1}} d\underline{b}, \tag{3.1.10}$$

where $G_{n,A}(a, \underline{b})$ may also be written as

$$\begin{aligned} G_{n,A}(a, \underline{b}) &= \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} \overline{\widehat{\rho_{(a,\underline{b})}\psi_n}(\underline{u})} e^{-i\langle A, \underline{u}\rangle} d\underline{u} \\ &= \frac{a^{m/2}}{(2\pi)^m} \int_{\mathbb{R}^m} e^{i\langle \underline{b}, \underline{u}\rangle} \overline{\widehat{\psi_n}(a\underline{u})} e^{-i\langle A, \underline{u}\rangle} d\underline{u} \\ &= \frac{1}{(2\pi)^m a^{m/2}} \int_{\mathbb{R}^m} \overline{\widehat{\psi_n}(\underline{u})} \exp\left(i \langle \frac{\underline{b} - A}{a}, \underline{u} \rangle \right) d\underline{u} \\ &= \overline{\frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} \widehat{\psi_n}(\underline{u}) \frac{1}{a^m/2} \exp\left(i \langle \frac{A - \underline{b}}{a}, \underline{u} \rangle \right) d\underline{u}}. \end{aligned} \tag{3.1.11}$$

So the Weyl calculus $\mathcal{W}_A(f)$ in (3.1.9) is reobtained from (3.1.10).

Denote by $L(L_{(m)}(X_{(m)}))$ the space of bounded operators from $L_{(m)}(X_{(m)})$ to $L_{(m)}(X_{(m)})$, and by $\mathcal{B}(U, L(L_{(m)}(X_{(m)})))$ be the mappings from U to $L(L_{(m)}(X_{(m)}))$. As $e^{i\langle A, \underline{u}\rangle} \in L_{(m)}(X_{(m)})$, by (3.1.11), $G_{n,A}(a, \underline{b}) \in \mathcal{B}(U, L(L_{(m)}(X_{(m)})))$.

Take $(a, \underline{b}) \in U$, and define

$$\tilde{\rho}_{(a,\underline{b})} e^{i\langle \underline{u}, A\rangle} = \frac{1}{a^{m/2}} \exp\left(i\langle \frac{A-\underline{b}}{a}, \underline{u}\rangle \right), \tag{3.1.12}$$

then $\tilde{\rho}_{(a,\underline{b})} e^{i\langle A,\underline{u}\rangle}$ is a continuous operator-valued wavelet in $\mathcal{B}(U, L(L_{(m)}(X_{(m)})))$.

Theorem 3.1.1 *$\tilde{\rho}$ is a representation of U in the set of all operator-valued wavelets $e^{i\langle A,\underline{u}\rangle}$.*

Proof. Take $(a, \underline{b}) \in U$, and $(c, \underline{d}) \in U$. Then

$$\begin{aligned}
\tilde{\rho}_{(c,\underline{d})} \tilde{\rho}_{(a,\underline{b})} \exp(i\langle A, \underline{u}\rangle) &= \frac{1}{c^{m/2}} \tilde{\rho}_{(a,\underline{b})} \exp(i\langle \frac{A-\underline{d}}{c}, \underline{u}\rangle) \\
&= \frac{1}{c^{m/2}} \frac{1}{a^{m/2}} \exp(i\langle \frac{\frac{A-\underline{d}}{c} - \underline{b}}{a}, \underline{u}\rangle) \\
&= \frac{1}{(ac)^{m/2}} \exp(i\langle \frac{A-\underline{d}-c\underline{b}}{ac}, \underline{u}\rangle).
\end{aligned}$$

On the other hand,

$$\tilde{\rho}_{(c,\underline{d})\cdot(a,\underline{b})} \exp(i\langle A, \underline{u}\rangle) = \tilde{\rho}_{(ac,\underline{d}+c\underline{b})} \exp(i\langle A, \underline{u}\rangle) = \frac{1}{(ac)^{m/2}} \exp(i\langle \frac{A-\underline{d}-c\underline{b}}{ac}, \underline{u}\rangle).$$

So we have $\tilde{\rho}_{(c,\underline{d})} \tilde{\rho}_{(a,\underline{b})} \exp(i\langle A, \underline{u}\rangle) = \tilde{\rho}_{(c,\underline{d})\cdot(a,\underline{b})} \exp(i\langle A, \underline{u}\rangle)$. ∎

Now know from (3.1.11), (3.1.12) and Theorem 3.1.1 it follows that

$$\overline{G_{n,A}(a, \underline{b})} = \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} \hat{\psi}_n(\underline{u}) \tilde{\rho}_{(a,\underline{b})} e^{i\langle A,\underline{u}\rangle} d\underline{u} = \tilde{\rho}_{(a,\underline{b})} \mathcal{W}_A(\psi_n) \tag{3.1.13}$$

and

$$\tilde{\rho}_{(c,\underline{d})} \overline{G_{n,A}(a, \underline{b})} = \tilde{\rho}_{(c,\underline{d})} \tilde{\rho}_{(a,\underline{b})} \mathcal{W}_A(\psi_n) = \tilde{\rho}_{(c,\underline{d})\cdot(a,\underline{b})} \mathcal{W}_A(\psi_n). \tag{3.1.14}$$

Definition 3.1.2 *Let $\underline{x} \in \mathbb{R}^m$ be embedded in U as $(1, \underline{x})$. Define*

$$\Psi_{n,A}(\underline{x}, \underline{y}) \triangleq \frac{1}{C_n} \int_U \overline{G_{n,A}(a, \underline{b})}\, \overline{(\rho_{(a,\underline{b})} \rho_{(1,\underline{x})} \psi_n)(\underline{y})} \frac{da}{a^{m+1}} d\underline{b}, \tag{3.1.15}$$

and

$$\mathbf{\Psi}_{n,A} : f \in L^2(\mathbb{R}^m) \to [\mathbf{\Psi}_{n,A} f](\underline{x}) = \int_{\mathbb{R}^m} \Psi_{n,A}(\underline{x}, \underline{y}) f(\underline{y}) d\underline{y}. \tag{3.1.16}$$

Then $[\mathbf{\Psi}_{n,A} f](\underline{x})$ is a functional calculus from $L^2(\mathbb{R}^m)$ to $\mathcal{B}(U, L(L_{(m)}(X_{(m)})))$ parametrized by $\underline{x}$ and A.

Theorem 3.1.3 *Let $\Psi_{n,A}(\underline{x}, \underline{y})$ be defined as in (3.1.15), and $(c, \underline{d}) \in U$. Then*

$$(\tilde{\rho}_{(c,\underline{d})} \Psi_{n,A})(\underline{x}, \underline{y}) = \frac{1}{C_n} \int_U \overline{G_{n,A}(a, \underline{b})}\, \overline{\rho_{(c,\underline{d})^{-1}\cdot(a,\underline{b})\cdot(1,\underline{x})} \psi_n(\underline{y})} \frac{da}{a^{m+1}} d\underline{b}. \tag{3.1.17}$$

Proof. Note that $a^{-(m+1)}dad\underline{b}$ is the left Haar measure of U, and so by Theorem 3.1.1, (3.1.13) and (3.1.14), we have

$$\begin{aligned}\tilde{\rho}_{(c,\underline{d})}\Psi_{n,A}(\underline{x},\underline{y}) &= \frac{1}{C_n}\int_U \tilde{\rho}_{(c,\underline{d})}\tilde{\rho}_{(a,\underline{b})}\mathcal{W}_A(\psi_n)\,\overline{\rho_{(a,\underline{b})\cdot(1,\underline{x})}\psi_n(\underline{y})}\frac{da}{a^{m+1}}d\underline{b}\\ &= \frac{1}{C_n}\int_U \tilde{\rho}_{(c,\underline{d})\cdot(a,\underline{b})}\mathcal{W}_A(\psi_n)\overline{\rho_{(a,\underline{b})\cdot(1,\underline{x})}\psi_n(\underline{y})}\frac{da}{a^{m+1}}d\underline{b}\\ &= \frac{1}{C_n}\int_U \tilde{\rho}_{(a,\underline{b})}\mathcal{W}_A(\psi_n)\overline{\rho_{(c,\underline{d})^{-1}\cdot(a,\underline{b})\cdot(1,\underline{x})}\psi_n(\underline{y})}\frac{da}{a^{m+1}}d\underline{b}.\end{aligned}$$ ■

Theorem 3.1.4 $\mathbf{\Psi}_{n,A}$ *is an intertwining transform between* ρ *and* $\tilde{\rho}$*, more precisely if* $(c,\underline{d}) \in U$*, and* $f \in L^2(\mathbb{R}^m)$*, one has*

$$\tilde{\rho}_{(c,\underline{d})}[\mathbf{\Psi}_{n,A}f(\underline{y})](\underline{x}) = [\mathbf{\Psi}_{n,A}(\rho_{(c,\underline{d})}f)(\underline{y})](\underline{x}). \tag{3.1.18}$$

Proof. By Theorem 3.1.3 and a direct calculation it is shown that

$$\begin{aligned}&[\mathbf{\Psi}_{n,A}(\rho_{(c,\underline{d})}f)(\underline{y})](\underline{x}) = \frac{1}{c^{m/2}}\int_{\mathbb{R}^m}\Psi_{n,A}(\underline{x},\underline{y})f(\frac{\underline{y}-\underline{d}}{c})d\underline{y}\\ &= \int_{\mathbb{R}^m} c^{m/2}\left[\frac{1}{C_n}\int_U \overline{G_{n,A}(a,\underline{b})}\,\overline{(\rho_{(a,\underline{b})\cdot(1,\underline{x})}\psi_n)(\underline{d}+c\underline{y})}\frac{da}{a^{m+1}}d\underline{b}\right]f(\underline{y})d\underline{y}\\ &= \int_{\mathbb{R}^m}\left[\frac{1}{C_n}\int_U \overline{G_{n,A}(a,\underline{b})}\,\overline{\rho_{(c,\underline{d})^{-1}}\rho_{(a,\underline{b})\cdot(1,\underline{x})}\psi_n(\underline{y})}\frac{da}{a^{m+1}}d\underline{b}\right]f(\underline{y})d\underline{y}\\ &= \int_{\mathbb{R}^m}\tilde{\rho}_{(c,\underline{d})}\Psi_{n,A}(\underline{x},\underline{y})f(\underline{y})d\underline{y}\\ &= \tilde{\rho}_{(c,\underline{d})}[\mathbf{\Psi}_{n,A}f(\underline{y})](\underline{x}).\end{aligned}$$ ■

Corollary 3.1.5 $\mathcal{W}_A(f)$ *is the value of* $[\mathbf{\Psi}_{n,A}f](\underline{x})$ *at the identity* $(1,\underline{0})$ *of the affine group* U*, i.e.,* $\mathcal{W}_A(f) = [\mathbf{\Psi}_{n,A}f](\underline{0})$*.*

Proof. By Definition 3.1.2, Theorem 2.2.1 and (3.1.10), we have

$$\begin{aligned}[\mathbf{\Psi}_{n,A}f](\underline{0}) &= \int_{\mathbb{R}^m}\left[\frac{1}{C_n}\int_U \overline{G_{n,A}(a,\underline{b})}\,\overline{(\rho_{(a,\underline{b})\cdot(1,\underline{0})}\psi_n)(\underline{y})}\frac{da}{a^{m+1}}d\underline{b}\right]f(\underline{y})d\underline{y}\\ &= \int_{\mathbb{R}^m}\left[\frac{1}{C_n}\int_U \overline{G_{n,A}(a,\underline{b})}\,\overline{(\rho_{(a,\underline{b})}\psi_n)(\underline{y})}\frac{da}{a^{m+1}}d\underline{b}\right]f(\underline{y})d\underline{y}\\ &= \frac{1}{C_n}\int_U \overline{G_{n,A}(a,\underline{b})}\left[\int_{\mathbb{R}^m}\overline{(\rho_{(a,\underline{b})}\psi_n)(\underline{y})}f(\underline{y})d\underline{y}\right]\frac{da}{a^{m+1}}d\underline{b}\\ &= \mathcal{W}_A(f).\end{aligned}$$ ■

3.2 Covariant Functional Calculus derived from V

From (2.1.7) and (2.3.4) it is easily seen that $\psi_{n,k}^{(a,\underline{b},s)}(\underline{x}) = \overline{\beta_{(a,\underline{b},s)}\psi_{n,k}(\underline{x})}$.

Let $f,g \in L^2(\mathbb{R}^m)$, then

$$F_{n,k}(a,\underline{b},s) = \langle\overline{\beta_{(a,\underline{b},s)}\psi_{n,k}}, f\rangle = \frac{1}{(2\pi)^m}\langle\widehat{\overline{\beta_{(a,\underline{b},s)}\psi_{n,k}}}, \hat{f}\rangle, \tag{3.2.1}$$

$$G_{n,k}(a,\underline{b},s) = \langle \overline{\beta_{(a,\underline{b},s)}\psi_{n,k}}, g\rangle = \frac{1}{(2\pi)^m}\langle \overline{\widehat{\beta_{(a,\underline{b},s)}\psi_{n,k}}}, \hat{g}\rangle, \tag{3.2.2}$$

are called the generalized CHCWT of f and g respectively.

Define the inner product of $L_2(V, C_{n,k}^{-1}a^{-(m+1)}dad\underline{b}ds)$ by

$$[G_{n,k}, F_{n,k}] = \frac{1}{C_{n,k}}\int_V \overline{G_{n,k}(a,\underline{b},s)}F_{n,k}(a,\underline{b},s)\frac{da}{a^{m+1}}d\underline{b}ds, \tag{3.2.3}$$

where $C_{n,k}$ is the admissibility constant given by

$$C_{n,k} = \int_{Spin(m)}\int_0^{+\infty} \overline{\widehat{\psi_{n,k}}}(a\bar{s}\underline{u}s)\,\widehat{\psi_{n,k}}(a\bar{s}\underline{u}s)\frac{da}{a}ds = \int_{\mathbb{R}^m} \overline{\widehat{\psi_{n,k}}(\underline{u})}\widehat{\psi_{n,k}}(\underline{u})\frac{d\underline{u}}{|\underline{u}|^m}. \tag{3.2.4}$$

If $P_k(\underline{x})$ is chosen such that $\overline{\hat{\psi}_{n,k}(\underline{u})}\hat{\psi}_{n,k}(\underline{u})$ is real-valued, then $C_{n,k} < +\infty$ (see [4]).

The following Parseval formula holds: (see [4])

$$[G_{n,k}(a,\underline{b},s), F_{n,k}(a,\underline{b},s)] = \langle g, f\rangle = \frac{1}{(2\pi)^m}\langle \hat{g}, \hat{f}\rangle. \tag{3.2.5}$$

The original function f may then be recovered in the $L^2(\mathbb{R}^m)$-norm from (see [4])

$$\begin{aligned} f(\underline{x}) &= [\beta_{(a,\underline{b},s)}\psi_{n,k}(\underline{x}), F_{n,k}(a,\underline{b},s)] \\ &= \frac{1}{C_{n,k}}\int_V \overline{\beta_{(a,\underline{b},s)}\psi_{n,k}(\underline{x})}F_{n,k}(a,\underline{b},s)\frac{da}{a^{m+1}}d\underline{b}ds. \end{aligned} \tag{3.2.6}$$

Define

$$\begin{aligned} \Psi_{n,k}(\underline{x},\underline{y}) &= [(\beta_{(a,\underline{b},s)}\psi_{n,k})(\underline{x}), (\beta_{(a,\underline{b},s)}\psi_{n,k})(\underline{y})] \\ &= \frac{1}{C_{n,k}}\int_V \overline{(\beta_{(a,\underline{b},s)}\psi_{n,k})(\underline{x})}(\beta_{(a,\underline{b},s)}\psi_{n,k})(\underline{y})\frac{da}{a^{m+1}}d\underline{b}ds \end{aligned} \tag{3.2.7}$$

and the integral operator $\mathbf{\Psi}_{n,k} : L^2(\mathbb{R}^m) \to L^2(\mathbb{R}^m)$ by

$$(\mathbf{\Psi}_{n,k}f)(\underline{x}) = \int_{\mathbb{R}^m} \Psi_{n,k}(\underline{x},\underline{y})f(\underline{y})d\underline{y}, \tag{3.2.8}$$

then (3.2.6) can be rewritten as $f(x) = (\mathbf{\Psi}_{n,k}f)(\underline{x})$. As β is an irreducible and unitary representation in $L^2(\mathbb{R}^m)$ (see Theorem 2.3.3), then obviously $\beta_{(c,\underline{d},s')}\mathbf{\Psi}_{n,k} = \mathbf{\Psi}_{n,k}\beta_{(c,\underline{d},s')}$.

If we substitute $\hat{g}(\underline{u}) = e^{-i\langle A,\underline{u}\rangle}$ into (3.2.5) and (3.2.2) respectively, then we have

$$\mathcal{W}_A(f) = \frac{1}{C_{n,k}}\int_V \overline{G_{n,k,A}(a,\underline{b},s)}F_{n,k}(a,\underline{b},s)\frac{da}{a^{m+1}}d\underline{b}ds \tag{3.2.9}$$

and

$$\begin{aligned}
G_{n,k,A}(a,\underline{b},s) &= \overline{\frac{1}{(2\pi)^m}\int_{\mathbb{R}^m}\widehat{\overline{\beta_{(a,\underline{b},s)}\psi_{n,k}}}(\underline{u})e^{-i\langle A,\underline{u}\rangle}d\underline{u}} \\
&= \overline{\frac{a^{m/2}}{(2\pi)^m}\int_{\mathbb{R}^m}e^{-i\langle \underline{b},\underline{u}\rangle}\widehat{\overline{\psi_{n,k}}}(a\overline{s}\underline{u}s)\overline{s}e^{i\langle A,\underline{u}\rangle}d\underline{u}} \\
&= \overline{\frac{1}{(2\pi)^m a^{m/2}}\int_{\mathbb{R}^m}\widehat{\overline{\psi_{n,k}}}(\overline{s}\underline{u}s)\overline{s}\exp\left(i\langle\frac{A-\underline{b}}{a},\underline{u}\rangle\right)d\underline{u}} \qquad (3.2.10)\\
&= \overline{\frac{1}{(2\pi)^m}\int_{\mathbb{R}^m}\widehat{\overline{\psi_{n,k}}}(\underline{u})s\frac{1}{a^{m/2}}\exp\left(i\langle\frac{A-\underline{b}}{a},s\underline{u}\overline{s}\rangle\right)d\underline{u}} \\
&= \overline{\frac{1}{(2\pi)^m}\int_{\mathbb{R}^m}\widehat{\overline{\psi_{n,k}}}(\underline{u})s\frac{1}{a^{m/2}}\exp\left(i\langle\frac{\overline{s}(A-\underline{b})s}{a},\underline{u}\rangle\right)d\underline{u}}.
\end{aligned}$$

Relation (3.2.9) shows that again the Weyl calculus $\mathcal{W}_A(f)$ is reobtained.

Theorem 3.2.1 *If for any $(a,\underline{b},s)\in V$, put*

$$\tilde{\beta}_{(a,\underline{b},s)}e^{i\langle A,\underline{u}\rangle} = \frac{1}{a^{m/2}}s\,\exp\left(i\langle\frac{\overline{s}(A-\underline{b})s}{a},\underline{u}\rangle\right), \qquad (3.2.11)$$

then $\tilde{\beta}_{(a,\underline{b},s)}$ is a representation of V in the set of all operator-valued wavelets $e^{i\langle A,\underline{u}\rangle}$.

Proof. The proof is similar to that of Theorem 3.1.1 and is omitted. ■

Now by (3.2.10), we have

$$\overline{G_{n,k,A}(a,\underline{b},s)} = \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m}\widehat{\overline{\psi_{n,k}}}(\underline{u})\tilde{\beta}_{(a,\underline{b},s)}e^{i\langle A,\underline{u}\rangle}d\underline{u} = \tilde{\beta}_{(a,\underline{b},s)}\mathcal{W}_A(\overline{\psi_{n,k}}) \qquad (3.2.12)$$

and

$$\tilde{\beta}_{(c,\underline{d},s')}\overline{G_{n,k,A}(a,\underline{b},s)} = \tilde{\beta}_{(c,\underline{d},s')}\tilde{\beta}_{(a,\underline{b},s)}\mathcal{W}_A(\overline{\psi_{n,k}}) = \tilde{\beta}_{(c,\underline{d},s')\cdot(a,\underline{b},s)}\mathcal{W}_A(\overline{\psi_{n,k}}). \qquad (3.2.13)$$

Theorem 3.2.2 *Let $f\in L^2(\mathbb{R}^m)$, let $(a,\underline{b},s)\in V$ and let $\underline{x}\in\mathbb{R}^m$ be embedded in V as $(1,\underline{x},e_0)$. Put*

$$\Psi_{n,k,A}(\underline{x},\underline{y}) = \frac{1}{C_{n,k}}\int_V\overline{G_{n,k,A}(a,\underline{b},s)}(\beta_{(a,\underline{b},s)\cdot(1,\underline{x},e_0)}\psi_{n,k})(\underline{y})\frac{da}{a^{m+1}}d\underline{b}ds \qquad (3.2.14)$$

and

$$[\mathbf{\Psi}_{n,k,A}f](\underline{x}) = \int_{\mathbb{R}^m}\Psi_{n,k,A}(\underline{x},\underline{y})f(\underline{y})d\underline{y}, \qquad (3.2.15)$$

then $\mathbf{\Psi}_{n,k,A}$ is an intertwining integral transform between β and $\tilde{\beta}$. More explicitly, for any $(c,\underline{d},s')\in V$, we have

$$\tilde{\beta}_{(c,\underline{d},s')}[\mathbf{\Psi}_{n,k,A}f](\underline{x}) = \mathbf{\Psi}_{n,k,A}[\beta_{(c,\underline{d},s')}f](\underline{x}). \qquad (3.2.16)$$

Proof. As $\frac{da}{a^{m+1}}d\underline{b}ds$ is the left Haar measure of V, then for any $(c,\underline{d},s')\in V$, we have

$$\begin{aligned}
\tilde{\beta}_{(c,\underline{d},s')}\Psi_{n,k,A}(\underline{x},\underline{y}) &= \frac{1}{C_{n,k}}\int_V\tilde{\beta}_{(c,\underline{d},s')}\overline{G_{n,k,A}(a,\underline{b},s)}(\beta_{(a,\underline{b},s)\cdot(1,\underline{x},e_0)}\psi_{n,k})(\underline{y})\frac{da}{a^{m+1}}d\underline{b}ds \\
&= \frac{1}{C_{n,k}}\int_V\tilde{\beta}_{(c,\underline{d},s')}\tilde{\beta}_{(a,\underline{b},s)}\mathcal{W}_A(\overline{\psi_{n,k}})(\beta_{(a,\underline{b},s)\cdot(1,\underline{x},e_0)}\psi_{n,k})(\underline{y})\frac{da}{a^{m+1}}d\underline{b}ds \\
&= \frac{1}{C_{n,k}}\int_V\tilde{\beta}_{(a,\underline{b},s)}\mathcal{W}_A(\overline{\psi_{n,k}})(\beta_{(c,\underline{d},s')^{-1}\cdot(a,\underline{b},s)\cdot(1,\underline{x},e_0)}\psi_{n,k})(\underline{y})\frac{da}{a^{m+1}}d\underline{b}ds.
\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathbf{\Psi}_{n,k,A}[\beta_{(c,\underline{d},s')}f](\underline{x}) &= \int_{\mathbb{R}^m} \Psi_{n,k,A}(\underline{x},\underline{y})(\beta_{(c,\underline{d},s')}f)(\underline{y})d\underline{y} \\ &= \int_{\mathbb{R}^m} \left[\frac{1}{C_{n,k}}\int_V \overline{G_{n,k,A}(a,\underline{b},s)}\ (\beta_{(c,\underline{d},s')^{-1}}\beta_{(a,\underline{b},s)\cdot(1,\underline{x},e_0)}\psi_{n,k})(\underline{y})\frac{da}{a^{m+1}}d\underline{b}ds\right] f(\underline{y})d\underline{y} \\ &= \int_{\mathbb{R}^m} \tilde{\beta}_{(c,\underline{d},s')}\Psi_{n,k,A}(\underline{x},\underline{y})f(\underline{y})d\underline{y} \\ &= \tilde{\beta}_{(c,\underline{d},s')}[\mathbf{\Psi}_{n,k,A}f](\underline{x}).\end{aligned}$$

∎

Corollary 3.2.3 *$\mathcal{W}_A(f)$ is the value of $[\mathbf{\Psi}_{n,k,A}f](\underline{x})$ at the identity $(1,\underline{0},e_0)$ of the affine group V, i.e., $\mathcal{W}_A(f) = [\mathbf{\Psi}_{n,k,A}f](\underline{0})$.*

Proof. Direct calculations show that

$$\begin{aligned}[\mathbf{\Psi}_{n,k,A}f](\underline{0}) &= \int_{\mathbb{R}^m} \left[\frac{1}{C_{n,k}}\int_V \overline{G_{n,k,A}(a,\underline{b},s)}(\beta_{(a,\underline{b},s)}\psi_{n,k})(\underline{y})\frac{da}{a^{m+1}}d\underline{b}ds\right] f(\underline{y})d\underline{y} \\ &= \frac{1}{C_{n,k}}\int_V \overline{G_{n,k,A}(a,\underline{b},s)}\left[\int_{\mathbb{R}^m}(\beta_{(a,\underline{b},s)}\psi_{n,k})(\underline{y})f(\underline{y})d\underline{y}\right]\frac{da}{a^{m+1}}d\underline{b}ds \\ &= \frac{1}{C_{n,k}}\int_V \overline{G_{n,k,A}(a,\underline{b},s)}F_{n,k}(a,\underline{b},s)\frac{da}{a^{m+1}}d\underline{b}ds \\ &= \mathcal{W}_A(f).\end{aligned}$$

∎

Remark 3.2.4 *As $\psi_n(\underline{x})$ and $\psi_{n,k}(\underline{x})$ are rapidly decreasing functions, they enjoy very nice properties and are possibly suitable to construct a functional calculus for both unbounded and non-commuting operators. In as far the results contained in this paper can be generalized to that kind of operators is a subject of further research.*

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References

[1] R. F. V. Anderson. *The Weyl functional calculus*. J. Funct. Anal., 1969, **4**: 240-267.

[2] F. Brackx, R. Delanghe and F. Sommen. *Clifford Analysis*. Research Notes in Mathematics **76**. London: Pitman, 1982.

[3] F. Brackx and F. Sommen. *Clifford-Hermite wavelets in Euclidean space*. Journal of Fourier analysis and Applications, 2000, **6**(3): 299-310.

[4] F. Brackx and F. Sommen. *The generalized Clifford-Hermite continuous wavelet transform*. Advances in Applied Clifford Algebras, 2001, **11(S1)**: 219-231.

[5] R. Delanghe, F. Sommen and V. Soucek. *Clifford Algebra and Spinor Valued Functions: A Function Theory for the Dirac Operator*. Math. and its Appl. **53**. Dordrecht: Kluwer Acad. Publ., 1992.

[6] R. Delanghe and Tao Qian. *Half Dirichlet Problems and Decompositions of Poisson Kernels*. Advances In Clifford Algebras. 2007, **17**: 383-393.

[7] Brian Jefferies. *Spectral Properties of Noncommuting operators*. Lecture Notes in Mathematics **1843**. Springer-Verlag, 2004.

[8] Vladimir V. Kisil. *Wavelets in Banach spaces*. Acta Appl. Math. 1999, **59**(1): 79-109.

[9] Vladimir V. Kisil. *Möbius transformations and monogenic functional calculus*. ERA Amer. Math. Soc., 1996, **2**: 26-33.

[10] Vladimir V. Kisil. *Spectrum as the support of functional calculus*. Proceedings of the S. Banach Conference (LVIV), Elsevier, 2003.

[11] Vladimir V. Kisil. *Monogenic calculus as an intertwing operator*. Bull. Belg. Math. Soc. Simon Stevin, 2005, **11**: 739-757.

[12] M. W. Wong. *Wavelet Transforms and Localization Operators*. Operator Theory: Advances and Applications, Vol. **136**. Berlin: Birkhäuser, 2002.

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