Crofton formulas and convexity condition for secantoptics

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Abstract

The aim of this paper is to study some properties of secantoptics, defined in [8]. We show that any evolutoid of a given oval C is a hedgehog and that any secantoptic of an oval C, is an isoptic of a pair of required evolutoids. We prove some Crofton-type formulas for secantoptics and give a necessary and sufficient condition for a secantoptic to be convex.

1 Introduction

An isoptic of a given curve C is a set of points from which this curve is seen under the fixed angle α, where $\alpha \in (0, \pi)$. In [6] A. Miernowski and W. Mozgawa define and study isoptics of pairs of nested closed strictly convex curves. Now, we consider curves which must not be nested. Instead of strictly convex curves we consider hedgehogs defined by R. Langevin, G. Levitt and H. Rosenberg in [3] and later studied by Y. Martinez-Maure in [4]. A hedgehog Γ is a curve which can be parametrized by the formula

$$z(t) = \psi(t)e^{it} + \dot{\psi}(t)ie^{it}, \tag{1.1}$$

where $h(\cos t, \sin t) = \psi(t)$, $h \in C^2(\mathbb{S}^1, \mathbb{R})$ is called the support function of Γ. The hedgehog is the envelope of the family of lines given by the equation

$$x \cos t + y \sin t = p(t). \tag{1.2}$$

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In [8] we defined secantoptics, a generalization of isoptic curves of ovals. Recall that a closed convex curve C of class C^2 with nonvanishing curvature is said to be an oval. To real this notion we choose a coordinate system with the origin O in the interior of the oval C and parametrize C by the equation

$$z(t) = p(t)e^{it} + \dot{p}(t)ie^{it} \quad \text{for} \quad t \in [0, 2\pi], \tag{1.3}$$

where $p \in C^3$ and the curvature radius $R(t) = p(t) + \ddot{p}(t) > 0$. Let $\beta \in [0, \pi)$, $\gamma \in [0, \pi - \beta)$ and $\alpha \in (\beta + \gamma, \pi)$ be fixed angles. We take the line $l_1(t)$ tangent to the oval C at a point $z(t)$. We construct a secant line $s_1(t)$ of C rotating $l_1(t)$ around the point $z(t)$ through the angle $-\beta$. Let us take another tangent line $l_2(t) = l_1(t + \alpha - \beta - \gamma)$ at the point $z(t + \alpha - \beta - \gamma)$ and let $s_2(t)$ be a secant obtained by rotating $l_2(t)$ around the tangency point through angle γ. Then $s_1(t)$ and $s_2(t)$ intersect forming a fixed angle α.

Definition 1.1. *The set of intersection points $z_{\alpha,\beta,\gamma}(t)$ of $s_1(t)$ and $s_2(t)$ for $t \in [0, 2\pi]$ form a curve which we call a secantoptic $C_{\alpha,\beta,\gamma}$ of an oval C.*

The equation of the secantoptic $C_{\alpha,\beta,\gamma}$ can be written as

$$z_{\alpha,\beta,\gamma}(t) = (p(t) + \lambda(t)\sin\beta + i(\dot{p}(t) + \lambda(t)\cos\beta))e^{it}, \tag{1.4}$$

where

$$\begin{aligned} \lambda(t) = & \frac{1}{\sin\alpha}(p(t + \alpha - \beta - \gamma)\cos\gamma + \dot{p}(t + \alpha - \beta - \gamma)\sin\gamma \\ & - \dot{p}(t)\sin(\alpha - \beta) - p(t)\cos(\alpha - \beta)). \end{aligned} \tag{1.5}$$

For the further purposes we introduce also a function

$$\begin{aligned} \mu(t) = & \frac{1}{\sin\alpha}(p(t + \alpha - \beta - \gamma)\cos(\alpha - \gamma) - \dot{p}(t + \alpha - \beta - \gamma)\sin(\alpha - \gamma) \\ & + \dot{p}(t)\sin\beta - p(t)\cos\beta). \end{aligned} \tag{1.6}$$

2 Isoptics of pairs of hedgehogs.

Let us fix a coordinate system with the origin O at a point in the plane and let Γ_1 and Γ_2 be two hedgehogs given by support functions $h_1(\cos t, \sin t) = \psi_1(t)$ and $h_2(\cos t, \sin t) = \psi_2(t)$, respectively. Therefore we know equations of these hedgehogs

$$\Gamma_1 : \; z_1(t) = \psi_1(t)e^{it} + \dot{\psi}_1(t)ie^{it}, \tag{2.1}$$

$$\Gamma_2 : \; z_2(t) = \psi_2(t)e^{it} + \dot{\psi}_2(t)ie^{it}. \tag{2.2}$$

Let $l(t)$ be the tangent line to the curve Γ_1 at a point $z_1(t)$ and let $m(t)$ be the tangent to Γ_2 at a point $z_2(t)$. Then, for a given $\alpha \in (0, \pi)$, lines $l(t)$ and $m(t + \alpha)$ form an angle α.

Definition 2.1. *The set of intersection points $z_\alpha^{\Gamma_1\Gamma_2}(t)$ of tangent lines $l(t)$ and $m(t + \alpha)$ for $t \in [0, 2\pi]$ form a curve which we call an α-isoptic $C_\alpha^{\Gamma_1\Gamma_2}$ of the pair Γ_1 and Γ_2.*

Note that we drop here the assumption from [6] that Γ_1 and Γ_2 have to be nested. Therefore we do not have to define isoptics of first and second kind, but we can consider isoptics of the pair Γ_2 and Γ_1. The lines $l(t)$ and $m(t+\alpha-\pi)$ form an angle α. If we put $t=\tau+\pi-\alpha$, then the set of intersection points of lines $m(t+\alpha-\pi)=m(\tau)$ and $l(t)=l(\tau+\pi-\alpha)$ for $\tau\in[0,2\pi]$ forms an isoptic $C_{\pi-\alpha}^{\Gamma_2\Gamma_1}$ of the pair Γ_2 and Γ_1.

We fix $\alpha\in(0,\pi)$ and define a vector

$$q(t)=z_1(t)-z_2(t+\alpha)=(B(t)-ib(t))e^{it}, \tag{2.3}$$

where

$$B(t)=\psi_1(t)-\psi_2(t+\alpha)\cos\alpha+\dot{\psi}_2(t+\alpha)\sin\alpha, \tag{2.4}$$

$$b(t)=\psi_2(t+\alpha)\sin\alpha+\dot{\psi}_2(t+\alpha)\cos\alpha-\dot{\psi}_1(t). \tag{2.5}$$

If we write the vector $q(t)$ by the formula

$$q(t)=M(t)ie^{i(t+\alpha)}-L(t)ie^{it}, \tag{2.6}$$

then we get $L(t)$ and $M(t)$ in terms of the support functions of the considered hedgehogs

$$L(t)=-\dot{\psi}_1(t)-\psi_1(t)\cot\alpha+\psi_2(t+\alpha)\frac{1}{\sin\alpha}, \tag{2.7}$$

$$M(t)=-\psi_1(t)\frac{1}{\sin\alpha}-\dot{\psi}_2(t+\alpha)+\psi_2(t+\alpha)\cot\alpha. \tag{2.8}$$

Any point on an α-isoptic of a pair of curves Γ_1 and Γ_2 can be expressed as

$$z_\alpha^{\Gamma_1\Gamma_2}(t)=z_1(t)+L(t)ie^{it}=z_2(t+\alpha)+M(t)ie^{i(t+\alpha)}, \tag{2.9}$$

where $L(t)$ and $M(t)$ are some real functions. We can write an equation of $C_\alpha^{\Gamma_1\Gamma_2}$ by the following formula

$$z_\alpha^{\Gamma_1\Gamma_2}(t)=\psi_1(t)e^{it}+(\psi_2(t+\alpha)\frac{1}{\sin\alpha}-\psi_1(t)\cot\alpha)ie^{it}. \tag{2.10}$$

If we introduce the notation

$$\rho(t)=\psi_1(t)+\dot{\psi}_2(t+\alpha)\frac{1}{\sin\alpha}-\dot{\psi}_1(t)\cot\alpha, \tag{2.11}$$

then the first derivative of $z_\alpha^{\Gamma_1\Gamma_2}(t)$ can be written as

$$\dot{z}_\alpha^{\Gamma_1\Gamma_2}(t)=-L(t)e^{it}+\rho(t)ie^{it}. \tag{2.12}$$

Note that

$$\left|\dot{z}_\alpha^{\Gamma_1\Gamma_2}(t)\right|^2=\frac{1}{\sin^2\alpha}|q(t)|^2, \tag{2.13}$$

hence $C_\alpha^{\Gamma_1\Gamma_2}$ can be a nonregular curve, for if $z_1(t)=z_2(t+\alpha)$ for some $t\in[0,2\pi]$, then $|\dot{z}_\alpha^{\Gamma_1\Gamma_2}(t)|=0$.

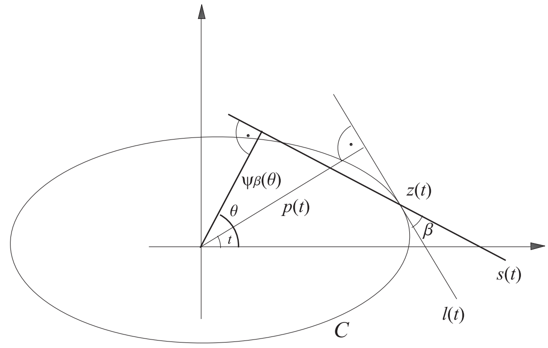

Figure 1:

3 Evolutoid of an oval is a hedgehog

Let C be a given oval. Note that C is an envelope of the family of its tangent lines. We define the family S_β of the secants $s(t)$ to oval C obtained by rotating the tangent line $l(t)$ about the tangency point $z(t)$ through angle β for each $t \in [0, 2\pi]$. The family S_β can be expressed by the formula

$$x \cos\theta + y \sin\theta = \psi_\beta(\theta), \tag{3.1}$$

where $\theta = t + \beta$ and $(x, y) \in s(t)$ is the point $z(t)$ on the oval C. Hence

$$(x, y) = (p(t)\cos t - \dot{p}(t)\sin t,\ p(t)\sin t + \dot{p}(t)\cos t), \quad \text{where} \quad t = \theta - \beta. \tag{3.2}$$

We can find the function $\psi_\beta(\theta)$ from (3.1)

$$\psi_\beta(\theta) = p(\theta - \beta)\cos\beta + \dot{p}(\theta - \beta)\sin\beta, \quad \theta \in [0, 2\pi]. \tag{3.3}$$

Note that $\psi_\beta \in C^\infty$ is a support function of an envelope Γ_β of the family S_β and Γ_β is parametrized by

$$z^\beta(t) = \psi_\beta(t)e^{it} + \dot{\psi}_\beta(t)ie^{it}. \tag{3.4}$$

Since $\psi_\beta(t)$ is at least of class $C^2(\mathbb{R})$, the curve Γ_β is a hedgehog.

We need to recall the definition of the evolutoid given in [2].

Definition 3.1. *The evolutoid of angle δ of a curve $f(s)$ is the envelope of the lines making a fixed angle δ with the normal vector at $f(s)$.*

Hence the curve Γ_β is the evolutoid of angle $\frac{\pi}{2} + \beta$ of an oval C as the envelope of the family of secants to this oval.

Corollary 3.1. *Any evolutoid of an oval is a hedgehog.*

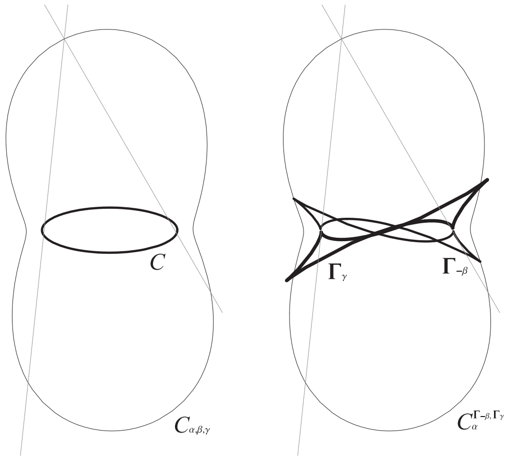

Figure 2:

4 Secantoptic as isoptic of pair of evolutoids

Theorem 4.1. *The isoptic $C_\alpha^{\Gamma_{-\beta}\Gamma_\gamma}$ and the secantoptic $C_{\alpha,\beta,\gamma}$ of a given oval C coincide if $\beta \in [0,\pi)$, $\gamma \in [0,\pi-\beta)$ and $\alpha \in (\beta+\gamma,\pi)$.*

Proof. Consider two evolutoids $\Gamma_{-\beta}$ and Γ_γ of an oval C, given by support functions

$$\psi_{-\beta}(t) = p(t+\beta)\cos\beta - \dot{p}(t+\beta)\sin\beta \tag{4.1}$$

and

$$\psi_\gamma(t) = p(t-\gamma)\cos\gamma + \dot{p}(t-\gamma)\sin\gamma, \tag{4.2}$$

respectively. The equation of the isoptic $C_\alpha^{\Gamma_{-\beta}\Gamma_\gamma}$, where $\beta \in [0,\pi)$, $\gamma \in [0,\pi-\beta)$ and $\alpha \in (\beta+\gamma,\pi)$ can be written as

$$z_\alpha^{\Gamma_{-\beta}\Gamma_\gamma}(t) = \psi_{-\beta}(t)e^{it} + \left(\psi_\gamma(t+\alpha)\frac{1}{\sin\alpha} - \psi_{-\beta}(t)\cot\alpha\right)ie^{it} \tag{4.3}$$

and it depends on the support function of C. Since

$$\begin{aligned} z_\alpha^{\Gamma_{-\beta}\Gamma_\gamma}(t-\beta) = {} & \frac{1}{\sin\alpha}[(p(t)\cos\beta\sin(\alpha-\beta) - \dot{p}(t)\sin\beta\sin(\alpha-\beta) \\ & + p(t+\alpha-\beta-\gamma)\sin\beta\cos\gamma + \dot{p}(t+\alpha-\beta-\gamma)\sin\beta\sin\gamma)e^{it} \\ & + (-p(t)\cos\beta\cos(\alpha-\beta) + \dot{p}(t)\sin\beta\cos(\alpha-\beta) \\ & + p(t+\alpha-\beta-\gamma)\cos\beta\cos\gamma + \dot{p}(t+\alpha-\beta-\gamma)\cos\beta\sin\gamma)ie^{it}] \\ = {} & z_{\alpha,\beta,\gamma}(t), \end{aligned} \tag{4.4}$$

for a given oval C the isoptic $C_\alpha^{\Gamma_{-\beta}\Gamma_\gamma}$ is the same curve as the secantoptic $C_{\alpha,\beta,\gamma}$ under the assumption that $\beta \in [0,\pi)$, $\gamma \in [0,\pi-\beta)$ and $\alpha \in (\beta+\gamma,\pi)$. ∎

We introduce the following notations which will simplify our next calculations

$$\begin{aligned} L(t) &= R(t)\sin\beta + \lambda(t), \\ M(t) &= \mu(t) - R(t+\alpha-\beta-\gamma)\sin\gamma, \\ Q(t) &= M(t)ie^{i(t+\alpha-\beta)} - L(t)ie^{i(t-\beta)}. \end{aligned} \tag{4.5}$$

Let us remind that all secantoptics of a curve C for a fixed $\beta \in [0,\pi)$, fixed $\gamma \in [0,\pi-\beta)$ and various $\alpha \in (\beta+\gamma,\pi)$ form two parameters family of curves $F_{\beta,\gamma}(\alpha,t)$ (see [8]). Hence, we have the mapping

$$F_{\beta,\gamma} : (\beta+\gamma,\pi)\times(0,2\pi) \mapsto \Omega\setminus\zeta, \tag{4.6}$$

where Ω denotes the exterior of the curve C and $\zeta(\alpha)$ is a set of points $F_{\beta,\gamma}(\alpha,0)$. Let us remind that the Jacobian of a secantoptic $F_{\beta,\gamma}(\alpha,t) = z_{\alpha,\beta,\gamma}(t)$ can be written as

$$J(F_{\beta,\gamma}) = \frac{-M(t)L(t)}{\sin\alpha} > 0. \tag{4.7}$$

It is easy to see that the mapping $F_{\beta,\gamma}(\alpha,t)$ is a diffeomorphism, similarly as it was for isoptics $F_{0,0}(\alpha,t)$. Note that $C = C^{\Gamma_{-\beta}\Gamma_\gamma}_{\beta+\gamma}$.

5 Crofton integral formulas for secantoptics

For all convex curves the following Crofton integral formula holds

$$\iint_\Omega \frac{\sin\omega}{t_1t_2}\mathrm{d}x\mathrm{d}y = 2\pi^2, \tag{5.1}$$

where t_1 and t_2 are lengths of segments of tangents to C passing through the point $(x,y)\in\Omega$ and $\omega = \angle(t_1,t_2)$. The formula (5.1) can be found in [7] but in [1] authors used the isoptics to prove it. Now, we are going to extend this formula to secantoptics.

Theorem 5.1. *For any oval the following integral formula holds*

$$\iint_{\Omega_1} \frac{\sin\omega}{t_1t_2}\mathrm{d}x\mathrm{d}y = 2\pi^2 - 2\pi(\beta+\gamma), \tag{5.2}$$

where

$$\begin{aligned} \omega &= \pi-\alpha, \\ t_1 &= -M(t), \\ t_2 &= L(t). \end{aligned} \tag{5.3}$$

Proof. We determine the integral (5.1) in our framework using the mapping $F_{\beta,\gamma}$

$$\begin{aligned} \iint_\Omega \frac{\sin\omega}{t_1t_2}\mathrm{d}x\mathrm{d}y &= \int_0^{2\pi}\int_{\beta+\gamma}^{\pi} \frac{\sin\alpha}{L(t)(-M(t))}\frac{-L(t)M(t)}{\sin\alpha}\mathrm{d}\alpha\mathrm{d}t \\ &= \int_0^{2\pi}\int_{\beta+\gamma}^{\pi}\mathrm{d}\alpha\mathrm{d}t = 2\pi^2 - 2\pi(\beta+\gamma). \end{aligned} \tag{5.4}$$

∎

To give a geometric interpretation of the value $2\pi(\beta+\gamma)$ consider the expression

$$\iint_{\Omega_1} \frac{\sin\omega}{t_1 t_2} \mathrm{d}x\mathrm{d}y, \tag{5.5}$$

where Ω_1 is the exterior of an isoptic $C_{\beta+\gamma}$ of an oval C. For $t_1 = |z_\alpha(t) - z(t)| = \lambda_1(\alpha, t)$, $t_2 = |z_\alpha(t) - z(t+\alpha)| = -\mu_1(\alpha, t)$ and $\omega = \pi - \alpha$, where $\alpha \in (\beta+\gamma, \pi)$ we obtain

$$\begin{aligned} \iint_{\Omega_1} \frac{\sin\omega}{t_1 t_2} \mathrm{d}x\mathrm{d}y &= \int_0^{2\pi}\int_{\beta+\gamma}^{\pi} \frac{\sin\alpha}{\lambda_1(t)(-\mu_1(t))} \frac{-\lambda_1(t)\mu_1(t)}{\sin\alpha} \mathrm{d}\alpha\mathrm{d}t \\ &= \int_0^{2\pi}\int_{\beta+\gamma}^{\pi} \mathrm{d}\alpha\mathrm{d}t = 2\pi^2 - 2\pi(\beta+\gamma). \end{aligned} \tag{5.6}$$

Hence $2\pi(\beta+\gamma)$ is the value of the integral of $\frac{\sin\omega}{t_1 t_2}$ over the annulus $C_{\beta+\gamma} \setminus C$.

Remark 5.1. Let C be a circle parametrized by $z(t) = re^{it}$ for $r > 0$ and $t \in [0, 2\pi]$. Let $\beta, \gamma \in [0, \pi/2)$ and $\beta < \gamma$. We can define evolutoids

$$\begin{aligned} \Gamma_{-\beta} &: \; z^{-\beta}(t) = r\cos\beta\, e^{it}, \\ \Gamma_{\gamma} &: \; z^{\gamma}(t) = r\cos\gamma\, e^{it}. \end{aligned} \tag{5.7}$$

Since they are nested, we can apply [6] and define the mapping $F_{\beta,\gamma}(\alpha, t)$ for isoptics of pair $\Gamma_{-\beta}$ and Γ_γ. If

$$\alpha \in \left(\arccos\left(\frac{\cos\gamma}{\cos\beta}\right), \pi\right) \tag{5.8}$$

and $t \in (0, 2\pi)$, then $F_{\beta,\gamma}$ is a diffeomorphism from $\left(\arccos\left(\frac{\cos\gamma}{\cos\beta}\right), \pi\right) \times (0, 2\pi)$ to the exterior of $\Gamma_{-\beta}$. Note that the equation of isoptic $C_{\beta+\gamma}^{\Gamma_{-\beta}\Gamma_\gamma}$ given by the formula

$$z_{\beta+\gamma}^{\Gamma_{-\beta}\Gamma_\gamma}(t) = r\cos\beta e^{it} + r\sin\beta i e^{it} = re^{i(t+\beta)} \tag{5.9}$$

describes the circle C. If we put

$$t_1 = |z^{-\beta}(t-\beta) - z(t)| = L(t), \quad t_2 = |z^{\gamma}(t+\alpha-\beta) - z(t)| = -M(t) \tag{5.10}$$

and $\omega = \pi - \alpha$, then we can write the Crofton integral formula for isoptics $C_\alpha^{\Gamma_{-\beta}\Gamma_\gamma}$ in exterior of C that is for $\alpha \in (\beta+\gamma, \pi)$. Hence we obtain

$$\iint_{\Omega} \frac{\sin\omega}{t_1 t_2} \mathrm{d}x\mathrm{d}y = \int_0^{2\pi}\int_{\beta+\gamma}^{\pi} \mathrm{d}\alpha\mathrm{d}t = 2\pi^2 - 2\pi(\beta+\gamma). \tag{5.11}$$

Let us recall, that $\int_0^{2\pi} p(t)\mathrm{d}t$ means the length of a given convex curve C, which we denote by L_C. In [4] the algebraic length of a hedgehog Γ is defined by the formula

$$L_\Gamma = \int_0^{2\pi} \psi(t)\mathrm{d}t, \tag{5.12}$$

where $\psi(t)$ is the support function of Γ. Hence, for $\Gamma_{-\beta}$ and Γ_γ we get

$$L_{\Gamma_{-\beta}} = \int_0^{2\pi} \psi_{-\beta}(t)\mathrm{d}t = \int_0^{2\pi} (p(t)\cos\beta - \dot{p}(t)\sin\beta)\mathrm{d}t = L_C \cos\beta \tag{5.13}$$

and

$$L_{\Gamma_\gamma} = \int_0^{2\pi} \psi_\gamma(t)\mathrm{d}t = \int_0^{2\pi} (p(t)\cos\gamma + \dot{p}(t)\sin\gamma)\mathrm{d}t = L_C \cos\gamma, \tag{5.14}$$

respectively. In [7] one can find the following Crofton-type formula for convex curves

$$\iint_\Omega \frac{\sin^2\omega}{t_1}\mathrm{d}x\mathrm{d}y = \iint_\Omega \frac{\sin^2\omega}{t_2}\mathrm{d}x\mathrm{d}y = \pi L_C. \tag{5.15}$$

Theorem 5.2. *For any oval the following integral formulas hold*

$$\iint_\Omega \frac{\sin^2\omega}{t_1}\mathrm{d}x\mathrm{d}y = L_{\Gamma_{-\beta}}(\pi - (\beta+\gamma)) + L_{\Gamma_\gamma}\sin(\beta+\gamma), \tag{5.16}$$

$$\iint_\Omega \frac{\sin^2\omega}{t_2}\mathrm{d}x\mathrm{d}y = L_{\Gamma_\gamma}(\pi - (\beta+\gamma)) + L_{\Gamma_{-\beta}}\sin(\beta+\gamma), \tag{5.17}$$

where

$$\begin{aligned} \omega &= \pi - \alpha, \\ t_1 &= -M(t), \\ t_2 &= L(t). \end{aligned} \tag{5.18}$$

Proof. For secantoptics we get

$$\begin{aligned} \iint_\Omega \frac{\sin^2\omega}{t_1}\mathrm{d}x\mathrm{d}y &= \int_0^{2\pi}\int_{\beta+\gamma}^{\pi} \frac{\sin^2\alpha}{L(t)} \frac{(-L(t)M(t))}{\sin\alpha}\,\mathrm{d}\alpha\mathrm{d}t \\ &= \int_0^{2\pi}\int_{\beta+\gamma}^{\pi} -M(t)\sin\alpha\,\mathrm{d}\alpha\mathrm{d}t \\ &= L_{\Gamma_{-\beta}}(\pi - (\beta+\gamma)) + L_{\Gamma_\gamma}\sin(\beta+\gamma) \end{aligned} \tag{5.19}$$

and

$$\begin{aligned} \iint_\Omega \frac{\sin^2\omega}{t_2}\mathrm{d}x\mathrm{d}y &= \int_0^{2\pi}\int_{\beta+\gamma}^{\pi} L(t)\sin\alpha\,\mathrm{d}\alpha\mathrm{d}t \\ &= L_{\Gamma_\gamma}(\pi - (\beta+\gamma)) + L_{\Gamma_{-\beta}}\sin(\beta+\gamma). \end{aligned} \tag{5.20}$$

■

6 The condition for secantoptic to be convex

The geometric condition for convexity of isoptics is given in [5]. We will use the similar method and give the conditions for convexity of secantoptics.

Note that

$$\begin{aligned} R_{\Gamma_{-\beta}}(t-\beta) &= R(t)\cos\beta - \dot{R}(t)\sin\beta, && (6.1)\\ R_{\Gamma_{\gamma}}(t+\alpha-\beta) &= R(t+\alpha-\beta-\gamma)\cos\gamma + \dot{R}(t+\alpha-\beta-\gamma)\sin\gamma && (6.2)\end{aligned}$$

are curvature radii of evolutoids $\Gamma_{-\beta}$ and Γ_{γ}, respectively. Denote curvatures of these evolutoids by

$$k_{\Gamma_{\gamma}}(t) = \frac{1}{R_{\Gamma_{\gamma}}(t)} \quad \text{and} \quad k_{\Gamma_{-\beta}}(t) = \frac{1}{R_{\Gamma_{-\beta}}(t)}. \tag{6.3}$$

Theorem 6.1. *The secantoptic $C_{\alpha,\beta,\gamma}$ is convex if and only if*

$$2|Q(t)|^2 > \sin\alpha\left(\frac{L(t)}{k_{\Gamma_{\gamma}}(t+\alpha-\beta)} - \frac{M(t)}{k_{\Gamma_{-\beta}}(t-\beta)}\right). \tag{6.4}$$

Proof. For secantoptic $C_{\alpha,\beta,\gamma}$ we have defined a mapping $F_{\beta,\gamma}(\alpha,t) = z_{\alpha,\beta,\gamma}(t)$. Let us calculate its partial derivatives

$$\begin{aligned} \frac{\partial F_{\beta,\gamma}}{\partial\alpha} &= \frac{-M(t)}{\sin\alpha}(\sin\beta + i\cos\beta)e^{it}, && (6.5)\\ \frac{\partial F_{\beta,\gamma}}{\partial t} &= \frac{1}{\sin\alpha}((-M(t)\sin\beta - L(t)\sin(\alpha-\beta) && (6.6)\\ &\quad + i(L(t)\cos(\alpha-\beta) - M(t)\cos\beta))e^{it}. \end{aligned}$$

Note that the curvature of the secantoptic

$$k(t) = \frac{\sin\alpha}{|Q(t)|^3}(2|Q(t)|^2 - [Q,\dot{Q}]) \tag{6.7}$$

can be written as

$$k(t) = \frac{\sin\alpha}{|Q(t)|^3}(2|Q(t)|^2 - R_{\Gamma_{\gamma}}(t+\alpha-\beta)L(t)\sin\alpha + R_{\Gamma_{-\beta}}(t-\beta)M(t)\sin\alpha). \tag{6.8}$$

Hence, the secantoptic is convex if and only if

$$2|Q(t)|^2 - \sin\alpha\left(\frac{L(t)}{k_{\Gamma_{\gamma}}(t+\alpha-\beta)} - \frac{M(t)}{k_{\Gamma_{-\beta}}(t-\beta)}\right) > 0. \tag{6.9}$$

∎

Theorem 6.2. *The secantoptic $C_{\alpha,\beta,\gamma}$ is convex if and only if*

$$2|Q(t)| > \sin\alpha\left(\frac{\sin\alpha_1}{k_{\Gamma_{\gamma}}(t+\alpha-\beta)} + \frac{\sin\alpha_2}{k_{\Gamma_{-\beta}}(t-\beta)}\right). \tag{6.10}$$

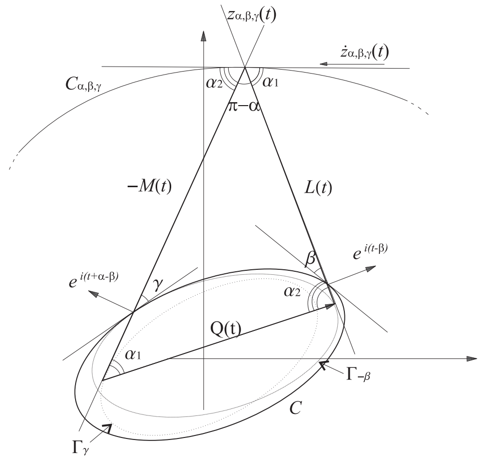

Figure 3:

Proof. In [8] we proved the sine theorem for secantoptics

$$\frac{|Q(t)|}{\sin\alpha} = \frac{L(t)}{\sin\alpha_1} = \frac{-M(t)}{\sin\alpha_2}, \tag{6.11}$$

where α_1 and α_2 are angles as in Fig.3. Now, we use it to (6.4)

$$2|Q(t)|^2 > \sin\alpha \left(\frac{|Q(t)|\sin\alpha_1}{k_{\Gamma_\gamma}(t+\alpha-\beta)} + \frac{|Q(t)|\sin\alpha_2}{k_{\Gamma_{-\beta}}(t-\beta)} \right) \tag{6.12}$$

and obtain the following condition for convexity

$$2|Q(t)| > \sin\alpha \left(\frac{\sin\alpha_1}{k_{\Gamma_\gamma}(t+\alpha-\beta)} + \frac{\sin\alpha_2}{k_{\Gamma_{-\beta}}(t-\beta)} \right). \tag{6.13}$$

■

We use the observation from [5] to find a geometric interpretation of this condition.

Theorem 6.3. *The secantoptic $C_{\alpha,\beta,\gamma}$ is convex if and only if the sum of the lengths of the projections of the curvature vectors of $\Gamma_{-\beta}$ at $z^{-\beta}(t-\beta)$ and Γ_γ at $z^\gamma(t+\alpha-\beta)$ in the direction of the vector $Q(t)$ is less than $2|Q(t)|$.*

Proof. Note that the length of the projection of the curvature vector of the curve $\Gamma_{-\beta}$ at a point $z^{-\beta}(t-\beta)$ in the direction of the vector $Q(t)$ is equal to $\frac{\sin\alpha_2}{k_{\Gamma_{-\beta}}(t-\beta)}$. Similarly, the length of the projection of the curvature vector of Γ_γ at $z^\gamma(t+\alpha-\beta)$ in the direction of $Q(t)$ is equal to $\frac{\sin\alpha_1}{k_{\Gamma_\gamma}(t+\alpha-\beta)}$. ■

We can, similarly as it was done for isoptics, give a condition which implies that all secantoptics of an oval C are convex. Let $\Gamma_{-\beta}$ and Γ_γ be two evolutoids of a given oval C. For each point $z^{-\beta}(t)$ for $t \in [0, 2\pi]$ we can choose a point $z^\gamma(t+\alpha)$, where $\alpha \in (\beta+\gamma, \pi)$. The vector $z^{-\beta}(t)z^\gamma(t+\alpha)$ will be denoted by $q(t, t+\alpha)$.

Theorem 6.4. *If the sum of the lengths of the projections of the curvature vectors of* $\Gamma_{-\beta}$ *at* $z^{-\beta}(t)$ *and of* Γ_{γ} *at* $z^{\gamma}(t+\alpha)$ *in the direction of* $q(t,t+\alpha)$ *is less than* $2|q(t,t+\alpha)|$ *for all* $t \in [0,2\pi]$ *and* $\alpha \in (\beta+\gamma,\pi)$*, then secantoptics* $C_{\alpha,\beta,\gamma}$ *of an oval* C *are convex for all* $\alpha \in (\beta+\gamma,\pi)$.

Proof. Let $\alpha \in (\beta+\gamma,\pi)$ be arbitrary, but fixed and let $t=\tau-\beta$, $\tau \in [0,2\pi]$. For each τ the sum of the lengths of the projections of the curvature vectors of $\Gamma_{-\beta}$ at $z^{-\beta}(\tau-\beta)$ and of Γ_{γ} at $z^{\gamma}(\tau+\alpha-\beta)$ in the direction of the vector $z^{-\beta}(\tau-\beta)z^{\gamma}(\tau+\alpha-\beta)$ is less than $2|z^{-\beta}(\tau-\beta)z^{\gamma}(\tau+\alpha-\beta)|$. Since $z^{-\beta}(\tau-\beta)z^{\gamma}(\tau+\alpha-\beta)=Q(\tau)$, then from the Theorem 6.3 the secantoptic $C_{\alpha,\beta,\gamma}$ is convex. From arbitrariness of $\alpha \in (\beta+\gamma,\pi)$ each secantoptic $C_{\alpha,\beta,\gamma}$ of an oval C is convex. ■

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