Common fixed point theorems in symmetric spaces employing a new implicit function and common property (E.A)

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Abstract

The aim of this paper is broadly two fold. Firstly, we define a new class of implicit function unifying a multitude of strict contractive conditions and utilize the same to prove a general common fixed point theorem for two pairs of weak compatible mappings satisfying common property $(E.A)$ when underlying space is not necessarily compact. Secondly, we show that common property $(E.A)$ relaxes the required containment of ranges of the involved mappings in common fixed point considerations up to two pairs of mappings.

1 Introduction

Banach contraction principle has been extended and generalized in several ways which include the noted article due to Jungck [15]. The paper due to Jungck [15] has inspired vigorous research activity around it since its appearance. Sessa [32] initiated the tradition of weakening the commutativity condition in such common fixed point theorems by introducing the notion of weak commutativity. After the appearance of this notion, several authors introduced the similar conditions of weak commutativity such as: R-weak commutativity, compatible mappings, compatible mappings of type (A), type (B), type (C), type (P) and weak compatibility whose systematic comparisons and illustrations are available in Murthy [22].

Received by the editors February 2008.

Communicated by F. Bastin.

2000 *Mathematics Subject Classification* : 47H10, 54H25.

Key words and phrases : Common fixed points; common property $(E.A)$; weakly compatible mappings; symmetric spaces and implicit function.

In the setting of metric as well as symmetric spaces, contractive conditions do not ensure the existence of fixed points unless the space is compact (cf. [4]) or the contractive condition is replaced by a relatively stronger condition. In recent years, noncompatible mappings have made it possible to prove results on strict contractions beyond compact metric space. The study of common fixed points of noncompatible mappings is a subject of investigation in the recent past and still continues to be an interesting aspect for further investigation. In this regard, the results contained in Pant [23] deserve special mention wherein author has shown the existence of common fixed points of an strict contraction when the underlying space is not essentially compact.

Rhoades [30,31] carried out an exhaustive comparative study of contraction conditions wherein he introduced some contraction conditions and also established the equivalence of several contraction conditions. In recent years, Popa [29] utilized implicit functions instead of contraction conditions to prove common fixed point theorems. Implicit functions are proving fruitful due to their unifying power besides admitting new contraction conditions. Imdad and Ali [12] also proved some results on common fixed points of self mappings using implicit function. In this paper, we define a new class of implicit function and utilize the same to prove our results because of their versatility of deducing several known and unknown contraction conditions in one go. One of the most striking feature of our implicit function (to be introduced in the next section) lies in the nonrequirement of triangular inequality in the course of the proofs of our results in this paper and this is why we opt to prove our results in symmetric spaces instead of metric spaces.

Let X be a nonempty set. A symmetric d is a nonnegative real function defined on $X \times X$ such that

(a) $d(x,y) = 0$ if and only if $x = y$,

(b) $d(x,y) = d(y,x)\ \forall\ x,y \in X$.

As expected by (X,d), we denote a nonempty set X equipped with a symmetric d on X and call it a symmetric space. The spaces (X,d) in which limiting points are defined in the usual way is also sometime called an E-space. The idea of E-spaces is due to Fréchet and Menger. For more details, one can see [2,9,11,13,35].

Most recently, Aamri and Moutawakil [1] introduced the notion of property $(E.A)$ which is a generalization of compatible (nontrivial) as well as noncompatible mappings and utilize the same to prove some common fixed point theorems for strict contractions in metric spaces. In this continuation, Imdad and Ali [12] also shown that the property $(E.A)$ relaxes the required containment of ranges of involved mappings up to a pair of mappings. Only recently, Liu et al. [20] introduced the notion of common property $(E.A)$ which is in fact an extension of property $(E.A)$ to two pairs of mappings and utilize the same to prove common fixed points for strict contractions.

Definition 1.1.[1] A pair (S,T) of self mappings of a symmetric space (X,d) is said to satisfy the property $(E.A)$ if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \qquad \text{for some } \ t \in X.$$

Remark 1.1. Recall that a pair (S,T) of self mappings of a symmetric space (X,d) is noncompatible if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$ but $\lim_{n\to\infty} d(STx_n, TSx_n)$ is either nonzero or nonexistent.

Definition 1.2.[20] Two pairs (A,S) and (B,T) of self mappings of a symmetric space (X,d) are said to satisfy the common property $(E.A)$ if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t$$

for some $t \in X$.

Definition 1.3. A pair (S,T) of self mappings of a nonempty set X is said to be weakly compatible if mappings commute at their coincidence points; i.e. $Sx = Tx \Rightarrow STx = TSx$.

The aim of this paper is to show that the common property $(E.A)$ relaxes the required containment of ranges of the involved mappings up to two pairs. We also show that numerous contractive conditions of the existing literature enjoy the format of our newly introduced implicit function besides admitting several new and natural contractive conditions. We prove a general common fixed point theorem for two pairs of strict contractive mappings satisfying our new implicit function when underlying space is not essentially compact. Some related results are also derived besides furnishing illustrative examples.

2 Implicit Function

In this section, we introduce a new class of implicit function which is different from the one considered in Popa [29] and furnish examples to substantiate the worth of this definition. To describe it, let Φ be the family of lower semi-continuous functions $F : \Re_+^6 \to \Re$ satisfying the following conditions.

$(F_1) : F(t,0,0,t,t,0) > 0$, for all $t > 0$,
$(F_2) : F(t,0,t,0,0,t) > 0$, for all $t > 0$,
$(F_3) : F(t,t,0,0,t,t) \geq 0$, for all $t > 0$.

Example 2.1. $F(t_1,\cdots,t_6) = t_1 - \max\left\{t_2, \dfrac{t_3+t_4}{2}, \dfrac{t_5+t_6}{2}\right\}$.

(F_1): $F(t,0,0,t,t,0) = \dfrac{t}{2} > 0$, for all $t > 0$,

(F_2): $F(t,0,t,0,0,t) = \dfrac{t}{2} > 0$, for all $t > 0$,

(F_3): $F(t,t,0,0,t,t) = 0$, for all $t > 0$.

Example 2.2. $F(t_1,\cdots,t_6) = t_1 - \max\left\{t_2, t_3t_5, t_4t_6\right\}$.

(F_1): $F(t,0,0,t,t,0) = t > 0$, for all $t > 0$,

(F_2): $F(t,0,t,0,0,t) = t > 0$, for all $t > 0$,

(F_3): $F(t,t,0,0,t,t) = 0$, for all $t > 0$.

Example 2.3. $F(t_1,\cdots,t_6) = t_1^2 - \alpha\max\{t_2^2, t_3^2, t_4^2\} - \beta\max\{t_3t_5, t_4t_6\} - \gamma t_5t_6$, where $\alpha,\beta,\gamma \geq 0$, $\alpha < 1$ and $\alpha+\gamma \leq 1$.

Example 2.4.
$$F(t_1,\cdots,t_6) = (1+\alpha t_2)t_1 - \alpha\max\{t_3t_4, t_5t_6\} - \max\left\{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\right\}, \text{ where } \alpha > 0.$$

Example 2.5. $F(t_1,\cdots,t_6) = t_1 - \max\left\{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}, \frac{t_3+t_5}{2}, \frac{t_4+t_6}{2}\right\}$.

Example 2.6.
$$F(t_1,\cdots,t_6) = t_1 - \alpha[\beta\max\{t_2, t_3, t_4, \tfrac{1}{2}(t_5+t_6)\} + (1-\beta)[\max\{t_2^2, t_3t_4, t_5t_6, t_3t_6, t_4t_5\}]^{\frac{1}{2}}],$$
where $\alpha \in (0,1)$ and $0 \leq \beta \leq 1$.

Example 2.7. $F(t_1,\cdots,t_6) = t_1 - \max\{t_2, t_3, t_4\} + \alpha(t_5+t_6)$, where $\alpha > 0$.

Example 2.8. $F(t_1,\cdots,t_6) = t_1^3 - \alpha t_1^2t_2 - \beta t_1t_3t_4 - \gamma t_5^2t_6 - \eta t_5t_6^2$, where α,β,γ, $\eta \geq 0$ and $\alpha+\gamma+\eta \leq 1$.

Example 2.9. $F(t_1,\cdots,t_6) = t_1 - \phi\left(\max\left\{t_2, t_3, t_4, t_5, t_6\right\}\right)$, where $\phi : \Re_+ \to \Re$ is upper semi-continuous function such that $\phi(0) = 0$ and $\phi(t) < t$ for all $t > 0$.

Example 2.10. $F(t_1,\cdots,t_6) = t_1 - \phi(t_2, t_3, t_4, t_5, t_6)$, where $\phi : \Re_+^5 \to \Re$ is upper semi-continuous function such that $\max\{\phi(0,t,0,0,t), \phi(0,0,t,t,0), \phi(t,0,0,t,t)\} < t$ for each $t > 0$.

Example 2.11. $F(t_1,\cdots,t_6) = t_1 - \dfrac{t_3t_4 + t_5t_6}{1+t_2}$.

Example 2.12. $F(t_1,\cdots,t_6) = t_1^2 - \alpha t_2^2 - \beta\dfrac{t_5t_6}{1+t_3+t_4}$, where $\alpha,\beta \geq 0$ and $\alpha+\beta \leq 1$.

Example 2.13. $F(t_1,\cdots,t_6) = t_1 - t_2 + \alpha(t_3+t_4) + \beta\left(\dfrac{t_5^2+t_6^2}{t_5+t_6}\right)$,

where $\alpha,\beta \geq 0$ and $\alpha+\beta < 1$.

Example 2.14. $F(t_1,\cdots,t_6) = t_1 - t_2 + \dfrac{t_3t_4 + t_5t_6}{t_5+t_6}$.

Example 2.15. $F(t_1,\cdots,t_6) = t_1 - t_2 - \alpha\dfrac{t_3+t_4}{1+t_5t_6}$, where $0 \leq \alpha < 1$.

The verification of Examples 2.3–2.15 is easy, hence details are omitted.

3 Main Results

Throughout this paper, we assume symmetric d to be continuous. We begin with the following observation.

Lemma 3.1. Let A, B, S and T be self mappings of a symmetric space (X, d) such that

(a) the pair (A, S) (or (B, T)) satisfies the property $(E.A)$,

(b) $A(X) \subset T(X)$ (or $B(X) \subset S(X)$), and

(c) for all $x \neq y \in X$ and $F \in \Phi$

$$F(d(Ax, By), d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Sx, By), d(Ty, Ax)) < 0. \quad (3.1.1)$$

Then the pairs (A, S) and (B, T) satisfy the common property $(E.A)$.

Proof. If the pair (A, S) enjoys property $(E.A)$, then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t, \quad \text{for some } t \in X.$$

Since $A(X) \subset T(X)$, hence for each $\{x_n\}$ there exists $\{y_n\}$ in X such that $Ax_n = Ty_n$. Therefore, $\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Ax_n = t$. Thus, in all we have $Ax_n \to t, Sx_n \to t$ and $Ty_n \to t$. Now, we assert that $By_n \to t$. If not, then using (3.1.1), we have

$$F(d(Ax_n, By_n), d(Sx_n, Ty_n), d(Ax_n, Sx_n), d(By_n, Ty_n), d(Sx_n, By_n), d(Ty_n, Ax_n)) < 0$$

which on making $n \to \infty$, reduces to

$$F(d(t, By_n), 0, 0, d(By_n, t), d(t, By_n), 0) \leq 0$$

a contradiction to (F_1). Hence $By_n \to t$ which shows that the pairs (A, S) and (B, T) satisfy the common property $(E.A)$.

Remark 3.1. The converse of Lemma 3.1 is not true in general. For a counter example, one can see Example 4.1.

Now we state and prove our main result as follows.

Theorem 3.1. Let A, B, S and T be self mappings of a symmetric space (X, d) which satisfy inequality (3.1.1). Suppose that

(a) the pairs (A, S) and (B, T) share the common property $(E.A)$,

(b) $S(X)$ and $T(X)$ are closed subsets of X.

Then the pair (A, S) as well as (B, T) has a point of coincidence. Moreover, if the pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.

Proof. In view of (a), there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t$$

for some $t \in X$.

Since $S(X)$ is a closed subset of X, hence $\lim_{n\to\infty} Sx_n = t \in S(X)$. Therefore, there exists $u \in X$ such that $Su = t$. Now, we assert that $Au = Su$. If it is not, then $d(Au, Su) > 0$. Using (3.1.1), we have

$$F(d(Au, By_n), d(Su, Ty_n), d(Au, Su), d(By_n, Ty_n), d(Su, By_n), d(Ty_n, Au)) < 0$$

which on making $n \to \infty$, reduces to

$$F(d(Au, t), d(Su, t), d(Au, Su), d(t, t), d(Su, t), d(t, Au)) \leq 0$$

$$or \quad F(d(Au, Su), 0, d(Au, Su), 0, 0, d(Su, Au)) \leq 0$$

which contradicts (F_2) as $d(Au, Su) > 0$. Hence $Au = Su$ which shows that u is a coincidence point of the pair (A, S).

Also $T(X)$ is a closed subset of X. Therefore $\lim_{n\to\infty} Ty_n = t \in T(X)$. Hence $Tw = t$ for some $w \in X$. Suppose $d(Tw, Bw) > 0$, then again using (3.1.1)

$$F(d(Ax_n, Bw), d(Sx_n, Tw), d(Ax_n, Sx_n), d(Bw, Tw), d(Sx_n, Bw), d(Tw, Ax_n)) < 0$$

which on making $n \to \infty$, reduces to

$$F(d(t, Bw), d(t, Tw), d(t, t), d(Bw, Tw), d(t, Bw), d(Tw, t)) \leq 0$$

$$or \quad F(d(Tw, Bw), 0, 0, d(Bw, Tw), d(Tw, Bw), 0) \leq 0$$

which contradicts (F_1) as $d(Tw, Bw) > 0$. This shows that w is a coincidence point of the pair (B, T).

Since the pairs (A, S) and (B, T) are weakly compatible, therefore

$$At = ASu = SAu = St, \text{ and } Bt = BTw = TBw = Tt.$$

Suppose that $d(At, t) > 0$. Using (3.1.1), we have

$$F(d(At, t), d(St, t), d(At, St), d(t, t), d(St, t), d(t, At)) < 0$$

$$or \quad F(d(At, t), d(At, t), 0, 0, d(At, t), d(t, At)) < 0$$

which is a contradiction to (F_3). Hence $At = t$. This shows that t is a common fixed point of the pair (A, S). Similarly, one can show that t is a common fixed point of the pair (B, T). Therefore, t is a common fixed of the mappings A, B, S and T. The uniqueness of the common fixed point is an easy consequence of the condition (F_3). This completes the proof.

Theorem 3.2. The conclusions of Theorem 3.1 remain true if the condition (b) of Theorem 3.1 is replaced by following.

(b′) $\overline{A(X)} \subset T(X)$ and $\overline{B(X)} \subset S(X)$.

As a corollary of Theorem 3.2, we can have the following result which is also a variant of Theorem 3.1.

Corollary 3.1. The conclusions of Theorems 3.1 and 3.2 remain true if the conditions (b) and (b′) are replaced by following.

(b″) $A(X)$ and $B(X)$ are closed subsets of X provided $A(X) \subset T(X)$ and $B(X) \subset S(X)$.

Theorem 3.3. Let A, B, S and T be self mappings of a symmetric space (X, d) which satisfy inequality (3.1.1). Suppose that

(a) the pair (A, S) (or (B, T)) has the property $(E.A)$,

(b) $A(X) \subset T(X)$ (or $B(X) \subset S(X)$), and

(c) $S(X)$ (or $T(X)$) is a closed subset of X.

Then the pairs (A, S) and (B, T) have a point of coincidence. If the pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point.

Proof. In view of Lemma 3.1, the pairs (A, S) and (B, T) share the common property $(E.A)$, i.e. there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t \in X.$$

If $S(X)$ is a closed subset of X, then on the lines of Theorem 3.1, the pair (A, S) has a point of coincidence, say u, i.e. $Au = Su$. Since $Au \in A(X)$ and $A(X) \subset T(X)$, there exists $w \in X$ such that $Au = Tw$. Now we assert that $Bw = Tw$. If not, then using (3.1.1), we have

$$F(d(Ax_n, Bw), d(Sx_n, Tw), d(Ax_n, Sx_n), d(Bw, Tw), d(Sx_n, Bw), d(Tw, Ax_n)) < 0$$

which on making $n \to \infty$, reduces to

$$F(d(t, Bw), d(t, Tw), d(t, t), d(Bw, Tw), d(t, Bw), d(Tw, t)) \leq 0$$

$$\text{or} \qquad F(d(Tw, Bw), 0, 0, d(Bw, Tw), d(Tw, Bw), 0) \leq 0$$

a contradiction to (F_1). Hence $Bw = Tw$, which shows that w is a coincidence point of the pair (B, T). Rest of the proof can be completed on the lines of Theorem 3.1.

By choosing A, B, S and T suitably, one can deduce corollaries for a pair as well as for a triod of mappings. The detail of two possible corollaries for triod of mappings are not included. As a sample, we outline the following natural result for a pair of self mappings.

Corollary 3.2. Let A and S be self mappings of a symmetric space (X,d). Suppose that

(a) the pair (A,S) has property $(E.A)$,

(b) for all $x \neq y \in X$ and $F \in \Phi$

$$F(d(Ax,Ay),d(Sx,Sy),d(Ax,Sx),d(Ay,Sy),d(Sx,Ay),d(Sy,Ax)) < 0 \quad (3.1.2)$$

(c) $S(X)$ is a closed subset of X.

Then A and S have a point of coincidence. Moreover, if the pair (A,S) is weakly compatible, then A and S have a unique common fixed point.

Corollary 3.3. The conclusions of Theorem 3.1 remain true if condition (3.1.1) is replaced by one of the following: (for all $x \neq y \in X$)

(a_1) $d(Ax,By) < \max\left\{d(Sx,Ty), \dfrac{d(Ax,Sx)+d(By,Ty)}{2}, \dfrac{d(Sx,By)+d(Ty,Ax)}{2}\right\}.$

(a_2) $d(Ax,By) < \max\{d(Sx,Ty), d(Ax,Sx)d(Sx,By), d(By,Ty)d(Ty,Ax)\}.$

(a_3) $d^2(Ax,By) < \alpha\max\{d^2(Sx,Ty), d^2(Ax,Sx), d^2(By,Ty)\}$
$+\beta\max\{d(Ax,Sx)d(Sx,By), d(By,Ty)d(Ty,Ax)\} + \gamma d(Sx,By)d(Ty,Ax)$

where $\alpha,\beta,\gamma \geq 0$, $\alpha < 1$ and $\alpha+\gamma \leq 1$.

(a_4) $(1+\alpha d(Sx,Ty))d(Ax,By) < \alpha\max\{d(Ax,Sx)d(By,Ty), d(Sx,By)d(Ty,Ax)\}$
$+\max\left\{d(Sx,Ty), \dfrac{d(Ax,Sx)+d(By,Ty)}{2}, \dfrac{d(Sx,By)+d(Ty,Ax)}{2}\right\}$

where $\alpha \geq 0$.

(a_5) $d(Ax,By) < \max\left\{d(Sx,Ty), \dfrac{d(Ax,Sx)+d(By,Ty)}{2}, \dfrac{d(Sx,By)+d(Ty,Ax)}{2}, \dfrac{d(Ax,Sx)+d(Sx,By)}{2}, \dfrac{d(By,Ty)+d(Ty,Ax)}{2}\right\}.$

(a_6) $d(Ax,By) < \alpha\Big[\beta\max\left\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), \dfrac{d(Sx,By)+d(Ty,Ax)}{2}\right\}$
$+(1-\beta)(\max\{d^2(Sx,Ty), d(Ax,Sx)d(By,Ty), d(Sx,By)d(Ty,Ax),$
$d(Ax,Sx)d(Ty,Ax), d(By,Ty)d(Sx,By)\})^{\frac{1}{2}}\Big],$

where $\alpha \in (0,1)$ and $0 \leq \beta \leq 1$.

(a_7) $d(Ax,By) < \max\{d(Sx,Ty), d(Ax,Sx), d(By,Ty)\} - \alpha(d(Sx,By)+d(Ty,Ax)),$

where $\alpha > 0$.

(a_8) $d^3(Ax,By) < \alpha d^2(Ax,By)d(Sx,Ty) + \beta d(Ax,By)d(Ax,Sx)d(By,Ty)$

$$+\gamma d^2(Sx,By)d(Ty,Ax)+\eta d(Sx,By)d^2(Ty,Ax),$$

where $\alpha,\beta,\gamma,\eta \geq 0$ and $\alpha+\gamma+\eta \leq 1$.

(a_9) $d(Ax,By) < \phi(\max\{d(Sx,Ty),d(Ax,Sx),d(By,Ty),d(Sx,By),d(Ty,Ax)\})$,

where $\phi : \Re_+ \to \Re$ is upper semi-continuous function such that $\phi(0)=0$ and $\phi(t) < t$ for all $t > 0$.

(a_{10}) $d(Ax,By) < \phi(d(Sx,Ty),d(Ax,Sx),d(By,Ty),d(Sx,By),d(Ty,Ax))$,

where $\phi : \Re_+^5 \to \Re$ is upper semi-continuous function such that $\max\{\phi(0,t,0,0,t), \phi(0,0,t,t,0),\phi(t,0,0,t,t)\} < t$ for each $t > 0$.

(a_{11}) $d(Ax,By) < \dfrac{d(Ax,Sx)d(By,Ty)+d(Sx,By)d(Ty,Ax)}{1+d(Sx,Ty)}$.

(a_{12}) $d^2(Ax,By) < \alpha d^2(Sx,Ty)+\beta\dfrac{d(Sx,By)d(Ty,Ax)}{1+d(Ax,Sx)+d(By,Ty)}$,

where $\alpha,\beta \geq 0$ and $\alpha+\beta \leq 1$.

(a_{13}) $d(Ax,By) < d(Sx,Ty)-\alpha(d(Ax,Sx)+d(By,Ty))-\beta\dfrac{d^2(Sx,By)+d^2(Ty,Ax)}{d(Sx,By)+d(Ty,Ax)}$,

where $\alpha,\beta \geq 0$ and $\alpha+\beta < 1$.

(a_{14}) $d(Ax,By) < d(Sx,Ty)-\dfrac{d(Ax,Sx)d(By,Ty)+d(Sx,By)d(Ty,Ax)}{d(Sx,By)+d(Ty,Ax)}$.

(a_{15}) $d(Ax,By) < d(Sx,Ty)+\alpha\dfrac{d(Ax,Sx)+d(By,Ty)}{1+d(Sx,By)d(Ty,Ax)}$,

where $0 \leq \alpha < 1$.

Proof. Proof follows from Theorem 3.1 and Examples 2.1-2.15.

Remark 3.1. Corollaries corresponding to contraction conditions (a_1) to (a_{15}) are new results as these never require any condition on containments of ranges of the involved mappings besides being proved in symmetric spaces instead of metric spaces. The majority of results corresponding to various above mentioned contraction conditions present generalized and improved versions of numerous existing results for metric spaces which include Aamri and Moutawakil [1], Edelstein [4], Chugh and Kumar [3], Fisher [5-7], Husain and Sehgal [10], Imdad et al. [11], Jeong and Rhoades [14], Jungck [16], Kasahara and Rhoades [19], Liu et al. [20], Meade and Singh [21], Pant [23], Pant and Pant [24], Park [26-28], Tas et al. [33], Telci et al. [24] and some others besides yielding some results which are seeming new to the literature (e.g. (a_2), (a_7), (a_8), (a_{11})-(a_{15})).

Corollary 3.4. Let A,B,S and T be self mappings of a symmetric space (X,d) which satisfy conditions (a) and (b) of Theorem 3.1 and

$$d(Ax,By) < \phi\left(\max\left\{d(Sx,Ty),d(Ax,Sx),d(By,Ty),\frac{d(Sx,By)+d(Ty,Ax)}{2}\right\}\right)$$

where $\phi : \Re^+ \to \Re$ is upper semi-continuous function such that $\phi(0) = 0$ and $\phi(t) < t$ for all $t > 0$.

Then the pairs (A, S) and (B, T) have a point of coincidence each. Moreover, A, B, S and T have a unique common fixed point provided the pairs (A, S) and (B, T) are weakly compatible.

Proof. Notice that

$$d(Ax, By) < \phi\left(\max\left\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), \frac{d(Sx, By) + d(Ty, Ax)}{2}\right\}\right)$$
$$\leq \phi(\max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Sx, By), d(Ty, Ax)\}).$$

Now proof follows from contractive condition (a_9) of Corollary 3.3.

4 Illustrative Examples

Here we present an example demonstrating the validity of the hypotheses and degree of generality of Theorem 3.1 over the majority of earlier results proved till date with rare possible exceptions.

Example 4.1. Consider $X = (-1, 1)$ equipped with the symmetric $d(x, y) = (x - y)^2$. Define self mappings A, B, S and T on X as follows:

$$A(x) = \begin{cases} \frac{3}{5} & \text{if } -1 < x < -1/2 \\ \frac{x}{4} & \text{if } -1/2 \leq x \leq 1/2 \\ \frac{3}{5} & \text{if } 1/2 < x < 1, \end{cases} \qquad S(x) = \begin{cases} \frac{1}{2} & \text{if } -1 < x < -1/2 \\ \frac{x}{2} & \text{if } -1/2 \leq x \leq 1/2 \\ \frac{-1}{2} & \text{if } 1/2 < x < 1, \end{cases}$$

$$B(x) = \begin{cases} \frac{3}{5} & \text{if } -1 < x < -1/2 \\ \frac{-x}{4} & \text{if } -1/2 \leq x \leq 1/2 \\ \frac{3}{5} & \text{if } 1/2 < x < 1, \end{cases} \qquad T(x) = \begin{cases} \frac{-1}{2} & \text{if } -1 < x < -1/2 \\ \frac{-x}{2} & \text{if } -1/2 \leq x \leq 1/2 \\ \frac{1}{2} & \text{if } 1/2 < x < 1. \end{cases}$$

Choose our sequences $\{x_n = \frac{1}{n+1}\}$ and $\{y_n = \frac{-1}{n+1}\}$ in X, then

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 0$$

which shows that the pairs (A, S) and (B, T) share the common property $(E.A)$. Also, $A(X) = B(X) = \{\frac{3}{5}\} \cup [\frac{-1}{8}, \frac{1}{8}] \not\subset S(X) = T(X) = \{\frac{-1}{2}, \frac{1}{2}\} \cup [\frac{-1}{4}, \frac{1}{4}]$ and $S(X)$ as well as $T(X)$ is a closed subspace of X. Define a continuous implicit function $F : \Re_+^6 \to \Re$ such that $F(t_1, t_2, ..., t_6) = t_1 - \max\{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\}$ where $F \in \Phi$. By a routine calculation, one can verify the inequality (3.1.1) for all $x \neq y \in X$. Thus, all the conditions of Theorem 3.1 are satisfied and 0 is a unique common fixed point the pairs (A, S) and (B, T) which also remains point of coincidence as well.

Here it is worth noting that none of the earlier theorems can be used in the context of this example as Theorem 3.1 never require any condition on the containment of ranges of mappings while compactness of the space is replaced by closeness of subspaces. Moreover, the continuity requirements of involved mappings are completely relaxed whereas all earlier theorems (prior to 2001) require the continuity of at least two involved mappings. Apart from above mentioned improvements, we are also able to enhance the domain of applicability of our results from metric spaces to symmetric spaces.

Finally, we furnish the following example in support of Theorem 3.2 whenever Theorems 3.1 and 3.3 are not applicable.

Example 4.2.[25] Consider $X = [2, 20]$ equipped with symmetric $d(x, y) = (x - y)^2$. Define self mappings A, B, S and T on X as follows:

$$A(x) = \begin{cases} 2 & \text{if } x = 2 \\ 3 & \text{if } x \in (2, 20], \end{cases} \qquad B(x) = \begin{cases} 2 & \text{if } x \in \{2\} \cup [5, 20] \\ 6 & \text{if } x \in (2, 5), \end{cases}$$

$$S(x) = \begin{cases} 2 & \text{if } x = 2 \\ \dfrac{x+8}{2} & \text{if } x \in (2, 5) \\ 8 & \text{if } x \in [5, 20], \end{cases} \qquad T(x) = \begin{cases} 2 & \text{if } x = 2 \\ x+8 & \text{if } x \in (2, 5) \\ \dfrac{x+1}{2} & \text{if } x \in [5, 20]. \end{cases}$$

Clearly, the pairs (A, S) and (B, T) satisfy the common property $(E.A)$. Also $A(X) = \{2, 3\} \subset \{2\} \cup [3, 13) = T(X)$ and $B(X) = \{2, 6\} \subset \{2, 8\} \cup (5, 13/2) = S(X)$. Define a continuous implicit function F as in Example 4.1. One can easily verify the inequality (3.1.1) for all $x \neq y \in X$. Thus, all the conditions of Theorem 3.2 are satisfied and 2 is a unique common fixed point of the pairs (A, S) and (B, T) which is their coincidence point as well. Here it may be noticed that all the mappings in this example are discontinuous even at their common fixed point 2.

Remark 4.1. It is evident from examples that one can also furnish an example substantiating the utility of Theorem 3.3 whenever Theorems 3.1 and 3.2 are not applicable.

Acknowledgment. Authors are grateful to learned referee for bringing out our attention to the article due to Turkoglu and Altun [35].

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