# New results of periodic solutions for a class of delay Rayleigh equation *

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**Abstract**

In this studies, we discuss the following Rayleigh equation with two delays:

$$x''(t) + f(t, x'(t)) + g_1(t, x(t - \tau_1)) + g_2(t, x(t - \tau_2)) = e(t).$$

By using Mawhin's continuation theorem and some new techniques, some criteria to guarantee the existence and uniqueness of periodic solutions of this equation is given. Our results are new and complement the known results in the literature.

## 1 Introduction

In this present paper, we investigate the existence and uniqueness of the periodic solutions of the following Rayleigh equation with two delays

$$x''(t) + f(t, x'(t)) + g_1(t, x(t - \tau_1)) + g_2(t, x(t - \tau_2)) = e(t), \tag{1.1}$$

where $\tau_1, \tau_2 \geq 0$ are two constants, $f, g_1, g_2 \in C(\mathbb{R}^2, \mathbb{R})$, $f(t, x), g_1(t, x), g_2(t, x)$ are $T$-periodic functions with respect to $t$, $T > 0$, $f(t, 0) = 0$ for all $t \in \mathbb{R}$, $e \in C(\mathbb{R}, \mathbb{R})$, and $e(t)$ is a $T$-periodic function.

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As we know, Rayleigh equation can be derived from many fields, such as physics, mechanics and engineering technique fields. The problem concerning the periodic solutions for this equation has been studied extensively by lots of authors. For example, in 1977, R.E. Gaines and J.L. Mawhin [1] introduced some continuation theorems and applied them to the existence of solutions of differential equations. In particular, a specific example is provided in [1, p. 99] on how $T$-periodic solutions can be obtained by means of these theorems for the Rayleigh equation

$$x''(t) + f(x'(t)) + g(t, x(t)) = 0. \tag{1.2}$$

In this direction, many researchers (see [4–9]) continued to discuss the Rayleigh equation and got some new results on the $T$-periodic solutions of Eq.(1.1), and generalized the results in [1]. However, to the best of our knowledge, there exist much fewer results for the existence and uniqueness of $T$-periodic solutions of Eq.(1.1). One of the significant reasons is that various methods to obtain some criteria for securing the uniqueness of $T$-periodic solutions in the case of Duffing equation and Liénard equation can not be adapted directly to the case of Rayleigh equation. Hence, it is still essential to study the $T$-periodic solutions of Eq.(1.1).

In this paper, we get around with these difficulties by using some new techniques and obtain some criteria for securing the existence and uniqueness of $T$-periodic solutions of Eq.(1.1), which can not be achieved in most of the previous papers. The results of this studies are new and complement the previously known results. An illustrative example will be provided to demonstrate the applications of our results in Section 4.

## 2 Lemmas

Let us start with some notations. Define

$$|x|_\infty = \max_{t\in[0,T]} |x(t)|, \quad |x'|_\infty = \max_{t\in[0,T]} |x'(t)|, \quad |x|_k = \left(\int_0^T |x(t)|^k dt\right)^{1/k}.$$

Let

$$C_T^1 := \{x \in C^1(\mathbb{R}, \mathbb{R}) : x \text{ is } T\text{-periodic}\}$$

and

$$C_T := \{x \in C(\mathbb{R}, \mathbb{R}) : x \text{ is } T\text{-periodic}\},$$

which are two Banach spaces with the norms

$$||x||_{C_T^1} = max\{|x|_\infty, |x'|_\infty\}, \quad ||x||_{C_T} = |x|_\infty.$$

The following conditions will be used later:

($H_1$) $(g_i(t,u) - g_i(t,v))(u - v) < 0$ for all $t, u, v \in \mathbb{R}, u \neq v$;

($H'_1$) $(g_i(t,u) - g_i(t,v))(u - v) > 0$ for all $t, u, v \in \mathbb{R}, u \neq v$, where $i = 1, 2$.

**Lemma 2.1.** *If* $x \in C^2(\mathbb{R}, \mathbb{R})$ *with* $x(t+T) = x(t)$*, then*

$$|x'|_2^2 \leq \left(\frac{T}{2\pi}\right)^2 |x''|_2^2.$$

*Proof.* Lemma 2.1 is a direct consequence of the Wirtinger inequality, and see [2, 3] for its proof.

**Lemma 2.2.** *Let* $(H_1)$ *or* $(H'_1)$ *hold. Suppose there exist some nonnegative constants* $C_0$,$C_1$ *and* $C_2$ such that

$(H_2)$ $|f(t,u) - f(t,v)| \leq C_0|u - v|$, *for all* $t, u, v \in \mathbb{R}$;

$(H_3)$ $|g_i(t,u) - g_i(t,v)| \leq C_i|u - v|$, *for all* $t, u, v \in \mathbb{R}, i = 1, 2$;

$(H_4)$ $C_0\frac{T}{2\pi} + (C_1 + C_2)\frac{T^2}{4\pi} < 1$,

*then (1.1) has at most one T-periodic solution.*

*Proof.* Suppose that $x_1(t)$ and $x_2(t)$ are two $T$-periodic solutions of (1.1). Then, we have

$$[x_1(t) - x_2(t)]'' + [f(t, x_1'(t)) - f(t, x_2'(t))] + [g_1(t, x_1(t-\tau_1)) - g_1(t, x_2(t-\tau_1))] \\ + [g_2(t, x_1(t-\tau_2)) - g_2(t, x_2(t-\tau_2))] = 0. \quad (2.1)$$

Set $Z(t) = x_1(t) - x_2(t)$, then, from (2.1), we obtain

$$Z''(t) + [f(t, x_1'(t)) - f(t, x_2'(t))] + [g_1(t, x_1(t-\tau_1)) - g_1(t, x_2(t-\tau_1))] \\ + [g_2(t, x_1(t-\tau_2)) - g_2(t, x_2(t-\tau_2))] = 0. \quad (2.2)$$

Since $Z(t) = x_1(t) - x_2(t)$ is a continuous $T$-periodic function in $\mathbb{R}$, there exist two constants $t_{max}, t_{min} \in \mathbb{R}$ such that

$$Z(t_{max}) = \max_{t\in[0,T]} Z(t) = \max_{t\in\mathbb{R}} Z(t), \quad Z(t_{min}) = \min_{t\in[0,T]} Z(t) = \min_{t\in\mathbb{R}} Z(t). \quad (2.3)$$

Then we have

$$Z'(t_{max}) = x_1'(t_{max}) - x_2'(t_{max}) = 0, \quad Z''(t_{max}) \leq 0, \quad (2.4)$$

and

$$Z'(t_{min}) = x_1'(t_{min}) - x_2'(t_{min}) = 0, \quad Z''(t_{min}) \geq 0. \quad (2.5)$$

In view of (2.2)–(2.5), we get

$$g_1(t_{max}, x_1(t_{max} - \tau_1)) - g_1(t_{max}, x_2(t_{max} - \tau_1)) \\ + g_2(t_{max}, x_1(t_{max} - \tau_2)) - g_2(t_{max}, x_2(t_{max} - \tau_2)) \\ = -Z''(t_{max}) - [f(t_{max}, x_1'(t_{max})) - f(t_{max}, x_2'(t_{max}))] = -Z''(t_{max}) \geq 0 \quad (2.6)$$

and

$$
\begin{aligned}
g_1(t_{min}, x_1(t_{min}-\tau_1)) &- g_1(t_{min}, x_2(t_{min}-\tau_1)) \\
&g_2(t_{min}, x_1(t_{min}-\tau_2)) - g_2(t_{min}, x_2(t_{min}-\tau_2)) \\
= -Z''(t_{min}) &- [f(t_{min}, x_1'(t_{min})) - f(t_{min}, x_2'(t_{min}))] = -Z''(t_{min}) \leq 0, \quad (2.7)
\end{aligned}
$$

which implies there exists a constant $t_0 \in \mathbb{R}$ such that

$$
\begin{aligned}
g_1(t_0, x_1(t_0-\tau_1)) - g_1(t_0, x_2(t_0-\tau_1)) + g_2(t_0, x_1(t_0-\tau_2)) \\
- g_2(t_0, x_2(t_0-\tau_2)) = 0. \quad (2.8)
\end{aligned}
$$

From ($H_1$) or ($H'_1$) and (2.8), we have

$$Z(t_0-\tau_1)Z(t_0-\tau_2) = (x_1(t_0-\tau_1) - x_2(t_0-\tau_1))(x_1(t_0-\tau_2) - x_2(t_0-\tau_2)) \leq 0,$$

which implies there exists a constant $t_{00} \in \mathbb{R}$, such that

$$Z(t_{00}) = 0$$

Set $t_{00} = nT + \widetilde{t}_0$, where $\widetilde{t}_0 \in [0, T]$ and $n$ is an integer. Noticing $Z(t+T) = Z(t)$, we get

$$Z(\widetilde{t}_0) = Z(nT + \widetilde{t}_0) = Z(t_{00}) = 0. \quad (2.9)$$

Hence, for any $t \in [\widetilde{t}_0, \widetilde{t}_0 + T]$, we obtain

$$|Z(t)| = \left| Z(\widetilde{t}_0) + \int_{\widetilde{t}_0}^{t} Z'(s)ds \right| \leq \int_{\widetilde{t}_0}^{t} |Z'(s)|ds$$

and

$$|Z(t)| = \left| Z(\widetilde{t}_0 + T) + \int_{\widetilde{t}_0+T}^{t} Z'(s)ds \right| = \left| -\int_{t}^{\widetilde{t}_0+T} Z'(s)ds \right| \leq \int_{t}^{\widetilde{t}_0+T} |Z'(s)|ds.$$

Combining above two inequalities, we get

$$|Z(t)| \leq \frac{1}{2}\int_0^T |Z'(s)|\, ds.$$

Using Schwartz inequality yields

$$|Z|_\infty = \max_{t\in[\widetilde{t}_0, \widetilde{t}_0+T]} |Z(t)| \leq \frac{1}{2}\int_0^T |Z'(s)|ds \leq \frac{1}{2}|1|_2|Z'|_2 = \frac{1}{2}\sqrt{T}|Z'|_2. \quad (2.10)$$

Multiplying $Z''(t)$ and (2.2) and then integrating it from 0 to $T$, by Lemma 2.1, ($H_2$), ($H_3$), (2.10) and Schwartz inequality, we obtain

$$\begin{aligned}
|Z''|_2^2 &= -\int_0^T [f(t,x_1'(t)) - f(t,x_2'(t))]Z''(t)dt \\
&\quad -\int_0^T [g_1(t,x_1(t-\tau_1)) - g_1(t,x_2(t-\tau_1))]Z''(t)dt \\
&\quad -\int_0^T [g_2(t,x_1(t-\tau_2)) - g_2(t,x_2(t-\tau_2))]Z''(t)dt \\
&\leq \int_0^T |f(t,x_1'(t)) - f(t,x_2'(t))||Z''(t)|dt \\
&\quad +\int_0^T |g_1(t,x_1(t-\tau_1)) - g_1(t,x_2(t-\tau_1))||Z''(t)|dt \\
&\quad +\int_0^T |g_2(t,x_1(t-\tau_2)) - g_2(t,x_2(t-\tau_2))||Z''(t)|dt \\
&\leq \int_0^T C_0|x_1'(t) - x_2'(t)||Z''(t)|dt \\
&\quad +\int_0^T C_1|x_1(t-\tau_1) - x_2(t-\tau_1)||Z''(t)|dt \\
&\quad +\int_0^T C_2|x_1(t-\tau_2) - x_2(t-\tau_2)||Z''(t)|dt \\
&\leq \int_0^T C_0|Z'(t)||Z''(t)|dt + \int_0^T C_1|Z(t-\tau_1)||Z''(t)|dt \\
&\quad +\int_0^T C_2|Z(t-\tau_2)||Z''(t)|dt
\end{aligned}$$

$$\begin{aligned}
&\leq C_0\left(\int_0^T |Z'(t)|^2dt\right)^{1/2}\left(\int_0^T |Z''(t)|^2dt\right)^{1/2} \\
&\quad +C_1\left(\int_0^T |Z(t-\tau_1)|^2dt\right)^{1/2}\left(\int_0^T |Z''(t)|^2dt\right)^{1/2} \\
&\quad +C_2\left(\int_0^T |Z(t-\tau_2)|^2dt\right)^{1/2}\left(\int_0^T |Z''(t)|^2dt\right)^{1/2} \\
&\leq C_0|Z'|_2|Z''|_2 + (C_1+C_2)\sqrt{T}|Z|_\infty|Z''|_2 \\
&\leq \left[C_0\frac{T}{2\pi} + (C_1+C_2)\frac{T^2}{4\pi}\right]|Z''|_2^2 \qquad (2.11)
\end{aligned}$$

Since $Z(t), Z'(t)$ and $Z''(t)$ are continuous $T$-periodic functions, by Lemma 2.1, ($H_4$) and (2.10), we get

$$Z(t) = Z'(t) = Z''(t) = 0, \quad \text{for all} \quad t \in \mathbb{R}.$$

Thus, $x_1(t) \equiv x_2(t)$, for all $t \in \mathbb{R}$. Hence, (1.1) has at most one $T$-periodic solution. This completes the proof.

For convenience of use, Mawhin's Continuation Theorem is introduced as follows.

**Lemma 2.3.** (Gaines and Mawhin [1]) *Let $X$ and $Y$ be two Banach spaces. Suppose that $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero and $N : X \to Y$ is L-compact on $\overline{\Omega}$, where $\Omega$ is an open bounded subset of X. Moreover, assume that all the following conditions are satisfied:*

*(i)* $Lx \neq \lambda Nx$, *for all* $x \in \partial\Omega \cap D(L), \lambda \in (0,1)$;

*(ii)* $Nx \overline{\in} ImL$, *for all* $x \in \partial\Omega \cap KerL$;

*(iii) the Brouwer degree* $deg\{JQN, \Omega \cap KerL, 0\} \neq 0$,

*where* $J : ImQ \to KerL$ *is an isomorphism.*

*Then equation* $Lx = Nx$ *has at least one solution on* $\overline{\Omega} \cap D(L)$.

## 3 Main results

Now we are in the position to give our main results.

**Theorem 1.** *Suppose* $(H_1)$*(or* $(H'_1)$*)–*$(H_4)$ *hold. Also suppose there exists a nonnegative constant* $D_1$ *such that*

$(H_5)$ $x(g_1(t,x) + g_2(t,x) - e(t)) < 0$, *for all* $|x| > D_1$ *and* $t \in \mathbb{R}$;

*Then Eq.(1.1) has a unique T-periodic solution.*

*Proof.* Consider the homotopic equation of Eq.(1.1) as follows:

$$x''(t) + \lambda f(t, x'(t)) + \lambda g_1(t, x(t-\tau_1)) + \lambda g_2(t, x(t-\tau_2)) = \lambda e(t). \tag{3.1}$$

By Lemma 2.2, it is easy to see that Eq.(1.1) has at most one $T$-periodic solution. Thus, to complete the proof of Theorem 1, it suffices to show that Eq.(1.1) has at least one $T$-periodic solution. to do this, Lemma 2.3 will be applied.

Firstly, we proof all possible $T$-periodic solutions of Eq.(3.1) are bounded in $C^1_T$. Let $S \subset C^1_T$ be the set of $T$-periodic solutions of (3.1). If $S = \emptyset$, the proof is ended. Suppose $S \neq \emptyset$, and let $x \in S$, then exist two constants $\bar{t}, \underline{t} \in \mathbb{R}$ such that

$$x(\bar{t}) = \max_{t\in\mathbb{R}} x(t) \text{ and } x(\underline{t}) = \min_{t\in\mathbb{R}} x(t),$$

which implies

$$x'(\bar{t}) = 0, x''(\bar{t}) \leq 0; \quad x'(\underline{t}) = 0, x''(\underline{t}) \geq 0. \tag{3.2}$$

In view of (3.1) and (3.2) and noticing $f(t,0) = 0$ for all $t \in \mathbb{R}$, we obtain

$$g_1(\bar{t}, x(\bar{t}-\tau_1)) + g_2(\bar{t}, x(\bar{t}-\tau_2)) - e(\bar{t}) \geq 0 \text{ and}$$
$$g_1(\underline{t}, x(\underline{t}-\tau_1)) + g_2(\bar{t}, x(\bar{t}-\tau_2)) - e(\underline{t}) \leq 0,$$

which implies there exists a constant $\hat{t} \in \mathbb{R}$ such that

$$g_1(\hat{t}, x(\hat{t}-\tau_1)) + g_2(\hat{t}, x(\hat{t}-\tau_2)) - e(\hat{t}) = 0. \tag{3.3}$$

Now we show that the following claim is true.

**Claim.** *There exists a constant* $\hat{t_0} \in \mathbb{R}$ *such that*

$$|x(\hat{t_0})| \leq D_1. \tag{3.4}$$

Assume, by way of contradiction, that (3.4) does not hold. Then

$$|x(t)| > D_1 \quad \text{for all } t \in \mathbb{R}, \tag{3.5}$$

which, together with ($H_5$) and (3.3), implies that one of the following relations holds:

$$x(\hat{t} - \tau_1) \geq x(\hat{t} - \tau_2) > D_1, \tag{3.6}$$

$$x(\hat{t} - \tau_2) \geq x(\hat{t} - \tau_1) > D_1, \tag{3.7}$$

$$x(\hat{t} - \tau_1) \leq x(\hat{t} - \tau_2) < -D_1, \tag{3.8}$$

$$x(\hat{t} - \tau_2) \leq x(\hat{t} - \tau_1) < -D_1. \tag{3.9}$$

Suppose that (3.6) holds, in view of ($H_1$), ($H'_1$), and ($H_5$), we will consider two cases as follows:

*Case* (i): If ($H_1$) and ($H_5$) hold, according to (3.6), we have

$$\begin{aligned} 0 &> g_1(\hat{t}, x(\hat{t} - \tau_2)) + g_2(\hat{t}, x(\hat{t} - \tau_2)) - e(\hat{t}) \\ &\geq g_1(\hat{t}, x(\hat{t} - \tau_1)) + g_2(\hat{t}, x(\hat{t} - \tau_2)) - e(\hat{t}), \end{aligned}$$

which contradicts that (3.3). This contradiction implies that (3.4) is true.

*Case* (ii): If ($H'_1$) and ($H_5$) hold, according to (3.6), we have

$$\begin{aligned} 0 &> g_1(\hat{t}, x(\hat{t} - \tau_1)) + g_2(\hat{t}, x(\hat{t} - \tau_1)) - e(\hat{t}) \\ &\geq g_1(\hat{t}, x(\hat{t} - \tau_1)) + g_2(\hat{t}, x(\hat{t} - \tau_2)) - e(\hat{t}), \end{aligned}$$

which contradicts that (3.3). This contradiction implies that (3.4) is true.

Suppose that (3.7)(or (3.8), or (3.9)) holds; using methods similar to those used in Case(i) and (ii), we can show that (3.4) is also true. This completes the proof of the above claim.

For any $t \in [\hat{t_0}, \hat{t_0} + T]$, we have

$$|x(t)| = \left| x(\hat{t_0}) + \int_{\hat{t_0}}^{t} x'(s)ds \right| \leq D_1 + \int_{\hat{t_0}}^{t} |x'(s)|ds$$

and

$$\begin{aligned} |x(t)| = \left| x(\hat{t_0} + T) + \int_{\hat{t_0}+T}^{t} x'(s)ds \right| &\leq D_1 + \left| - \int_{t}^{\hat{t_0}+T} x'(s)ds \right| \\ &\leq D_1 + \int_{t}^{\hat{t_0}+T} |x'(s)|ds. \end{aligned}$$

Combining above two inequalities, we get

$$|x(t)| \leq D_1 + \frac{1}{2} \int_0^T |x'(s)|ds.$$

Using Schwartz inequality yields

$$|x|_\infty = \max_{t \in [\hat{t_0}, \hat{t_0}+T]} |x(t)| \leq D_1 + \frac{1}{2} \int_0^T |x'(s)|ds \leq D_1 + \frac{1}{2} |1|_2 |x'|_2 = D_1 + \frac{1}{2} \sqrt{T} |x'|_2. \tag{3.10}$$

Since ($H_2$) and $f(t,0)=0$ imply that $|f(t,x)|\le C_0|x|$, by Lemma 2.1, ($H_2$), ($H_3$), (3.1), (3.10) and Schwartz inequality, we obtain

$$
\begin{aligned}
|x''|_2^2 &= -\lambda\int_0^T f(t,x'(t))x''(t)dt-\lambda\int_0^T g_1(t,x(t-\tau_1))x''(t)dt\\
&\quad -\lambda\int_0^T g_2(t,x(t-\tau_2))x''(t)dt+\lambda\int_0^T e(t)x''(t)dt\\
&= -\lambda\int_0^T f(t,x'(t))x''(t)dt-\lambda\int_0^T [g_1(t,x(t-\tau_1))-g_1(t,0)+g_1(t,0)]x''(t)dt\\
&\quad -\lambda\int_0^T [g_2(t,x(t-\tau_2))-g_2(t,0)+g_2(t,0)]x''(t)dt+\lambda\int_0^T e(t)x''(t)dt\\
&\le \int_0^T C_0|x'(t)||x''(t)|dt+\int_0^T |g_1(t,x(t-\tau_1))-g_1(t,0)||x''(t)|dt+G_1\int_0^T |x''(t)|dt\\
&\quad +\int_0^T |g_2(t,x(t-\tau_2))-g_2(t,0)||x''(t)|dt+G_2\int_0^T |x''(t)|dt+|e|_\infty\int_0^T |x''(t)|dt\\
&\le C_0\int_0^T |x'(t)||x''(t)|dt+C_1\int_0^T |x(t-\tau_1)||x''(t)|dt\\
&\quad +C_2\int_0^T |x(t-\tau_2)||x''(t)|dt+(G_1+G_2+|e|_\infty)\int_0^T |x''(t)|dt\\
&\le C_0\left(\int_0^T |x'(t)|^2dt\right)^{1/2}\left(\int_0^T |x''(t)|^2dt\right)^{1/2}\\
&\quad +C_1\left(\int_0^T |x(t-\tau_1)|^2dt\right)^{1/2}\left(\int_0^T |x''(t)|^2dt\right)^{1/2}\\
&\quad +C_2\left(\int_0^T |x(t-\tau_2)|^2dt\right)^{1/2}\left(\int_0^T |x''(t)|^2dt\right)^{1/2}\\
&\quad +(G_1+G_2+|e|_\infty)\left(\int_0^T 1^2dt\right)^{1/2}\left(\int_0^T |x''(t)|^2dt\right)^{1/2}\\
&\le C_0|x'|_2|x''|_2^2+(C_1+C_2)\sqrt{T}|x|_\infty|x''|_2+(G_1+G_2+|e|_\infty)|1|_2|x''|_2\\
&\le \left[C_0\frac{T}{2\pi}+(C_1+C_2)\frac{T^2}{4\pi}\right]|x''|_2^2+G_3\sqrt{T}|x''|_2 \qquad (3.11)
\end{aligned}
$$

where $G_1=max\{|g_1(t,0)|:t\in[0,T]\}$,$G_2=max\{|g_2(t,0)|:t\in[0,T]\}$ and $G_3=G_1+G_2+|e|_\infty+C_1D_1+C_2D_1$.

By ($H_4$), there exists a constant $M_0>0$ such that

$$|x''|_2<M_0. \qquad (3.12)$$

Since $x(0)=x(T)$, there exists a constant $\tilde{t}\in[0,T]$ such that $x'(\tilde{t})=0$. For any $t\in[\tilde{t},\tilde{t}+T]$, by Schwartz inequality, we have

$$|x'(t)|=\left|x'(\tilde{t})+\int_{\tilde{t}}^t x''(s)ds\right|\le\int_0^T |x''(s)|ds\le|1|_2|x''|_2=\sqrt{T}|x''|_2,$$

which implies

$$|x'|_\infty = \max_{t\in[\bar{t},\bar{t}+T]} |x'(t)| \le \sqrt{T}|x''|_2. \tag{3.13}$$

By Lemma 2.1, (3.10), (3.12) and (3.13), there exists a constant $M > max\{M_0, D_1\}$ such that

$$|x|_\infty < M \text{ and } |x'|_\infty < M,$$

hence, all possible $T$-periodic solutions of Eq.(3.1) are bounded in $C_T^1$.

Secondly, we proof the existence of $T$-periodic solutions to Eq.(1.1).
Set

$$\Omega = \{x : x \in C_T^1, |x|_\infty < M, |x'|_\infty < M\}. \tag{3.14}$$

Define a linear operator $L : D(L) \subset C_T^1 \to C_T$ by setting

$$D(L) = \{x : x \in C_T^1, x'' \in C(\mathbb{R}, \mathbb{R})\}$$

and for $x \in D(L)$,

$$Lx = x''. \tag{3.15}$$

We also define a nonlinear operator $N : C_T^1 \to C_T$ by setting

$$Nx = -f(t, x'(t)) - g_1(t, x(t-\tau_1)) - g_2(t, x(t-\tau_2)) + e(t), \tag{3.16}$$

then, Eq.(3.1) is equivalent to the following operator equation

$$Lx = \lambda Nx, \quad \lambda \in (0,1). \tag{3.17}$$

It is easy to see that

$$KerL = \mathbb{R} \text{ and } ImL = \left\{ x : x \in C_T, \int_0^T x(t)dt = 0 \right\},$$

thus $L$ is a Fredholm operator with index zero.
Also let projectors $P : C_T^1 \to KerL$ and $Q : C_T \to C_T/ImL$ defined by

$$Px = x(0) \quad \text{where } x \in C_T^1$$

and

$$Qx = \frac{1}{T}\int_0^T x(t)dt \quad \text{where } x \in C_T,$$

hence, Im$P$=Im$Q$=Ker$L$=$\mathbb{R}$ and Ker$Q$=Im$L$.
Define the isomorphism as follows

$$J : ImQ \to KerL, \quad J(x) = x. \tag{3.18}$$

Let

$$L_P := L_{D(L)\cap Ker\,P} : D(L) \cap Ker\,P \to ImL,$$

then, from [5], $L_P$ has a continuous inverse $L_P^{-1}$ on Im$L$ defined by

$$(L_P^{-1}x)(t) = -\frac{t}{T}\int_0^T (t-s)x(s)ds + \int_0^t (t-s)x(s)ds, \tag{3.19}$$

In view of (3.14) and (3.19), $N$ is $L$-compact on $\overline{\Omega}$. By (3.14) and (3.17), the condition (i) of Lemma 2.3 is satisfied.
Since

$$QNx = \frac{1}{T}\int_0^T [-f(t,x'(t)) - g_1(t,x(t-\tau_1)) - g_2(t,x(t-\tau_2)) + e(t)]dt,$$

for any $x \in \partial\Omega \cap KerL$, $x = M$ or $x = -M$, then in view of ($H_5$) and $f(t,0) = 0$ for all $t \in \mathbb{R}$, we obtain

$$QN(M) = -\frac{1}{T}\int_0^T [g_1(t,M) + g_2(t,M) - e(t)]dt > 0 \tag{3.20}$$

and

$$QN(-M) = -\frac{1}{T}\int_0^T [g_1(t,-M) + g_2(t,-M) - e(t)]dt < 0, \tag{3.21}$$

which implies the condition (ii) of Lemma 2.3 is satisfied.
Moreover, define

$$H(x,\mu) = \mu x + (1-\mu)QNx = \mu x + (1-\mu)\frac{1}{T}\int_0^T [-f(t,x'(t)) - g_1(t,x(t-\tau_1)) \\ - g_2(t,x(t-\tau_2)) + e(t)]dt,$$

in view of (3.20) and (3.21), we get

$$xH(x,\mu) > 0, \quad \text{for all } x \in \partial\Omega \cap KerL, \mu \in (0,1).$$

Hence, $H(x,\mu)$ is a homotopic transformation, together with (3.18) and by using the homotopic invariance theorem, we have

$$deg\{JQN, \Omega \cap KerL, 0\} = deg\{QN, \Omega \cap KerL, 0\} = deg\{x, \Omega \cap KerL, 0\} \neq 0,$$

so condition (iii) of Lemma 2.3 is satisfied. In view of previous Lemma 2.3, there exists at least one solution with period $T$. This completes the proof.

**Remark 1.** If $f(t,0) \neq 0$, the problem of the existence and uniqueness of $T$-periodic solutions to Eq.(1.1) can be converted to the following equation

$$x''(t) + f_1(t,x'(t)) + g_1(t,x(t-\tau_1)) + g_2(t,x(t-\tau_2)) = e_1(t), \tag{3.22}$$

where $f_1(t,x'(t)) = f(t,x'(t)) - f(t,0), e_1(t) = e(t) - f(t,0)$. As $f_1(t,0) = 0$ for all $t \in \mathbb{R}$, Eq.(3.22) can be studied by Theorem 1 in this paper.

## 4 Example and remark

In this section, we apply the main results obtained in previous sections to an example.

**Example 4.1**. Consider the following Rayleigh equation with two delays

$$x''(t) + f(t, x'(t)) + g_1(t, x(t - \tau_1)) + g_2(t, x(t - \tau_2)) = e(t), \tag{4.1}$$

where $\tau_1 \geq 0$ and $\tau_2 \geq 0$ are two constants, $T = 2\pi$, $e(t) = cos^2 \frac{1}{2}t$ and

$$\begin{aligned} f(t, x'(t)) &= -\frac{1}{40}(1 + sin^2 t)x'(t)arctan x'(t), \\ g_1(t, x(t - \tau_1)) &= -\frac{1}{40}(1 + cos^2 t)x(t - \tau_1), \\ g_2(t, x(t - \tau_2)) &= -\frac{1}{60}e^{cos^2 t} arctan(x(t - \tau_2) + 1) \end{aligned}$$

Set $C_0 = \frac{3}{20}, C_1 = \frac{1}{20}, C_2 = \frac{1}{20}$, and let $D_1$ be big enough. Then it is easy to check that all the conditions of Theorem 1 in this paper hold, which implies Eq.(4.1) has a unique $2\pi$-periodic solution.

**Remark 2.** Eq.(4.1) is a very simple version of Rayleigh equation with two delays, all the results in [1,3–9] and the references therein cannot be applicable to Eq.(4.1) for securing the existence and uniqueness of $2\pi$-periodic solutions, which implies the results in this paper are new and they complement previously known results.

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