Sequences of some meromorphic function spaces

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Abstract

Our goal in this paper is to introduce some new sequences of some meromorphic function spaces, which will be called b_q and q_K-sequences. Our study is motivated by the theories of normal, $Q_K^{\#}$ and meromorphic Besov functions. For a non-normal function f the sequences of points $\{a_n\}$ and $\{b_n\}$ for which

$$\lim_{n\to\infty}(1-|a_n|^2)f^{\#}(a_n) = +\infty \text{ and}$$

$$\lim_{n\to\infty}\iint_{\Delta}(f^{\#}(z))^q(1-|z|^2)^{q-2}(1-|\varphi_{a_n}(z)|^2)^s dA(z) = +\infty$$

or

$$\lim_{n\to\infty}\iint_{\Delta}\left(f^{\#}(z)\right)^2 K(z,a_n)dA(z) = +\infty$$

are considered and compared with each other. Finally, non-normal meromorphic functions are described in terms of the distribution of the values of these meromorphic functions.

1 Introduction

Let $\Delta = \{z : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and let $dA(z)$ be the Euclidean area element on Δ. Let $M(\Delta)$ denote the class of functions meromorphic in Δ. The pseudohyperbolic distance between z and a is given by $\sigma(z,a) = |\varphi_a(z)|$, where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ is the Möbius transformation of Δ. For $0 < r < 1$, let $\Delta(a,r) = \{z \in \Delta : \sigma(z,a) < r\}$ be the pseudohyperbolic disk with

Received by the editors February 2008.
Communicated by F. Brackx.
1991 *Mathematics Subject Classification* : 30D45, 46E15.
Key words and phrases : b_q, q_K -sequences, meromorphic functions , Besov classes.

center $a \in \Delta$ and radius r. For $0 < q < \infty$ and $0 < s < \infty$, the classes $M^{\#}(p,q,s)$ are defined in [15] as follows:

$$M^{\#}(p,q,s) = \left\{ f \in M(\Delta) : \sup_{a\in\Delta} \iint_{\Delta} (f^{\#}(z))^{p} \left(1-|z|^2\right)^{q} \left(1-|\varphi_a(z)|^2\right)^{s} dA(z) < \infty \right\}, \tag{1}$$

where $f^{\#}(z) = \frac{|f'(z)|}{1+|f(z)|^2}$ is the spherical derivative of f. The classes $M^{\#}(q,q-2,0)$ are called the Besov-type classes, they are denoted by $B^{\#}_{q}$, where

$$B^{\#}_{q} = \left\{ f \in M(\Delta) : \sup_{a\in\Delta} \iint_{\Delta} \left(f^{\#}(z)\right)^{q} \left(1-|z|^2\right)^{q-2} dA(z) < \infty \right\}.$$

But in this paper the meromorphic Besov-type classes always refer to the classes $M^{\#}(q,q-2,s)$. Let $0 < q < \infty$ and $0 < s < \infty$. Then the Besov-type classes are defined by:

$$B^{\#}_{q,s} = \left\{ f \in M(\Delta) : \sup_{a\in\Delta} \iint_{\Delta} \left(f^{\#}(z)\right)^{q} \left(1-|z|^2\right)^{q-2} \left(1-|\varphi_a(z)|^2\right)^{s} dA(z) < \infty \right\}, \tag{2}$$

where the weight function is $\left(1-|z|^2\right)^{q-2}\left(1-|\varphi_a(z)|^2\right)^{s}$ and $z \in \Delta$.
For more information about holomorphic and meromorphic Besov classes, we refer to [5, 6, 10, 11, 12, 14, 17, 18, 23] and others.
Recently Wulan [20] gave the following definition:

Definition 1.1. *Let $K : [0,\infty) \to [0,\infty)$ be a nondecreasing function. A function f meromorphic in Δ is said to belong to the class $Q^{\#}_{K}$ if*

$$\sup_{a\in\Delta} \iint_{\Delta} \left(f^{\#}(z)\right)^{2} K\big(g(z,a)\big)\, dA(z) < \infty,$$

where, the function $g(z,a) = \ln\left|\frac{1-\bar{a}z}{a-z}\right|$ is defined as the composition of the Möbius transformation φ_a and the fundamental solution of the two-dimensional real Laplacian.

$Q^{\#}_{K}$ space has been studied during the last few years (see e.g [8, 9] and others). The meromorphic counterpart of the Bloch space is the class of normal functions $\mathcal{N}$ (see [1, 2, 15, 16, 21]), which is defined as follows:

Definition 1.2. *Let f be a meromorphic function in Δ. If*

$$\|f\|_{\mathcal{N}} = \sup_{z\in\Delta} (1-|z|^2) f^{\#}(z) < \infty, \tag{3}$$

then f belongs to the class $\mathcal{N}$ of normal functions.

Definition 1.3. *([4]) Let f be a meromorphic function in Δ. A sequence of points $\{a_n\}$ $(|a_n| \to 1)$ in Δ is called a $q_{\mathcal{N}}-$sequence if*

$$\lim_{n\to\infty} f^{\#}(a_n)(1-|a_n|^2) = +\infty. \tag{4}$$

Now, we will introduce the following definitions:

Definition 1.4. *Let f be a meromorphic function in Δ, $2 < q < \infty$ and $0 < s < \infty$. A sequence of points $\{a_n\}(|a_n| \to 1)$ in Δ is called a $bq-$sequence if*

$$\lim_{n\to\infty} \iint_{\Delta} (f^{\#}(z))^{q} (1-|z|^2)^{q-2} \left(1-|\varphi_{a_n}(z)|^2\right)^{s} dA(z) = +\infty. \tag{5}$$

Definition 1.5. *Let f be a meromorphic function in Δ. For a function K, $K : [0,\infty) \longrightarrow [0,\infty)$. A sequence of points $\{a_n\}(|a_n| \to 1)$ in Δ is called a q_K-sequence if*

$$\lim_{n\to\infty} \iint_{\Delta} (f^{\#}(z))^{2} K(g(z,a_n)) dA(z) = +\infty. \tag{6}$$

2 $\mathbf{b_q}$ and $\mathbf{q}_{\mathcal{N}}-$sequences

In this section, we study some new sequences of some meromorphic function spaces such as b_q and $q_{\mathcal{N}}-$sequences. Our study is motivated by the theories of normal and meromorphic Besov functions. We prove various results about these sequences. For example, if $\{a_n\}$ is a q_N sequence for the meromorphic function f and $\{b_n\}$ is a sequence with $\sigma(a_n,b_n) \to 0$ as $n \to \infty$, where $\sigma(a_n,b_n)$ denotes the pseudohyperbolic distance, then $\{b_n\}$ is a b_q sequence for f for every $q > 2$.
We will need the following definition in the sequel:

Definition 2.1. *[19] Let f be a meromorphic function in $\mathbb{C}$. If the family $\{f(z+a_n)\}$ is normal for any sequence $\{a_n\}$ of complex numbers, then f is a Yosida function $y(z)$.*

Theorem 2.1. *Let f be a meromorphic function in Δ. If $\{a_n\}$ is a $q_{\mathcal{N}}-$sequence, then any sequence of points $\{b_n\}$ in Δ for which $\sigma(a_n,b_n) \to 0$ is a $bq-$sequence for all q, $2 < q < \infty$.*

Proof. By([19], theorem 4.4.1) with $\beta = 0$ and $\alpha = 1$, there exist sequences $\{b_n\} \subset \Delta$ and $\{p_n\} \subset \mathbb{R}^+$, with $\sigma(a_n,b_n) \longrightarrow 0$ and

$$\frac{p_n}{(1-|b_n|^2)} \longrightarrow 0, \tag{7}$$

where the sequence of functions $\{f_n(t)\} = \{f(b_n + p_n t)\}$ converges uniformly on each compact subset of $\mathbb{C}$ to a nonconstant Yosida function $y(t)$. Then

$$\begin{aligned}
&\sup_{b_n\in\Delta} \iint_{\Delta} (f^{\#}(z))^{q} (1-|z|^2)^{q-2} \left(1-|\varphi_{b_n}(z)|^2\right)^{s} dA(z)\\
\geq\ & \iint_{\Delta(b_n,\frac{1}{e})} (f^{\#}(z))^{q} (1-|z|^2)^{q-2} \left(1-|\varphi_{b_n}(z)|^2\right)^{s} dA(z)\\
\geq\ & \iint_{\Delta(0,r)} (y_n^{\#}(t))^{q} (1-|b_n+p_nt|^2)^{q-2} \left(1-|\varphi_{b_n}(b_n+p_nt)|^2\right)^{s} p_n^{2-q} dA(t)\\
=\ & \iint_{\Delta(0,r)} |y_n^{\#}(t)|^{q} \left(\frac{1-|b_n+p_nt|^2}{p_n}\right)^{q-2} \times \left(1-\left|\frac{b_n-(b_n+p_nt)}{1-\bar{b}_n(b_n+p_nt)}\right|^2\right)^{s} dA(t)\\
=\ & \iint_{\Delta(0,r)} (y_n^{\#}(t))^{q} \left(\frac{1-|b_n+p_nt|^2}{p_n}\right)^{q-2} \times \left(1-\left|\frac{1}{\frac{1-|b_n|^2}{p_nt}-\bar{b}_n}\right|^2\right)^{s} dA(t)\,.
\end{aligned}$$

By the uniformly convergence, we have

$$\iint_{\Delta(0,r)} (f_n^{\#}(t))^q dA(t) \longrightarrow \iint_{\Delta(0,r)} (y^{\#}(t))^q dA(t),$$

and this last integral is positive, because $y(t)$ is a nonconstant meromorphic function. Moreover, using (7) as $n \to \infty$, we obtain that

$$1 - \left| \frac{1}{\frac{1-|b_n|^2}{p_n t} - \bar{b}_n} \right|^2 \longrightarrow 1.$$

Then, we conclude that

$$\iint_{\Delta(0,r)} (y_n^{\#}(t))^q \left(\frac{1 - |a_n + p_n t|^2}{p_n} \right)^{q-2} \times \left(1 - \left| \frac{1}{\frac{1-|a_n|^2}{p_n t} - \bar{a}_n} \right|^2 \right)^s dA(t) \longrightarrow \infty,$$

and it follows for all q, where $2 < q < \infty$ that

$$\iint_{\Delta} (f^{\#}(z))^q (1 - |z|^2)^{q-2} \left(1 - |\varphi_{b_n}(z)|^2\right)^s dA(z) \longrightarrow \infty,$$

then $\{b_n\} \in \Delta$ is a b_q-sequence for all q, $2 < q < \infty$. Thus the proof of Theorem 2.1 is established.

Theorem 2.2. *There exist a non-normal function f and $\{a_n\}$ in Δ which is a b_q-sequence for all q, $2 < q < \infty$, but $\{a_n\}$ is not a $q_{\mathcal{N}}-$sequence.*

Proof. By ([4] theorem 2), we can consider a function $f(z) = \exp\left(\frac{i}{1-z}\right)$ be not normal, $i = \sqrt{-1}$. Choose a sequence $\{b_n\} = \{\frac{n^2}{1+n^2}\}$ and by a computation, we obtain that

$$\lim_{n \to \infty} (1 - |b_n|^2) f^{\#}(b_n) = +\infty.$$

By Theorem 2.1 for any sequence of points $\{a_n\}$ in Δ for which $\sigma(a_n, b_n) \to 0$,

$$\lim_{n \to \infty} \iint_{\Delta} (f^{\#}(z))^q (1 - |z|^2)^{q-2} \left(1 - |\varphi_{a_n}(z)|^2\right)^s dA(z) = +\infty,$$

for all q, $2 < q < \infty$. Now we choose $\{a_n\} = \{\frac{n^2}{1+n^2} - \frac{i}{n+n^3}\}$ and notice that $\sigma(a_n, b_n) \to 0$. But

$$\lim_{n \to \infty} (1 - |a_n|^2) f^{\#}(a_n) = 0.$$

Thus $\{a_n\}$ is just one we need.

Theorem 2.3. *Let f be a meromorphic function in Δ and let $2 < q' < q < \infty$ and $0 < s' < s < \infty$. If, for a sequence of points $\{a_n\}$ in Δ,*

$$\lim_{n \to \infty} \iint_{\Delta} (f^{\#}(z))^q (1 - |z|^2)^{q-2} \left(1 - |\varphi_{a_n}(z)|^2\right)^s dA(z) = +\infty, \tag{8}$$

then

$$\lim_{n \to \infty} \iint_{\Delta} (f^{\#}(z))^{q'} (1 - |z|^2)^{q'-2} \left(1 - |\varphi_{a_n}(z)|^2\right)^{s'} dA(z) = +\infty. \tag{9}$$

Proof. If assumption (8) holds for $2 < q' < q < \infty$ and $0 < s' < s < \infty$, then by Hölder's inequality, we have that

$$\begin{aligned}
&\iint_\Delta (f^\#(z))^{q'}(1-|z|^2)^{q'-2}\left(1-|\varphi_{a_n}(z)|^2\right)^{s'} dA(z)\\
\leq\ & \left(\iint_\Delta (f^\#(z))^{q}(1-|z|^2)^{q-2}\left(1-|\varphi_{a_n}(z)|^2\right)^{s} dA(z)\right)^{\frac{q'}{q}}\\
\times\ & \left(\iint_\Delta (1-|\varphi_{a_n}(z)|^2)^{(s'-\frac{sq'}{q})(\frac{q}{q-q'})}(1-|z|^2)^{-2} dA(z)\right)^{(1-\frac{q'}{q})}\\
=\ & \left(\iint_\Delta (f^\#(z))^{q}(1-|z|^2)^{q-2}\left(1-|\varphi_{a_n}(z)|^2\right)^{s} dA(z)\right)^{\frac{q'}{q}}\\
\times\ & \left(\iint_\Delta (1-|w|^2)^{(\frac{s'q-sq'}{q-q'}-2)} dA(w)\right)^{(1-\frac{q'}{q})}.
\end{aligned}$$

Since it is easy to check $(\frac{s'q-sq'}{q-q'}-2) = (\kappa-2) > -1$, for $\kappa = \frac{s'q-sq'}{q-q'} > 1$, then we obtain that

$$\iint_\Delta (1-|w|^2)^{(\frac{s'q-sq'}{q-q'}-2)} dA(w) = \iint_\Delta (1-|w|^2)^{(\kappa-2)} dA(w) < C < \infty, \quad \text{for } C > 0.$$

Thus,

$$M^\#(q,q-2,s) \subset M^\#(q',q'-2,s'),$$

Hence, we obtain that

$$\begin{aligned}
&\iint_\Delta (f^\#(z))^{q'}(1-|z|^2)^{q'-2}\left(1-|\varphi_{a_n}(z)|^2\right)^{s'} dA(z)\\
\geq\ & \iint_\Delta (f^\#(z))^{q}(1-|z|^2)^{q-2}\left(1-|\varphi_{a_n}(z)|^2\right)^{s} dA(z) = +\infty.
\end{aligned}$$

Then assumption (9) holds. Hence the proof of Theorem 2.3 is completed.

Remark 2.1. *By assumption (8), we know that $f \notin M^\#(q,q-2,s)$. Since the function classes $M^\#(q,q-2,s)$ have a nesting property, $f \notin M^\#(q',q'-2,s')$, for $q' < q$ and $0 < s' < s < \infty$. However, Theorem 2.3 gives more information about this situation showing that the same sequence $\{a_n\}$, which breaks the $M^\#(q,q-2,s)-$condition, also breaks $M^\#(q',q'-2,s')-$condition.*

Remark 2.2. *In fact, from the proof of Theorem 2.3, we can see that if for a fixed r_0, $0 < r_0 < 1$ and $R > 0$,*

$$\lim_{n\to\infty} \iint_{\Delta(a_n,r_0)} (f^\#(z))^{q}(1-|z|^2)^{q-2}\left(1-|\varphi_{a_n}(z)|^2\right)^{s} dA(z) = +\infty,$$

then there exists a sequence of points $\{b_n\}$ in $U_R^n = \{z : (1-|\varphi_a(z)|^2) > R\}$, such that

$$\lim_{n\to\infty}(1-|b_n|^2)f^\#(b_n) = +\infty.$$

Theorem 2.4. *Let f be a meromorphic function in Δ. If, for a sequence of points $\{a_n\}$ in Δ,*

$$\lim_{n\to\infty}(1-|a_n|^2)f^{\#}(a_n) = +\infty, \tag{10}$$

then for the same sequence $\{a_n\}$

$$\lim_{n\to\infty}\iint_{\Delta(a_n,r)}(f^{\#}(z))^q(1-|z|^2)^{q-2}\left(1-|\varphi_{a_n}(z)|^2\right)^s dA(z) = +\infty,$$

holds for all q, s, $2 < q < \infty$, $0 < s < \infty$ and all r, $0 < r < 1$.

Proof. Suppose that (10) holds. If there exists an r_0, $0 < r_0 < 1$ and p, $1 < p < \infty$, such that

$$\limsup_{n\to\infty}\iint_{\Delta(a_n,r_0)}(f^{\#}(z))^q(1-|z|^2)^{q-2}\left(1-|\varphi_{a_n}(z)|^2\right)^s dA(z) = M < +\infty,$$

then there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$, such that

$$\iint_{\Delta(a_{n_k},r_0)}(f^{\#}(z))^q(1-|z|^2)^{q-2}\left(1-|\varphi_{a_{n_k}}(z)|^2\right)^s dA(z) \le M+1,$$

for k sufficiently large.
Now, choose an r_1, $0 < r_1 < r_0$, $\Delta(a_{n_k}, r_1) = \{z \in \Delta : |\ \varphi_{a_{n_k}}(z)\ | < r_1\}$, satisfying

$$\frac{M+1}{(1-r_1^2)^{s+q-2}} < \frac{\pi}{2}.$$

It follows that

$$\iint_{\Delta(a_{n_k},r_1)}(f^{\#}(z))^q dA(z) \le \frac{M+1}{(1-r_1^2)^{s+q-2}} < \frac{\pi}{2},$$

for $(1-|\varphi_{a_{n_k}}(z)|^2) \ge (1-r_1^2)$.
By Dufresnoy's theorem (see [16] pp.83), we have $(1-|a_{n_k}|^2)f^{\#}(a_{n_k}) \le \frac{1}{r_1}$, which contradicts our assumption. Hence the proof of Theorem 2.4 is completed.

Theorem 2.5. *Let f be a meromorphic function in Δ. Suppose for $0 < p < \infty$, there exists a sequence of points $\{a_n\} \subset \Delta$, such that*

$$\lim_{n\to\infty}\iint_{\Delta}(f^{\#}(z))^q(1-|z|^2)^{q-2}(1-|\varphi_{a_n}(z)|^2)^s dA(z) = +\infty.$$

Then, for any sequence of points $\{b_n\}$ in Δ for which $\sigma(a_n, b_n) \to 0$,

$$\lim_{n\to\infty}\iint_{\Delta}(f^{\#}(z))^q(1-|z|^2)^{q-2}\left(1-|\varphi_{b_n}(z)|^2\right)^s dA(z) = +\infty.$$

Proof. Choose positive constants M_1 and M_2 such that $M_2 < M_1$. Let

$$U^n_{M_1} = \{z : (1-|\varphi_{a_n}(z)|^2) > M_1\} \text{ and } U^n_{M_2} = \{z : (1-|\varphi_{a_n}(z)|^2) > M_2\}.$$

Then if $w \in U^n_{M_1}$, $z \in \Delta\backslash U^n_{M_2}$ and $C(1-|\varphi_{a_n}(z)|^2) \leq (1-|\varphi_{b_n}(w)|^2)$ for some constant $C > 0$.
This means for all n that,

$$\iint_{\Delta\backslash U^n_{M_2}} (f^{\#}(z))^q(1-|z|^2)^{q-2}\left(1-|\varphi_{b_n}(z)|^2\right)^s dA(z)$$
$$\geq C^s \iint_{\Delta\backslash U^n_{M_2}} (f^{\#}(z))^q(1-|z|^2)^{q-2}\left(1-|\varphi_{a_n}(z)|^2\right)^s dA(z), \qquad (11)$$

for any sequence of points $\{b_n\}$ in Δ for which $\sigma(a_n, b_n) \to 0$. If

$$\limsup_{n\to\infty} \iint_{\Delta\backslash U^n_{M_2}} (f^{\#}(z))^q(1-|z|^2)^{q-2}\left(1-|\varphi_{a_n}(z)|^2\right)^s dA(z) = +\infty,$$

Then, by (11)

$$\limsup_{n\to\infty} \iint_{\Delta\backslash U^n_{M_2}} (f^{\#}(z))^q(1-|z|^2)^{q-2}\left(1-|\varphi_{b_n}(z)|^2\right)^s dA(z) = +\infty.$$

Also, if

$$\limsup_{n\to\infty} \iint_{U^n_{M_2}} (f^{\#}(z))^q(1-|z|^2)^{q-2}\left(1-|\varphi_{a_n}(z)|^2\right)^s dA(z) = +\infty,$$

then, we have two different cases:
Either (i) there exists a sequence of points $\{c_n\}$ in $U^n_{M_2}$ for which $\sigma(a_n, c_n) \to 0$, such that

$$\lim_{n\to\infty}(1-|c_n|^2)f^{\#}(c_n) = +\infty,$$

or (ii) there exists r_0, $0 < r_0 < e^{-M_2}$ and $k > 0$, such that

$$(1-|z|^2)f^{\#}(z) \leq k, \text{ for all } z \in \Delta(a_n, r_0).$$

If (i) is true, then, by Theorem 2.1, for above $\{b_n\}$, for which $\sigma(a_n, b_n) \to 0$,

$$\limsup_{n\to\infty} \iint_{\Delta} (f^{\#}(z))^q(1-|z|^2)^{q-2}\left(1-|\varphi_{b_n}(z)|^2\right)^s dA(z) = +\infty,$$

since $\sigma(b_n, c_n) \to 0$. On the other hand, if (ii) holds, then using the same conclusions for weight functions we see that necessarily for any sequence of points $\{b_n\}$ for which $\sigma(a_n, b_n) \to 0$,

$$\limsup_{n\to\infty} \iint_{\Delta} (f^{\#}(z))^q(1-|z|^2)^{q-2}\left(1-|\varphi_{b_n}(z)|^2\right)^s dA(z) = +\infty.$$

This completes the proof.
Now, we consider the following question.
Question 2.1 Let $1 < q < \infty$ for any sequence of points $\{a_n\}$ and suppose that

$$\lim_{n\to\infty} \iint_{\Delta} (f^{\#}(z))^q(1-|z|^2)^{q-2}\left(1-|\varphi_{a_n}(z)|^2\right)^s dA(z) = +\infty.$$

Is it true for q', where $q < q'$,

$$\limsup_{n\to\infty} \iint_{\Delta} (f^{\#}(z))^{q'}(1-|z|^2)^{q'-2}\left(1-|\varphi_{a_n}(z)|^2\right)^s dA(z) = +\infty?$$

We answer the question by Theorem 2.6.

Definition 2.2. *Let $2 < q < \infty$. For any sequence of points $\{a_n\}$ in Δ is a m_q-sequence if*

$$\lim_{n\to\infty} \sup \iint_\Delta (f^\#(z))^q (1-|z|^2)^{q-2} (1-|\varphi_{a_n}(z)|^2)^q dA(z) = +\infty.$$

Our answer to Question 2.1 is naturally as follows:

Theorem 2.6. *Let $2 < q < \infty$ and suppose that*

$$\lim_{n\to\infty} \iint_\Delta (f^\#(z))^q (1-|z|^2)^{q-2} (1-|\varphi_{a_n}(z)|^2)^s dA(z) = +\infty.$$

If the sequence of points $\{a_n\}$ in Δ is not a m_q-sequence, then for any q' and $q < q'$ with $q' + s > 1$, then we have

$$\lim_{n\to\infty} \iint_\Delta (f^\#(z))^{q'} (1-|z|^2)^{q'-2} (1-|\varphi_{a_n}(z)|^2)^s dA(z) = +\infty.$$

Proof. Since,

(i) $M^\#(q, q-2, s) \subset \mathcal{N}$ for all q, $2 < q < \infty$ and $0 < s < 1$ (see [15] theorem 3.3.3).

(ii) $\bigcup_{2<q<q'} M^\#(q, q-2, s) \subsetneq M^\#(q', q'-2, s)$ for all q, where $2 < q < \infty$

and $0 < s < 1$ with $q' + s > 1$, the proof of this result can be found in [15]. So, it is easy to see that

$$\lim_{n\to\infty} \iint_\Delta (f^\#(z))^{q'} (1-|z|^2)^{q'-2} (1-|\varphi_{a_n}(z)|^2)^s dA(z) = +\infty.$$

3 q_K and $q_{\mathcal{N}}$-sequences

Now, we study b_q and $q_{\mathcal{N}}$-sequences. We prove many results about these sequences. Our results are obtained by the help of normal and $Q_K^\#$ functions. For example, if $\{a_n\}$ is a q_N sequence for the meromorphic function f and $\{b_n\}$ is a sequence with $\sigma(a_n, b_n) \to 0$ as $n \to \infty$, then $\{b_n\}$ is a q_K sequence for f.
Now, we give the following theorem:

Theorem 3.1. *Let f be a meromorphic function in Δ. If $\{a_n\}$ is a $q_{\mathcal{N}}-$sequence, then any sequence of points $\{b_n\}$ in Δ for which $\sigma(a_n, b_n) \to 0$ is a q_K-sequence for all K, $K(t) \to \infty$ as $t \to \infty$.*

Proof. By ([7], theorem 7.2), there exist sequences $\{c_n\} \subset \Delta$ and $\{p_n\} \subset \mathbb{R}^+$, with

$$\sigma(a_n, c_n) \longrightarrow 0 \quad \text{and} \quad \frac{p_n}{(1-|c_n|^2)} \longrightarrow 0, \tag{12}$$

where the sequence of functions $\{f_n(t)\} = \{f(c_n + p_n t)\}$ converges uniformly on each compact subset of $\mathbb{C}$ to a nonconstant meromorphic function $y(t)$.
For a fixed $R > 0$ set $\Delta_n = \{z : z = c_n + p_n t, |t| < R\}$. Now, for any sequence

of points $\{b_n\} \subset \Delta$, for which $\sigma(a_n, c_n) \longrightarrow 0$, we have $\sigma(b_n, c_n) \longrightarrow 0$ since $\sigma(a_n, c_n) \longrightarrow 0$. Thus, for n sufficiently large, we obtain that

$$\Delta_n = \{z : z = c_n + p_n t, |t| < R\} \subset \Omega_n = \{z : |\varphi_{b_n}(z)| < \frac{1}{e}\}.$$

Therefore, we get by change of variables

$$\begin{aligned} & \iint_{\Omega_n} (f^{\#}(z))^2 K(g(z, b_n) dA(z) \\ \geq & \iint_{\Delta_n} (f^{\#}(z))^2 K(g(z, b_n) dA(z) \\ = & \iint_{|t|<R} (f^{\#}(z))^2 K(g(c_n + p_n t, b_n) dA(z). \end{aligned}$$

By the uniformly convergence, we have

$$\iint_{|t|<R} (f_n^{\#}(t))^2 dA(t) \longrightarrow \iint_{|t|<R} (y^{\#}(t))^2 dA(t),$$

and the last integral is finite and non-zero, because $y(t)$ is a non-constant meromorphic function. However, $g(c_n + p_n t, b_n) \to +\infty$ as $n \to +\infty$ uniformly, for $|t| < R$, we obtain that

$$\iint_{|t|<R} (y^{\#}(z))^2 K(g(c_n + p_n t, b_n) dA(z) \longrightarrow \infty.$$

In fact,

$$g(c_n + p_n t, b_n) = \log \left| \frac{1 - \overline{b_n}(c_n + p_n t)}{c_n + p_n t - b_n} \right|$$

Moreover, using (12) as $n \to \infty$, we obtain that

$$\begin{aligned} \left| \frac{c_n + p_n t - b_n}{1 - \overline{b_n}(c_n + p_n t)} \right| & \leq \frac{|c_n - b_n| + p_n |t|}{|1 - \overline{b_n} c_n| - p_n |b_n t|} \\ & \leq \frac{\left| \frac{c_n - b_n}{1 - \overline{b_n} c_n} \right| + \frac{p_n |t|}{|1 - \overline{b_n} c_n|}}{1 - \frac{p_n |t|}{|1 - \overline{b_n} c_n|}} \longrightarrow 0. \end{aligned}$$

For all K, $K(t) \to \infty$ as $t \to \infty$, it follows that

$$\iint_{\Delta} (f^{\#}(z))^2 K(g(z, b_n) dA(z) \longrightarrow \infty,$$

then $\{b_n\} \in \Delta$ is a q_K-sequence for all K. Thus the proof of Theorem 3.1 is therefore established.

Theorem 3.2. *There exist a non-normal function f and $\{a_n\}$ in Δ which is a q_K-sequence for all K, $K : [0, \infty) \to [0, \infty)$, but $\{a_n\}$ is not a $q_{\mathcal{N}}-$sequence.*

Proof. The proof of this theorem is much akin to the proof of Theorem 2.2. So, it will be omitted.

4 Non-normal functions and $\rho_{\mathcal{N}}-$sequences.

In this section we define the concept of $\rho_{\mathcal{N}}-$sequences of meromorphic functions which allows one to describe non-normal functions. We give the necessary and sufficient condition for the sequence of points $\{z_n\}$, where $\lim_{n\to\infty}|z_n| = 1$ to be a $\rho_{\mathcal{N}}-$sequence in terms of the growth of f.
Makhmutov defined the concept of $\rho_{\mathcal{B}}-$sequences of holomorphic functions $f(z)$ in the unit disk Δ (see [13], pp. 9 definition 5.2.) as follows:

Definition 4.1. *A sequence of points $\{z_n\}$, $\lim_{n\to\infty}|z_n| = 1$, is a $\rho_{\mathcal{B}}-$sequence of holomorphic functions $f(z) \in \Delta$, if there are two sequences of numbers $\{\varepsilon_n\}$, where $\lim_{n\to\infty}|\varepsilon_n| = 0$ and $\{M_n\}$, $\lim_{n\to\infty} M_n = \infty$, for which the diameter of $f(\Delta(z_n, \varepsilon_n))$ exceeds $\{M_n\}$ for each n.*

Now, we define $\rho_{\mathcal{N}}-$sequences of meromorphic functions.

Definition 4.2. *A sequence of points $\{z_n\}$ with $\lim_{n\to\infty}|z_n| = 1$, is a $\rho_{\mathcal{N}}-$sequence of meromorphic functions f, if there are two sequences of numbers $\{\varepsilon_n\}$, where $\lim_{n\to\infty}|\varepsilon_n| = 0$ and $\{M_n\}$, $\lim_{n\to\infty} M_n = \infty$, for which the diameter of $f(\Delta(a_n, \varepsilon_n))$ exceeds $\{M_n\}$ for each n.*

Now, we let

$$A_f(a,r) = \iint_{\Delta(a,r)} (f^{\#}(z))^2 dxdy$$

be the area of the Riemann image of $\Delta(a,r)$ by f and

$$L(a,r) = \iint_{\Delta(a,r)} f^{\#}(z)\,|dz|$$

be the length of the Riemann image of the pseudohyperbolic circle $\Gamma(a,r)$ by f.
Let $F(a,r)$ be the Riemann image of $\Delta(a,r)$ by f and $\mathcal{F}(a,r)$ be the projection of $F(a,r)$ to $\mathbb{C}$. Let $\mathcal{A}_f(a,r)$ be the Euclidean area of $\mathcal{F}(a,r)$ and $\mathcal{L}(a,r)$ be the length of the outer boundary of $\mathcal{F}(a,r)$. It is clear that

$$\mathcal{A}_f(a,r) \le A_f(a,r) \quad \text{and} \quad \mathcal{L}_f(a,r) \le L_f(a,r)$$

for each $a \in \Delta$ and each $0 < r < 1$.

Yamashita proved in [22] that, for any holomorphic function $f(z)$ or a meromorphic function f in Δ, any $a \in \Delta$ and $0 < r < 1$,

$$(1 - |a|^2)f^{\#}(a) \le \left(\frac{\mathcal{A}_f(a,r)}{\pi r^2}\right)^{\frac{1}{2}},$$

$$(1 - |a|^2)f^{\#}(a) \le \frac{\mathcal{L}_f(a,r)}{2\pi r}.$$

Now, we give the following important proposition.

Proposition 4.1. *If f is a meromorphic function in Δ and $\{z_n\}$, $\lim_{n\to\infty}|z_n| = 1$, is such that*

$$\lim_{n\to\infty} (1 - |z_n|^2)f^{\#}(z_n) = +\infty,$$

then $\{z_n\}$ is a $\rho_{\mathcal{N}}-$sequence of the meromorphic function f.

Proof. Suppose that f is a meromorphic function in Δ and $\{z_n\}$, $\lim_{n\to\infty}|z_n| = 1$,

$$\lim_{n\to\infty}(1-|z_n|^2)f^{\#}(z_n) = +\infty,$$

let

$$(1-|z_n|^2) = \varepsilon_n \quad \textit{and} \quad M_n = f^{\#}(z_n),$$

then there are two sequences of numbers $\{\varepsilon_n\}$, $\{M_n\}$, where

$$\lim_{n\to\infty}|\varepsilon_n| = 0 \text{ and } \lim_{n\to\infty} M_n = 0.$$

By Definition 4.2, we have a sequence of points $\{z_n\}$ which is a $\rho_{\mathcal{N}}-$sequence.
If the sequence of points $\{z_n\}$ is a $\rho_{\mathcal{N}}-$sequence of the meromorphic function f, then there are two sequences $\lim_{n\to\infty}(1-|z_n|^2) = 0$ as $\lim_{n\to\infty}|z_n| = 1$ and $\lim_{n\to\infty} f^{\#}(z_n) = +\infty$. Our proposition is therefore proved.

Theorem 4.1. *A meromorphic function f is not a normal function if and only if it has a $\rho_{\mathcal{N}}-$sequence of points.*

Proof. **Necessity.** If $f \notin \mathcal{N}$, then there exists a sequence $\{z_n\}$ which satisfies the condition

$$\lim_{n\to\infty}(1-|z_n|^2)f^{\#}(z_n) = +\infty.$$

By Proposition 4.1, the sequence $\{z_n\}$ is a $\rho_{\mathcal{N}}-$sequence of the meromorphic function f.
Sufficiency. Let $\{a_n\}$ be a $\rho_{\mathcal{N}}-$sequence of the meromorphic function f.
If $f \in \mathcal{N}$ by ([13] theorem 3.4) we have $\mathcal{L}_f(a,r)$ and $\mathcal{A}_f(a,r)$ are bounded for any $0 < r < 1$, i.e. the diameters of $f(\Delta(a_n, r))$ don't tend to infinity. This contradicts our assumption that $\{a_n\}$ is a $\rho_{\mathcal{N}}-$sequence of f.

Theorem 4.2. *Let $\{a_n\}$ be a $\rho_{\mathcal{N}}-$sequence of the meromorphic function f and $\{b_n\}$ be such that*

$$\lim_{n\to\infty}\sigma(a_n, b_n) = 0, \tag{13}$$

then $\{b_n\}$ is a $\rho_{\mathcal{N}}-$sequence of f too.

Proof. Let $\{a_n\}$ be a $\rho_{\mathcal{N}}-$sequence of the meromorphic function f and $\{b_n\}$ be not a $\rho_{\mathcal{N}}-$sequence of f. Then by Definition 4.2 for each $\delta > 0$, we have

$$\lim_{n\to\infty}\mathcal{A}_f(b_n,\delta) < \infty,$$

and

$$\lim_{n\to\infty}\mathcal{L}_f(b_n,\delta) < \infty.$$

Suppose $\varepsilon = \frac{\delta}{2}$. As $\lim_{n\to\infty}\sigma(a_n, b_n) = 0$, then beginning with some N for any $n > N$, we obtain

$$\Delta(a_n,\varepsilon) \subset \Delta(b_n,\delta) \quad \text{and hence},$$

$$f(\Delta(a_n,\varepsilon)) \subset f(\Delta(b_n,\delta)).$$

Thus,

$$\dim f(\Delta(a_n,\varepsilon)) \to \infty \text{ as } n \to \infty,$$

which implies that,

$$\dim f(\Delta(b_n,\delta)) \to \infty.$$

This is a contradiction from our hypothesis.

Remark 4.1. *We need to remind the reader that the pseudohyperbolic circle* $\Gamma(z_n,\rho_n)$ *with center* z_n *and radius* ρ_n *is the same as Euclidean circle* $\{z : |z-\hat{z}_n| = r_n$ *with* $r_n = \frac{1-|z_n|^2}{1-|z_n|^2\rho_n^2}$ *and* $\hat{z}_n = z_n \frac{1-|\rho_n|^2}{1-|z_n|^2\rho_n^2}$. *In particular,* $\rho_n \to 0$ *if and only if* $\frac{r_n}{1-|z_n|^2} \to 0$.

Now we prove the next theorem :

Theorem 4.3. *A sequence* $\{z_n\}$, $(|z_n| \to 1)$, *is a* $\rho_{\mathcal{N}}-$*sequence of the meromorphic function* f *if and only if there is a sequence of positive numbers* $\{\varepsilon_n\}$, $(\varepsilon_n \to 0)$ *such that*

$$\lim_{n\to\infty} \sup_{z\in\Delta(z_n,\varepsilon_n)} (1-|z|^2)f^{\#}(z) = +\infty. \tag{14}$$

Proof. **Necessity.** Let $\{z_n\}$ be a $\rho_{\mathcal{N}}-$sequence of the meromorphic function f. Then by ([3], Lemma 2), there are sequences $\{a_n\}$ and $\{b_n\}$ such that

$$\lim_{n\to\infty} \sigma(a_n,z_n) = 0, \quad \lim_{n\to\infty} \sigma(b_n,z_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} |f^{\#}(a_n) - f^{\#}(b_n)| \geq \frac{1}{2}.$$

Suppose $\delta_n = \max\{|z_n - a_n|, |z_n - b_n|\}$ and L_n is a segment connecting the points a_n and b_n. Since a_n and b_n lie in a disk with hyperbolic radius tending to zero then by Remark 4.1, $\frac{\delta_n}{1-|z_n|^2}$ must also tend to zero. For some $w_n \in L_n$, we have that

$$|a_n - b_n| f^{\#}(w_n) \geq \int_{L_n} f^{\#}(z)|dz| \geq \left| \int_{L_n} f^{\#}(z)dz \right| = \left|f^{\#}(a_n) - f^{\#}(b_n)\right| \geq \frac{1}{2}\,.$$

On the other hand for sufficiently large n, we have that

$$\begin{aligned}(1-|w_n|^2)f^{\#}(w_n) \geq (1-|w_n|^2)\frac{1}{2|a_n-b_n|} &\geq \frac{1-(|z_n|+\delta_n)^2}{4\delta_n} \\ &= \frac{1-|z_n|^2}{4\delta_n} - \frac{|z_n|}{2} - \frac{\delta_n}{4}.\end{aligned}$$

The last expression tends to ∞ and condition (14) is proved .
Sufficiency. Let $\{z_n\}$ be such sequence of points that

$$\lim_{n\to\infty} (1-|z_n|^2)f^{\#}(z_n) = +\infty,$$

$\{\varepsilon_n\}$ be a sequence of positive numbers, $\lim_{n\to\infty}\varepsilon_n = 0$ and $z_n \in \Delta(z_n,\varepsilon_n)$ for each n. Then by Proposition 4.1 the sequence $\{z_n\}$ is a $\rho_{\mathcal{N}}-$sequence of f and by the Theorem 4.2 the sequence $\{z_n\}$, which satisfies condition (13), i.e. $\lim_{n\to\infty}\sigma(z_n,z_n) = 0$, is also a $\rho_{\mathcal{N}}-$sequence of the meromorphic function f.

Acknowledgements. The first author thanks Professor Klaus Gürlebeck from Bauhaus-University Weimar Germany for introducing him to the subject of function spaces and for several valuable discussions on the subject.

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