Derivations on certain matrix algebras with applications to compact groups

(with an Appendix by A. Valette)

H. Samea M. Abolghasemi

Abstract

In this paper we characterize the derivations from $\mathfrak{E}_p(\widehat{G})$ into $\mathfrak{E}_q(\widehat{G})$, $1 \leq p, q \leq \infty$. We find necessary and sufficient conditions for weak amenability of Banach algebras $\mathfrak{E}_p(\widehat{G})$, $1 \leq p \leq \infty$. Moreover we find a necessary condition for the weak amenability of the convolution Banach algebra $A(G)$ on a compact group G, where its sufficiency was proved earlier by Ghahramani and Lau.

Introduction

Throughout this paper G is a compact group with dual $\widehat{G}$. The Banach algebras $\mathfrak{E}_p(\widehat{G})$, where $p \in [1, \infty] \cup \{0\}$, and multipliers on these Banach algebras were introduced and extensively studied in Section 28 of [4]. The purpose of the present paper is to investigate the derivations on these Banach algebras. Also we find necessary and sufficient conditions for weak amenability of Banach algebras $\mathfrak{E}_p(\widehat{G})$, where $1 \leq p \leq \infty$, and we give applications to a number of convolution Banach algebras on compact groups.

The organization of this paper is as follows. The preliminaries and notations are given in section 1. In section 2, we state and prove a number of results on derivations between the Banach algebras $\mathfrak{E}_p(\widehat{G})$, where $1 \leq p \leq \infty$; these are then

Received by the editors January 2008 – In revised form in May 2008.
Communicated by A. Valette.
2000 *Mathematics Subject Classification* : 46H20, 43A20.
Key words and phrases : Multiplier, Derivation, Weak amenable Banach algebra.

applied in investigating the weak amenability of Banach algebras $\mathfrak{E}_p(\widehat{G})$ ($1 \leq p \leq \infty$). In section 3, we give some applications of the previous section. We prove that for each $1 < p < \infty$, $\mathfrak{E}_p(\widehat{G})$ is approximately weakly amenable, and is weakly amenable if and only if the group G is finite or abelian (however it is well-known that $\mathfrak{E}_\infty(\widehat{G})$ is weakly amenable [6, Theorem 4.2.4]). Also $\mathfrak{E}_1(\widehat{G})$ is approximately weakly amenable, and is weakly amenable if and only if $\sup\{d_\pi : \pi \in \widehat{G}\} < \infty$. Indeed we prove that the converse of Theorem 4.2(x) of [2] is valid.

1 Preliminaries

Let H be an n-dimensional Hilbert space, and let $B(H)$ be the space of all linear operators on H. Clearly we can identify $B(H)$ with $\mathbb{M}_n(\mathbb{C})$ (the space of all $n \times n$-matrices on $\mathbb{C}$). For $A \in \mathbb{M}_n(\mathbb{C})$, let $A^* \in \mathbb{M}_n(\mathbb{C})$ be defined by $(A^*)_{ij} = \overline{A_{ji}}$ ($1 \leq i, j \leq n$), and let $|A|$ denote the unique positive-definite square root of AA^*. The matrix A is *unitary*, if $A^*A = AA^* = I$, where I is the $n \times n$-identity matrix. For $E \in B(H)$, let $(\lambda_1, \ldots, \lambda_n)$ be the sequence of eigenvalues of the operator $|E|$, written in any order. Define $\|E\|_{\varphi_\infty} = \max_{1 \leq i \leq n} |\lambda_i|$, and $\|E\|_{\varphi_p} = (\sum_{i=1}^n |\lambda_i|^p)^{\frac{1}{p}}$ ($1 \leq p < \infty$). For more details see Definition D.37 and Theorem D.40 of [4].

Let G be a compact group with dual $\widehat{G}$ (the set of all irreducible representations of G). Let H_π be the representation space of π with dimension d_π, for each $\pi \in \widehat{G}$. The $*$-algebra $\prod_{\pi \in \widehat{G}} B(H_\pi)$ will be denoted by $\mathfrak{E}(\widehat{G})$; scalar multiplication, addition, multiplication, and the adjoint of an element are defined coordinate-wise.

Let $E = (E_\pi)$ be an element of $\mathfrak{E}(\widehat{G})$. We define $\|E\|_p := \left(\sum_{\pi \in \widehat{G}} d_\pi \|E_\pi\|_{\varphi_p}^p\right)^{\frac{1}{p}}$ ($1 \leq p < \infty$), and $\|E\|_\infty = \sup_{\pi \in \widehat{G}} \|E_\pi\|_{\varphi_\infty}$. For $1 \leq p \leq \infty$, $\mathfrak{E}_p(\widehat{G})$ is defined as the set of all $E \in \mathfrak{E}(\widehat{G})$ for which $\|E\|_p < \infty$, and $\mathfrak{E}_0(\widehat{G})$ is defined as the set of all $E \in \mathfrak{E}(\widehat{G})$ such that $\{\pi \in \widehat{G} : \|E_\pi\|_{\varphi_\infty} \geq \epsilon\}$ is finite for all $\epsilon > 0$. By Theorems 28.25, 28.27, and 28.32(v) of [4], both $(\mathfrak{E}_p(\widehat{G}), \|.\|_p)$ ($1 \leq p \leq \infty$), and $(\mathfrak{E}_0(\widehat{G}), \|.\|_\infty)$ are Banach algebras.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be subsets of $\mathfrak{E}(\widehat{G})$. An element E in $\mathfrak{E}(\widehat{G})$ is an $(\mathfrak{A}, \mathfrak{B})$*-multiplier* if $EA \in \mathfrak{B}$ for all $A \in \mathfrak{A}$. The set of all $(\mathfrak{A}, \mathfrak{B})$-multipliers will be denoted by $\mathcal{M}(\mathfrak{A}, \mathfrak{B})$. For more details, see Definition 35.1 of [4].

For a Banach algebra A, an A-bimodule will always refer to a *Banach A-bimodule X*, that is a Banach space which is algebraically an A-bimodule and for which there is a constant $C_{A,X} \geq 0$ such that

$$\|a.x\|, \|x.a\| \leq C_{A,X}\|a\|\|x\| \quad (a \in A, x \in X).$$

By renorming X, we may suppose that $C_{A,X} = 1$. A linear map $D : A \to X$ is called an *algebraic X-derivation*, if

$$D(ab) = D(a).b + a.D(b) \quad (a, b \in A).$$

The map D is a *derivation* if D is a continuous algebraic derivation. For every $x \in X$, we define ad_x by $ad_x(a) = a.x - x.a$ ($a \in A$). It is easily seen that ad_x is a derivation. Derivations of this form are *inner derivations*. The set of all derivations from A into X is denoted by $\mathcal{Z}^1(A,X)$, and the set of all inner X-derivations is denoted by $\mathcal{B}^1(A,X)$. Clearly, $\mathcal{Z}^1(A,X)$ is a linear subspace of the space of all linear operators of A into X, and $\mathcal{B}^1(A,X)$ is a linear subspace of $\mathcal{Z}^1(A,X)$. We denote by $\mathcal{H}^1(A,X)$ the quotient space of $\mathcal{Z}^1(A,X)$ modulo $\mathcal{B}^1(A,X)$.

A derivation $D : A \to X$ is *approximately inner* if there exists a net (x_α) in X such that, for every $a \in A$, $D(a) = \lim_\alpha ad_{x_\alpha}(a)$ (for more details see [3]).

The Banach space X^* with the *dual* module multiplications defined by

$$(f.a)(x) = f(a.x), (a.f)(x) = f(x.a) \quad (a \in A, f \in X^*, x \in X)$$

is a Banach A-bimodule, called the *dual* Banach A-bimodule X^*.

Every Banach algebra A with the product of A giving the two module multiplications defines a Banach A-bimodule. Let A^* be the dual A-bimodule. A Banach algebra A is *weakly amenable* if $\mathcal{H}^1(A,A^*) = \{0\}$. A Banach algebra A is *approximately weakly amenable* if each $D \in \mathcal{Z}^1(A,A^*)$ is approximately inner.

A Banach algebra A is *amenable* if for any A-bimodule X, every derivation from A into the dual Banach A-bimodule X^* is inner.

For a locally compact group G and a function $f : G \to \mathbb{C}$, $\check{f}$ is defined by $\check{f}(x) = f(x^{-1})$ $(x \in G)$. Let $A(G)$ (or with the notation $\mathfrak{K}(G)$ defined in 35.16 of [4]) consist of all functions h in $C_0(G)$ that can be written in at least one way as $\sum_{n=1}^\infty f_n * \check{g}_n$, where $f_n, g_n \in L^2(G)$, and $\sum_{n=1}^\infty \|f_n\|_2\|g_n\|_2 < \infty$. For $h \in A(G)$, define

$$\|h\|_{A(G)} = \inf\left\{\sum_{n=1}^\infty \|f_n\|_2\|g_n\|_2 : h = \sum_{n=1}^\infty f_n * \check{g}_n\right\}.$$

With this norm $A(G)$ is a Banach space. It is a commutative Banach algebra, called the *Fourier algebra*, with respect to pointwise product. For more details see 35.16 of [4]. In the case where G is a compact group, $A(G)$ with convolution product and the norm $\|.\|_{A(G)}$ defines a Banach algebra which is isometrically algebraically isomorphic with $\mathfrak{E}_1(\widehat{G})$ (see Theorem 34.35 of [4]).

2 Derivations on $\mathfrak{E}_p(I)$ $(1 \le p \le \infty)$

Throughout this paper for $A \in B(H_\pi)$, define A^π as an element of $\mathfrak{E}(\widehat{G})$ given by

$$(A^\pi)_\eta = \begin{cases} A & \text{for } \eta = \pi, \\ 0 & \text{otherwise.} \end{cases}$$

We denote the identity $d_\pi \times d_\pi$-matrix (i.e. the identity operator in $B(H_\pi)$) by I_π.

For each $\pi \in \widehat{G}$, and $1 \le m,n \le d_\pi$, let $\mathcal{E}^\pi_{mn}$ be the elementary $d_\pi \times d_\pi$-matrix such that for $1 \le k,l \le d_\pi$,

$$(\mathcal{E}^\pi_{mn})_{kl} = \begin{cases} 1 & \text{if } k = m, l = n, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that by Theorem 28.32(ii),(iii) of [4], for each $p,q \in [1,\infty] \cup \{0\}$, $\mathfrak{E}_p(\widehat{G})$ is a Banach $\mathfrak{E}_q(\widehat{G})$-bimodule with the product of $\mathfrak{E}(\widehat{G})$ giving the two module multiplications.

Proposition 2.1. *Let $1 \leq p,q \leq \infty$. Then $\mathcal{M}(\mathfrak{E}_p(\widehat{G}), \mathfrak{E}_q(\widehat{G}))$ with the norm $\|.\|_{p,q}$ that defined by*

$$\|L\|_{p,q} = \sup_{\|A\|_p=1} \|LA\|_q \quad (L \in \mathcal{M}(\mathfrak{E}_p(\widehat{G}), \mathfrak{E}_p(\widehat{G})),$$

is a Banach $\mathfrak{E}_\infty(\widehat{G})$-bimodule, with the product of $\mathfrak{E}(\widehat{G})$ giving the two module multiplications.

Proof. If $L \in \mathcal{M}(\mathfrak{E}_p(\widehat{G}), \mathfrak{E}_q(\widehat{G}))$, then, by Lemma 35.2(i) of [4], $\|E\|_{p,q} < \infty$. It is easy to see that $(\mathcal{M}(\mathfrak{E}_p(\widehat{G}), \mathfrak{E}_q(\widehat{G})), \|.\|_{p,q})$ is a Banach space. If $L \in \mathcal{M}(\mathfrak{E}_p(\widehat{G}), \mathfrak{E}_q(\widehat{G}))$ and $E \in \mathfrak{E}_\infty(\widehat{G})$, then, by Theorem 28.32 of [4],

$$\begin{aligned} \|EL\|_{p,q} &= \sup_{\|A\|_p=1} \|(EL)A\|_q = \sup_{\|A\|_p=1} \|E(LA)\|_q \\ &\leq \sup_{\|A\|_p=1} \|E\|_\infty \|LA\|_q = \|E\|_\infty \|L\|_{p,q}. \end{aligned}$$

Similarly one can prove that $\|LE\|_{p,q} \leq \|E\|_\infty \|L\|_{p,q}$. Therefore $(\mathcal{M}(\mathfrak{E}_p(\widehat{G}), \mathfrak{E}_q(\widehat{G})), \|.\|_{p,q})$ is a Banach $\mathfrak{E}_\infty(\widehat{G})$-bimodule. ∎

Notation: Throughout the rest of the paper we denote by $Z(\mathfrak{E}(\widehat{G}))$ the set of all $E \in \mathfrak{E}(\widehat{G})$ such that there exists a set $\{\lambda_\pi : \pi \in \widehat{G}\}$ in $\mathbb{C}$ such that, for each $\pi \in \widehat{G}$, $E_\pi = \lambda_\pi I_\pi$.

Proposition 2.2. *Let $1 \leq p,q \leq \infty$, and let D be a derivation from $\mathfrak{E}_p(\widehat{G})$ into $\mathfrak{E}_q(\widehat{G})$. Then $D(AI_\pi^\pi) = D(A)I_\pi^\pi$, $(A \in \mathfrak{E}_p(\widehat{G}), \pi \in \widehat{G})$. Moreover there exists $E \in \mathfrak{E}(\widehat{G})$ such that $D(A) = AE - EA \quad (A \in \mathfrak{E}_p(\widehat{G}))$, and D is inner if and only if $E \in \mathfrak{E}_q(\widehat{G}) + Z(\mathfrak{E}(\widehat{G}))$*

Proof. Let $\pi \in \widehat{G}$. Since by Proposition 1.8.2(ii) of [1], $D(I_\pi^\pi) = 0$, so for each $A \in \mathfrak{E}_p(\widehat{G})$,

$$D(AI_\pi^\pi) = D(A)I_\pi^\pi + AD(I_\pi^\pi) = D(A)I_\pi^\pi.$$

Define $D_\pi : B(H_\pi) \to B(H_\pi)$ through $D_\pi(A) = (D(A^\pi))_\pi$ $(A \in B(H_\pi))$. By Theorem 1.9.21(a) of [1], D_π is inner. So there exists $E_\pi \in B(H_\pi)$ such that $D_\pi = ad_{E_\pi}$. Define $E \in \mathfrak{E}(\widehat{G})$ by $E = (E_\pi)_{\pi \in \widehat{G}}$. Now, for each $A \in \mathfrak{E}_p(\widehat{G})$ and each $\pi \in \widehat{G}$

$$\begin{aligned} D(A)_\pi &= (D(A)I_\pi^\pi)_\pi = (D(AI_\pi^\pi))_\pi = (D(A_\pi^\pi))_\pi \\ &= D_\pi(A_\pi) = A_\pi E_\pi - E_\pi A_\pi = (AE - EA)_\pi . \end{aligned}$$

Therefore $D(A) = AE - EA$.

If D is inner, then there exists $E' \in \mathfrak{E}_q(\widehat{G})$, such that $D = ad_{E'}$. So for each $A \in \mathfrak{E}_p(\widehat{G})$, $A(E - E') = (E - E')A$. Hence, for each $\pi \in \widehat{G}$ and for each $d_\pi \times d_\pi$-matrix A,

$$A(E - E')_\pi = \left(A^\pi(E - E')\right)_\pi = \left((E - E')A^\pi\right)_\pi = (E - E')_\pi A.$$

Therefore, by Corollary 27.10 of [4], there exists $\lambda_\pi \in \mathbb{C}$ such that $(E-E')_\pi = \lambda_\pi I_\pi$. It follows that $E - E' \in Z(\mathfrak{E}(\widehat{G}))$ or, equivalently, $E \in \mathfrak{E}_q(\widehat{G}) + Z(\mathfrak{E}(\widehat{G}))$.

Conversely if $E \in \mathfrak{E}_q(\widehat{G}) + Z(\mathfrak{E}(\widehat{G}))$, then there exist $E' \in \mathfrak{E}_q(\widehat{G})$ and a set $\{\lambda_\pi : \pi \in \widehat{G}\}$ in $\mathbb{C}$ such that $E_\pi = E'_i + \lambda_\pi I_\pi$ for $\pi \in \widehat{G}$. Clearly, $D = ad_{E'}$ and hence D is inner. ∎

The following is the main result of this section.

Theorem 2.3. *Let $1 \le p, q \le \infty$, and let D be a derivation from $\mathfrak{E}_p(\widehat{G})$ into $\mathfrak{E}_q(\widehat{G})$. Then there exists a derivation $\overline{D}$ from $\mathfrak{E}_\infty(\widehat{G})$ into $(\mathcal{M}(\mathfrak{E}_p(\widehat{G}), \mathfrak{E}_q(\widehat{G})), \|.\|_{p,q})$ such that $\overline{D}_{|\mathfrak{E}_p(\widehat{G})} = D$.*

Proof. By Proposition 2.1, $(\mathcal{M}(\mathfrak{E}_p(\widehat{G}), \mathfrak{E}_q(\widehat{G}), \|.\|_{p,q})$ is a Banach $\mathfrak{E}_\infty(\widehat{G})$-bimodule. Define $\overline{D} : \mathfrak{E}_\infty(\widehat{G}) \to \mathcal{M}(\mathfrak{E}_p(\widehat{G}), \mathfrak{E}_q(\widehat{G})$ by

$$\left(\overline{D}(E)\right)_\pi = (D(EI^\pi_\pi))_\pi \quad (E \in \mathfrak{E}_\infty(\widehat{G}), \pi \in \widehat{G}).$$

We claim that $\overline{D}$ is a well-defined derivation. Let $E \in \mathfrak{E}_\infty(\widehat{G})$ and $A \in \mathfrak{E}_p(I)$. Since $EA \in \mathfrak{E}_p(\widehat{G})$, so by Proposition 2.2,

$$\begin{aligned}\left(\overline{D}(E)A\right)_\pi &= (D(EI^\pi_\pi)A)_\pi = (D(EI^\pi_\pi A) - EI^\pi_\pi D(A))_\pi \\ &= (D(EA)I^\pi_\pi - EI^\pi_\pi D(A))_\pi = (D(EA) - ED(A))_\pi,\end{aligned}$$

and hence $\overline{D}(E)A = D(EA) - ED(A) \in \mathfrak{E}_q(\widehat{G})$. Therefore $\overline{D}(E) \in \mathcal{M}(\mathfrak{E}_p(\widehat{G}), \mathfrak{E}_q(\widehat{G}))$.

Now, if $E, F \in \mathfrak{E}_\infty(\widehat{G})$, then for each $\pi \in \widehat{G}$

$$\begin{aligned}\left(\overline{D}(EF)\right)_\pi &= (D((EF)I^\pi_\pi))_\pi = (D((EI^\pi_\pi)(FI^\pi_\pi)))_\pi \\ &= (D(EI^\pi_\pi)FI^\pi_\pi + EI^\pi_\pi D(FI^\pi_\pi))_\pi = (D(EI^\pi_\pi))_\pi F_\pi + E_\pi (D(FI^\pi_\pi))_\pi \\ &= \left(\overline{D}(E)F + E\overline{D}(F)\right)_\pi.\end{aligned}$$

Hence $\overline{D}$ is an algebraic derivation.

Let (A_n) be a sequence in $\mathfrak{E}_p(\widehat{G})$ such that $\|A_n\|_p \to 0$ and $\|D(A_n) - B\|_q \to 0$, where $B \in \mathfrak{E}_q(\widehat{G})$. Let $\pi \in \widehat{G}$. Since $\mathfrak{E}_p(\widehat{G})I^\pi_\pi$ is finite dimensional, so by Lemma 1.20 of [5] the linear mapping $D_\pi : \mathfrak{E}_p(\widehat{G})I^\pi_\pi \to \mathfrak{E}_q(\widehat{G}); A \mapsto D(A)$ is continuous. Now, from $D(AI^\pi_\pi) = D(A)I^\pi_\pi$ $(A \in \mathfrak{E}_p(\widehat{G}))$, we obtain

$$\begin{aligned}B_\pi I^\pi_\pi &= BI^\pi_\pi = \lim_{n\longrightarrow\infty} D(A_n)I^\pi_\pi = \lim_{n\longrightarrow\infty} D(A_n I^\pi_\pi) \\ &= \lim_{n\longrightarrow\infty} D_\pi(A_n I^\pi_\pi) = D_\pi(\lim_{n\longrightarrow\infty} A_n I^\pi_\pi) = 0,\end{aligned}$$

and so $B_\pi = 0$. Hence $B = 0$ and so by the Closed Graph Theorem, D is continuous. It is clear that if $A \in \mathfrak{E}_p(\widehat{G})$, then $\overline{D}(A) = D(A)$. ∎

3 Weak amenability of Banach algebras $\mathfrak{E}_p(\widehat{G})$, $1 \leq p \leq \infty$

We start this section with the following result.

Proposition 3.1. *Let $1 \leq p, q \leq \infty$, $p \neq \infty$, and let D be a derivation from $\mathfrak{E}_p(\widehat{G})$ into $\mathfrak{E}_q(\widehat{G})$. Then D is approximately inner.*

Proof. For each finite subset F of $\widehat{G}$ define $E_F = \sum_{\pi \in F} I_\pi^\pi$. It is easy to prove that $(E_F)_F$ is an approximate identity for $\mathfrak{E}_p(\widehat{G})$. Hence, for each $A \in \mathfrak{E}_p(\widehat{G})$

$$\begin{aligned} D(A) &= \lim_F D(A)E_F = \lim_F (AE - EA)E_F \\ &= \lim_F (A(E_F E) - (E_F E)A) = \lim_F ad_{E_F E} A. \end{aligned}$$

Now, since $E_F E \in \mathfrak{E}_{00}(\widehat{G}) \subseteq \mathfrak{E}_q(\widehat{G})$, so D is approximately inner. ∎

Corollary 3.2. *Let G be an abelian compact group, and let $1 \leq p, q \leq \infty$, $p \neq \infty$. If D is a derivation from $\mathfrak{E}_p(\widehat{G})$ into $\mathfrak{E}_q(\widehat{G})$. Then $D = 0$.*

Proof. Since G is abelian, so by Remark 27.4 of [4] for each $\pi \in \widehat{G}$, $d_\pi = 1$. Hence for each finite set F of $\widehat{G}$, and each $A \in \mathfrak{E}_p(\widehat{G})$, $(AE - EA)E_F = 0$. Therefore, by the proof of the Proposition 3.1, $D = 0$. ∎

Notation: Throughout the rest of the paper for $1 < p < \infty$, let p' denote the exponent conjugate to p, that is $\frac{1}{p} + \frac{1}{p'} = 1$, for $p = 1$, let $p' = 0$, and for $p = \infty$, let $p' = 1$.

Lemma 3.3. *Let $1 \leq p \leq \infty$. Then the Banach space $\mathfrak{E}_p(\widehat{G})$ is a Banach $\mathfrak{E}_\infty(\widehat{G})$-bimodule ($\mathfrak{E}_0(\widehat{G})$-bimodule, respectively) with the product of $\mathfrak{E}(\widehat{G})$ giving the two module multiplications. Moreover it can be identified with the dual Banach $\mathfrak{E}_\infty(\widehat{G})$-bimodule ($\mathfrak{E}_0(\widehat{G})$-bimodule, respectively) $\mathfrak{E}_{p'}(\widehat{G})^*$ with the product of $\mathfrak{E}(\widehat{G})$ giving the two module multiplications.*

Proof. By Theorem 28.31 of [4], the mapping $T : \mathfrak{E}_p(\widehat{G}) \to \mathfrak{E}_{p'}(\widehat{G})^*$ given by

$$\langle B, T(A) \rangle = \sum_{\pi \in \widehat{G}} d_\pi \mathrm{tr}(B_\pi A_\pi) \quad (A \in \mathfrak{E}_p(\widehat{G}), B \in \mathfrak{E}_{p'}(\widehat{G})),$$

is an isometric Banach space isomorphism.

Let $A \in \mathfrak{E}_\infty(\widehat{G})$ and $X \in \mathfrak{E}_p(\widehat{G})$. For each $B \in \mathfrak{E}_{p'}(\widehat{G})$, we have

$$\begin{aligned} \langle B, T(X).A \rangle &= \langle AB, T(X) \rangle = \sum_{\pi \in \widehat{G}} d_\pi \mathrm{tr}((AB)_\pi X_\pi) \\ &= \sum_{\pi \in I} d_\pi \mathrm{tr}(X_\pi (AB)_\pi)) = \sum_{\pi \in \widehat{G}} d_\pi \mathrm{tr}((XA)_\pi B_\pi) \\ &= \langle B, T(XA) \rangle. \end{aligned}$$

So $T(X).A = T(XA)$. Similarly $A.T(X) = T(AX)$. ∎

By Lemma 3.3, for $1 \leq p < \infty$, $\mathfrak{E}_{p'}(G)$ is the dual module of $\mathfrak{E}_p(\widehat{G})$. So, by Proposition 3.1, we have the following result.

Corollary 3.4. *If $1 \leq p < \infty$, then the Banach algebra $\mathfrak{E}_p(\widehat{G})$ is approximately weakly amenable.*

Remark 3.5. By Theorem 28.26 of [4], $\mathfrak{E}_\infty(\widehat{G})$ is a C^*-algebra. But by Theorem 4.2.4 of [6], each C^*-algebra is weakly amenable. Therefore $\mathfrak{E}_\infty(\widehat{G})$ is weakly amenable.

Proposition 3.6. *The Banach algebra $\mathfrak{E}_0(\widehat{G})$ is amenable.*

Proof. Clearly $\mathfrak{E}_0(\widehat{G}) = c_0 - \oplus_{\pi\in\widehat{G}} B(H_\pi)$ where $B(H_\pi)$ is equipped with the norm $\|.\|_{\varphi_\infty}$. By Remark D.42 of [4] and Example 2.3.16 of [6], for each $\pi \in \widehat{G}$, the Banach algebra $B(H_\pi)$ with the norm $\|.\|_{\varphi_\infty}$ is 1-amenable. So by Corollary 2.3.19 of [6], $\mathfrak{E}_0(\widehat{G})$ is amenable. ∎

Proposition 3.7. *Let $1 \leq p, q \leq \infty$, $p \neq \infty$. If the set $\{\pi \in \widehat{G} : d_\pi \gneqq 1\}$ is finite, then $H^1(\mathfrak{E}_p(\widehat{G}), \mathfrak{E}_q(\widehat{G})) = \{0\}$.*

Proof. Let D be a derivation from $\mathfrak{E}_p(\widehat{G})$ into $\mathfrak{E}_q(\widehat{G})$. By Proposition 2.2, there exists $E \in \mathfrak{E}(\widehat{G})$ such that $D(A) = AE - EA$ $(A \in \mathfrak{E}_p(\widehat{G})$. Let

$$E' = \sum_{\pi\in\widehat{G}, d_\pi \gneqq 1} EI_\pi^\pi = \sum_{\pi\in\widehat{G}, d_\pi \gneqq 1} E_\pi^\pi I_\pi^\pi.$$

Since the set $\{\pi \in \widehat{G} : d_\pi \gneqq 1\}$ is finite, so $E' \in \mathfrak{E}_q(\widehat{G})$, and for each $A \in \mathfrak{E}_p(\widehat{G})$, $AE - EA = AE' - E'A$. Hence D is inner. ∎

The following proposition is used frequently in the rest of the paper.

Proposition 3.8. *If the set $\{\pi \in I : d_\pi > 1\}$ is infinite, then for $p, q \in [1, \infty]$, $\mathfrak{E}_p(\widehat{G}) \subseteq \mathfrak{E}_q(\widehat{G}) + Z(\mathfrak{E}(\widehat{G}))$ if and only if $p \leq q$.*

Proof. If $p \leq q$, then by Theorem 28.32(iv) of [4], $\mathfrak{E}_p(\widehat{G}) \subseteq \mathfrak{E}_q(\widehat{G}) \subseteq \mathfrak{E}_q(\widehat{G}) + Z(\mathfrak{E}(\widehat{G}))$.

Let $p > q$. Since the set $\{\pi \in \widehat{G} : d_\pi \gneqq 1\}$ is infinite, there exists a sequence $\{\pi_n\}_{n=1}^\infty$ of distinct elements in $\widehat{G}$ such that $d_{\pi_n} > 1$ for each n. Define $A_{\pi_n} = d_{\pi_n}^{-\frac{1}{p}} n^{-\frac{1}{q}} \mathcal{E}_{11}^{\pi_n}$ for each n, and $A_\pi = 0$ for all other π's. Since $\frac{p}{q} > 1$, so

$$\|A\|_p = \left(\sum_{\pi\in\widehat{G}} d_\pi \|A_\pi\|_{\varphi_p}^p\right)^{\frac{1}{p}} = \left(\sum_{n\in\mathbb{N}} d_{\pi_n} \|A_{\pi_n}\|_{\varphi_p}^p\right)^{\frac{1}{p}} = \left(\sum_{n\in\mathbb{N}} n^{-\frac{p}{q}}\right)^{\frac{1}{p}} < \infty,$$

and hence $A \in \mathfrak{E}_p(\widehat{G})$. We show that $A \notin \mathfrak{E}_q(\widehat{G}) + Z(\mathfrak{E}(\widehat{G}))$. Assume to the contrary that $A \in \mathfrak{E}_q(\widehat{G}) + Z(\mathfrak{E}(\widehat{G}))$. So there exist $A' \in \mathfrak{E}_q(\widehat{G})$ and a set $\{\lambda_\pi : \pi \in \widehat{G}\}$ in $\mathbb{C}$ such that, for each $\pi \in \widehat{G}$, $A_\pi = A'_\pi + \lambda_\pi I_\pi$. Since the eigenvalues of $|A_{\pi_n} - \lambda_{\pi_n} I_{\pi_n}|$ are $|\lambda_{\pi_n}|$ with multiplicity $d_{\pi_n} - 1$, and $\left| d_{\pi_n}^{-\frac{1}{p}} n^{-\frac{1}{q}} - \lambda_{\pi_n} \right|$ with multiplicity 1, so

$$\begin{aligned} \|A'_{\pi_n}\|_{\varphi_q} &\geq \|A'_{\pi_n}\|_{\varphi_\infty} = \|A_{\pi_n} - \lambda_{\pi_n} I_{\pi_n}\|_{\varphi_\infty} \\ &= \max\left(|\lambda_{\pi_n}|, \left| d_{\pi_n}^{-\frac{1}{p}} n^{-\frac{1}{q}} - \lambda_{\pi_n} \right| \right) \geq \frac{1}{2} d_{\pi_n}^{-\frac{1}{p}} n^{-\frac{1}{q}}. \end{aligned}$$

It follows that

$$\begin{aligned} \|A'\|_q &= \left(\sum_{\pi \in \widehat{G}} d_\pi \|A'_\pi\|_{\varphi_q}^q \right)^{\frac{1}{q}} \geq \left(\sum_{n \in \mathbb{N}} d_{\pi_n} \|A'_{\pi_n}\|_{\varphi_q}^q \right)^{\frac{1}{q}} \\ &\geq \frac{1}{2} \left(\sum_{n \in \mathbb{N}} d_{\pi_n}^{(1-\frac{q}{p})} n^{-1} \right)^{\frac{1}{q}} \geq \frac{1}{2} \left(\sum_{n \in \mathbb{N}} n^{-1} \right)^{\frac{1}{q}} = \infty. \end{aligned}$$

This contradiction shows that $\mathfrak{E}_p(\widehat{G}) \nsubseteq \mathfrak{E}_q(\widehat{G}) + Z(\mathfrak{E}(\widehat{G}))$. ■

Theorem 3.9. *Let $1 \leq p < \infty$ and $D : \mathfrak{E}_p(\widehat{G}) \to \mathfrak{E}_p(\widehat{G})$ be a derivation. Then there is an element $L \in \mathfrak{E}_\infty(\widehat{G})$ such that*

$$D(A) = AL - LA \quad (A \in \mathfrak{E}_p(\widehat{G})).$$

Moreover $\mathcal{H}^1(\mathfrak{E}_p(\widehat{G}), \mathfrak{E}_p(\widehat{G})) = \{0\}$ if and only if the set $\{\pi \in \widehat{G} : d_\pi > 1\}$ is finite.

Proof. By Theorem 35.4 of [4], $\mathcal{M}(\mathfrak{E}_p(\widehat{G}), \mathfrak{E}_p(\widehat{G})) = \mathfrak{E}_\infty(\widehat{G})$. By Theorem 28.32 of [4], $\|LA\|_p \leq \|L\|_\infty \|A\|_p$ $(L \in \mathfrak{E}_\infty(\widehat{G}), A \in \mathfrak{E}_p(\widehat{G}))$. Hence, by the Closed Graph Theorem, the identity mapping $id : (\mathcal{M}(\mathfrak{E}_p(\widehat{G}), \mathfrak{E}_p(\widehat{G})), \|.\|_{p,p}) \to \mathfrak{E}_\infty(\widehat{G})$; $L \mapsto L$ is a Banach space isomorphism. Now, by Lemma 3.3, $\mathfrak{E}_\infty(\widehat{G})$ is a dual Banach $\mathfrak{E}_0(\widehat{G})$-bimodule with the product of $\mathfrak{E}(\widehat{G})$ giving the two module multiplications. By Theorem 2.3, $\overline{D}_{|\mathfrak{E}_0(\widehat{G})}$ is a derivation that extends D. By Proposition 3.6, there exists $L \in \mathfrak{E}_\infty(\widehat{G})$ such that for each $A \in \mathfrak{E}_0(\widehat{G})$, $\overline{D}(A) = AL - LA$. Therefore, for each $A \in \mathfrak{E}_p(\widehat{G})$, $D(A) = AL - LA$.

If $\{\pi \in \widehat{G} : d_\pi > 1\}$ is finite, then by Proposition 3.7, D is inner.

Let $\{\pi \in \widehat{G} : d_\pi > 1\}$ be infinite. By Proposition 3.8, $\mathfrak{E}_\infty(\widehat{G}) \nsubseteq \mathfrak{E}_p(\widehat{G}) + Z(\mathfrak{E}(\widehat{G}))$ and hence, by Proposition 2.2, $\mathcal{H}^1(\mathfrak{E}_p(I), \mathfrak{E}_p(I)) \neq \{0\}$. ■

Proposition 3.10. *If G is an infinite non-abelian compact group, then the set $\{\pi \in \widehat{G} : \dim \pi > 1\}$ is infinite.*

Proof. Assume that the set $\{\pi \in \widehat{G} : \dim \pi > 1\}$ is finite. Hence, by Theorem 3.9, each derivation from $\mathfrak{E}_2(\widehat{G})$ into itself is inner. Now, by Peter-Weyl theorem [4], the convolution Banach algebra $L^2(G)$ is isometrically algebra isomorphic with $\mathfrak{E}_2(\widehat{G})$. So $\mathcal{H}^1(L^2(G), L^2(G)) = \{0\}$.

We claim that G is finite or abelian. Assume to the contrary that G is infinite and non-abelian. So there exist $x, y \in G$ such that $xy \neq yx$. The mapping $D_x : L^2(G) \to L^2(G)$ defined by

$$D_x(f) = \delta_x * f - f * \delta_x \quad (f \in L^2(G)),$$

is a non-inner derivation. To see this, let $D_x = ad_g$ for some $g \in L^2(G)$. Then for each $f \in L^2(G)$, $f * (\delta_x - g) = (\delta_x - g) * f$. Since $L^2(G)$ is dense in $L^1(G)$, so for each $f \in L^1(G)$, $f * (\delta_x - g) = (\delta_x - g) * f$. Let (e_α) be a bounded approximate identity for $L^1(G)$. With the weak*-topology on $M(G)$:

$$\begin{aligned}
\delta_{xy} - \delta_{yx} &= \text{weak}^* - \lim_\alpha (\delta_x * (e_\alpha * \delta_y) - (e_\alpha * \delta_y) * \delta_x) \\
&= \text{weak}^* - \lim_\alpha D_x(e_\alpha * \delta_y) = \text{weak}^* - \lim_\alpha ad_g(e_\alpha * \delta_y) \\
&= g * \delta_y - \delta_y * g \in L^2(G) \subseteq L^1(G).
\end{aligned}$$

Since G is compact and infinite, it is not discrete, and hence $\delta_{xy} - \delta_{yx} \notin L^1(G)$. This contradiction proves that G must be abelian or finite. ∎

A different proof of this result, not appealing to derivations, appears in the Appendix.

By Propositions 2.2 and 3.10, and a method similar to Theorem 3.9, we have the following result.

Proposition 3.11. *Let $1 \le p \le q < \infty$ and suppose that $D : \mathfrak{E}_p(\widehat{G}) \to \mathfrak{E}_q(\widehat{G})$ is a derivation. There is an element $L \in \mathfrak{E}(\widehat{G})$ such that*

$$D(A) = AL - LA \quad (A \in \mathfrak{E}_p(\widehat{G})).$$

Moreover each derivation from $\mathfrak{E}_p(\widehat{G})$ into $\mathfrak{E}_q(\widehat{G})$ is inner if and only if G is finite or abelian. ∎

Proposition 3.12. *Let $1 \le q < p < \infty$ and $D : \mathfrak{E}_p(\widehat{G}) \to \mathfrak{E}_q(\widehat{G})$ be a derivation. There is an element $L \in \mathfrak{E}_r(\widehat{G})$, where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, such that*

$$D(A) = AL - LA \quad (A \in \mathfrak{E}_p(\widehat{G})).$$

Moreover $\mathcal{H}^1(\mathfrak{E}_p(\widehat{G}), \mathfrak{E}_q(\widehat{G})) = \{0\}$ if and only if G is finite or abelian.

Proof. By Theorem 35.4 of [4], $\mathcal{M}(\mathfrak{E}_p(\widehat{G}), \mathfrak{E}_q(\widehat{G})) = \mathfrak{E}_r(\widehat{G})$. By Theorem 28.33 of [4], $\|LA\|_q \le \|L\|_r \|A\|_p$ $(L \in \mathfrak{E}_r(\widehat{G}), A \in \mathfrak{E}_p(\widehat{G}))$. Hence, by the Closed Graph Theorem, the identity mapping $id : (\mathcal{M}(\mathfrak{E}_p(\widehat{G}), \mathfrak{E}_q(\widehat{G})), \|.\|_{p,q}) \to \mathfrak{E}_r(\widehat{G}); L \mapsto L$ is a Banach space isomorphism. Now, by Lemma 3.3, $\mathfrak{E}_r(\widehat{G})$ is a dual Banach $\mathfrak{E}_0(\widehat{G})$-bimodule with the product of $\mathfrak{E}(\widehat{G})$ giving the two module multiplications. By Theorem 2.3, $\overline{D}_{|\mathfrak{E}_0(\widehat{G})}$ is a derivation that extends D. By Proposition 3.6, there exists $L \in \mathfrak{E}_r(\widehat{G})$ such that for each $A \in \mathfrak{E}_0(\widehat{G})$, $\overline{D}(A) = AL - LA$. Therefore, for each $A \in \mathfrak{E}_p(\widehat{G})$, $D(A) = AL - LA$.

If G is finite or abelian, then, by Proposition 3.7 and Corollary 3.2, D is inner.

Suppose that G is infinite. So by Proposition 3.10, the set $\{\pi \in \widehat{G} : d_\pi \gneqq 1\}$ is infinite. Since $p \neq \infty$, so $r > q$ and hence, by Proposition 3.8, $\mathfrak{E}_r(\widehat{G}) \nsubseteqq \mathfrak{E}_q(\widehat{G}) + Z(\mathfrak{E}(\widehat{G}))$. Therefore, by Proposition 2.2, $\mathcal{H}^1(\mathfrak{E}_p(\widehat{G}), \mathfrak{E}_q(\widehat{G}) \neq \{0\}$. ∎

A combination of Lemma 3.3, and Propositions 3.11 and 3.12 yields the following result.

Theorem 3.13. *Let G be a compact group. For $1 < p < \infty$, $\mathfrak{E}_p(\widehat{G})$ is weakly amenable if and only if G is finite or abelian.* ■

Finally, we find necessary and sufficient conditions for weak amenability of $\mathfrak{E}_1(\widehat{G})$.

Theorem 3.14. *Let G be a compact group. The Banach algebra $\mathfrak{E}_1(\widehat{G})$ is weakly amenable if and only if $\sup_{\pi\in\widehat{G}} d_\pi < \infty$.*

Proof. Let $\sup_{\pi\in\widehat{G}} d_\pi < \infty$, and $D : \mathfrak{E}_1(\widehat{G}) \to \mathfrak{E}_\infty(\widehat{G})$ be a derivation. Since $\sup_{\pi\in I'} d_\pi < \infty$, so by Theorem 35.4 of [4], $\mathcal{M}(\mathfrak{E}_1(\widehat{G}), \mathfrak{E}_\infty(\widehat{G})) = \mathfrak{E}_\infty(\widehat{G})$. Now, by an argument similar to the proof of Propositions 3.9 and 3.12, one can prove that there exists $L \in \mathfrak{E}_\infty(\widehat{G})$ such that $D(A) = AL - LA$ $(A \in \mathfrak{E}_1(\widehat{G}))$. Therefore $\mathfrak{E}_1(\widehat{G})$ is weakly amenable.

Let $\sup_{\pi\in\widehat{G}} d_\pi = \infty$. Define $E \in \mathfrak{E}(\widehat{G})$ by $E_\pi = d_\pi \mathcal{E}^\pi_{11}$ for all $\pi \in \widehat{G}$. Note that $\|E_i\|_{\varphi_\infty} = d_\pi$. Define $D : \mathfrak{E}_1(\widehat{G}) \to \mathfrak{E}_\infty(\widehat{G})$ by $D(A) = AE - EA$ $(A \in \mathfrak{E}_1(\widehat{G}))$. We claim that D is a non-inner derivation. Note that $\|E_\pi\|_{\varphi_\infty} = d_\pi$ $(\pi \in \widehat{G})$. For $A \in \mathfrak{E}_1(\widehat{G})$, use (D.51.(i)) and (D.52.(i)) of [4] to write

$$\begin{aligned}
\|D(A)\|_\infty &= \|AE - EA\|_\infty = \sup_{\pi\in\widehat{G}} \|A_\pi E_\pi - E_\pi A_\pi\|_{\varphi_\infty} \\
&\le \sup_{\pi\in\widehat{G}} \left(\|A_\pi E_\pi\|_{\varphi_\infty} + \|E_\pi A_\pi\|_{\varphi_\infty}\right) \le 2\sup_{\pi\in\widehat{G}} \|E_\pi\|_{\varphi_\infty} \|A_\pi\|_{\varphi_\infty} \\
&= 2\sup_{\pi\in\widehat{G}} d_\pi \|A_\pi\|_{\varphi_\infty} \le 2\sup_{\pi\in\widehat{G}} d_\pi \|A_\pi\|_{\varphi_1} \le 2\sum_{\pi\in\widehat{G}} d_\pi \|A_\pi\|_{\varphi_1} = 2\|A\|_1.
\end{aligned}$$

Therefore D is well-defined and continuous. Clearly D is a derivation. We claim that D is not inner. Assume to the contrary that D is inner. Hence, by Proposition 2.2, $E \in \mathfrak{E}_\infty(\widehat{G}) + Z(\mathfrak{E}(\widehat{G}))$. So there exist $E' \in \mathfrak{E}_\infty(\widehat{G})$ and a set $\{\lambda_\pi : \pi \in \widehat{G}\}$ such that for all $\pi \in \widehat{G}$, we have $E_\pi = E'_\pi + \lambda_\pi I_\pi$. Since $\sup_{\pi\in\widehat{G}} d_\pi = \infty$, there exists a subset $\{\pi_n : n \in \mathbb{N}\}$ of $\widehat{G}$ such that $\pi_m \neq \pi_n$ for $m \neq n$ and $\lim_n d_{\pi_n} = \infty$. The eigenvalues of $|E_{\pi_n} - \lambda_{\pi_n} I_{\pi_n}|$ are $|\lambda_{\pi_n}|$ with multiplicity $d_{\pi_n} - 1$ and $|d_{\pi_n} - \lambda_{\pi_n}|$ with multiplicity 1. We have

$$\|E'{}_{\pi_n}\|_{\varphi_\infty} = \max\{|\lambda_{\pi_n}|, |d_{\pi_n} - \lambda_{\pi_n}|\} \ge \frac{1}{2} d_{\pi_n},$$

and hence

$$\|E'\|_\infty \ge \sup_{n\in\mathbb{N}} \|E'_{\pi_n}\|_{\varphi_\infty} \ge \frac{1}{2}\sup_{n\in\mathbb{N}} d_{\pi_n} = \frac{1}{2}\lim_n d_{\pi_n} = \infty.$$

That is $E' \notin \mathfrak{E}_\infty(\widehat{G})$. This contradiction proves our claim. Therefore D is not inner, and so $\mathfrak{E}_1(\widehat{G})$ is not weakly amenable. ■

As a consequence of the above theorem, we have the following theorem. We find a necessary and sufficient condition for weak amenability of the convolution Banach algebra $A(G)$, whenever G is a compact group. Indeed we prove that the converse of Theorem 4.2(x) of [2] is also valid.

Theorem 3.15. *Let G be a compact group. Then the convolution Banach algebra $A(G)$ is approximately weak amenable. Moreover $A(G)$ is weakly amenable if and only if $\sup_{\pi \in \widehat{G}} d_\pi < \infty$.* ∎

Acknowledgements. The authors would like to thank the referee of the paper for his/her invaluable comments. The first author also would like to thank the University of Bu-Ali Sina (Hamedan) for its support.

Appendix, by A. Valette: characterizing finite or abelian compact groups

In Proposition 3.10 of the preceding paper, the authors use derivations of group algebras to establish that: **a compact group is either abelian or finite, if and only if it has finitely many irreducible representations of degree at least 2**. The aim of this appendix is to give a self-contained proof of this fact, using only classical results in harmonic analysis on compact groups.

We fix some notation. Let G be a compact group, and $\hat{G}$ be its dual. Set

$$\hat{G}_{\geq 2} = \{\pi \in \hat{G} : \dim \pi \geq 2\};$$

$$\hat{G}_1 = \{\chi \in \hat{G} : \dim \chi = 1\}.$$

We view $\hat{G}_1$ as the dual group of the abelianized group $G/\overline{[G,G]}$; let $\hat{G}_1$ act on $\hat{G}$ by $\chi \cdot \pi =: \chi \otimes \pi$.

Proposition 3.16. *The action of $\hat{G}_1$ on $\hat{G}$ is proper, i.e. it has finite stabilizers.*

Proof: Fix $\pi \in \hat{G}$; let $\hat{G}_{1,\pi}$ be its stabilizer in $\hat{G}_1$. Let us prove that $\hat{G}_{1,\pi}$ is finite.

Set $U =: \{g \in G : Tr\, \pi(g) \neq 0\}$, an open subset of G, containing 1, and which is a union of conjugacy classes of G; let N be the subgroup generated by U, so that N is an open, normal subgroup; by compactness of G, the subgroup N has finite index.

Since $\chi \cdot \pi = \pi$ for every $\chi \in \hat{G}_{1,\pi}$, taking characters we have for every $g \in G$:

$$Tr\, \pi(g) \;=\; Tr(\chi \cdot \pi)(g) \;=\; \chi(g)\, Tr\, \pi(g);$$

in particular, $\chi(g) = 1$ for every $g \in U$. This implies that $\chi(g) = 1$ for every $g \in N$.

We have shown that every element in $\hat{G}_{1,\pi}$ factors through the finite group G/N; therefore $|\hat{G}_{1,\pi}| \leq |G/N|$, concluding the proof.

Alternate proof of Proposition 3.10: Assume that $\hat{G}_{\geq 2}$ is finite. Two cases may occur.

- $\hat{G}_{\geq 2} = \emptyset$, i.e. $\hat{G} = \hat{G}_1$; then G is abelian, as irreducible representations separate points of G, by the Gelfand-Raikov theorem.

- $\hat{G}_{\geq 2} \neq \emptyset$; then $\hat{G}_1$ has at least one finite orbit in $\hat{G}$, so $\hat{G}_1$ is finite, by the previous Proposition. By the Peter-Weyl theorem, the left regular representation λ_G on $L^2(G)$ decomposes as

$$\bigoplus_{\pi\in\hat{G}}(\dim\,\pi)\pi = (\bigoplus_{\chi\in\hat{G}_1}\chi)\oplus(\bigoplus_{\pi\in\hat{G}_{\geq 2}}(\dim\,\pi)\,\pi);$$

So $\dim\, L^2(G) = |\hat{G}_1| + \sum_{\pi\in\hat{G}_{\geq 2}}(\dim\,\pi)^2 < +\infty$; this implies that G is finite.

References

[1] H. G. Dales, *Banach algebras and automatic continuity*, Clarendon Press, Oxford, 2000.

[2] F. Ghahramani and A. T. M. Lau; *Approximate weak amenability, derivations and Arens regularity of Segal algebras*, Studia Mathematica. **169** (2) (2005), 189-205.

[3] F. Ghahramani and R. J. Loy, *Generalized notions of amenability*, J. Funct. Anal. **208** (2004), 229 - 260.

[4] E. Hewitt and K. A. Ross, *Abstract harmonic analysis, Vol. II*, Springer, Berlin, 1970.

[5] W. Rudin, *Functional analysis*, McGraw Hill, New York, 1991.

[6] V. Runde, *Lectures on amenability*, Lecture Notes in Mathematics, Vol. **1774**, Springer, Berlin, 2002.

Department of Mathematics, Bu-Ali Sina University, Hamedan, Iran.
e-amil:h.samea@basu.ac.ir

Department of mathematics, Razi University, Kermanshah, Iran.
email:m.abolghasemi@razi.ac.ir