Quasinormable weighted Fréchet spaces of entire functions

Elke Wolf

Abstract

We give a characterization of quasinormable weighted Fréchet spaces of entire functions under the assumption that the system of weights belongs to the class $(E)_{a,A}$ of Bierstedt-Bonet-Taskinen, see [5].

1 Introduction and Notation

Quasinormable Fréchet spaces were introduced by Grothendieck as a class "which contains the most usual Fréchet function spaces" (see [12] p. 107) such as normed spaces, Schwarz spaces and many classical spaces of functions and distributions. The class of quasinormable spaces has played an important role in the study of tensor products ([11], [19]), the lifting of bounded sets ([8], [21], [17]) and the splitting of exact sequences of Fréchet spaces ([16], [24]).
Several authors studied quasinormable Köthe echelon spaces, see e.g. Bierstedt-Meise-Summers [7], Meise-Vogt [16], Valdivia [20], [22] and Vogt [23]. In the setting of weighted Fréchet spaces of continuous functions Bierstedt-Meise [6] and Bastin-Ernst [1] obtained a characterization of quasinormability in terms of the involved weights.
The aim of this article is to get such a characterization in the case of weighted Fréchet spaces $HW(\mathbb{C})$ resp. $HW_0(\mathbb{C})$ of entire functions. In [25] we were able to give a necessary condition for quasinormability in terms of the sequence of weights (which are considered as growth conditions in the sense of [4]) and their

Received by the editors December 2007.
Communicated by F. Bastin.
2000 *Mathematics Subject Classification :* primary 46E10; secondary 46A04, 46A13.
Key words and phrases : Fréchet spaces, weighted spaces of holomorphic functions, quasinormable spaces.

associated growth conditions in a rather general setting. In case of weighted Fréchet spaces of holomorphic functions on the unit disk we could show that under some restrictions on the weights the necessary condition is also sufficient, see [25] and [26].

For entire functions the sufficiency is obtained under the assumption that the systems of weights belong to the class $(E)_{a,A}$ of Bierstedt-Bonet-Taskinen (see [5]), see the details below. The class $(E)_{a,A}$ is based on methods developed by Lusky in [15].

Let us first describe the setting of the weighted Fréchet spaces of entire functions. Let $H(\mathbb{C})$ denote the set of all entire functions. We consider an increasing sequence $W = (w_n)_{n\in\mathbb{N}}$ of strictly positive continuous functions (weights) on $\mathbb{C}$. For every $n \in \mathbb{N}$ the spaces

$$\begin{aligned} Hw_n(\mathbb{C}) &:= \{f \in H(\mathbb{C});\ \|f\|_n := \sup_{z\in\mathbb{C}} w_n(z)|f(z)| < \infty\} \text{ and} \\ H(w_n)_0(\mathbb{C}) &:= \{f \in H(\mathbb{C}); w_n f \text{ vanishes at } \infty \text{ on } \mathbb{C}\} \end{aligned}$$

endowed with the norm $\|.\|_n$ are Banach spaces. The weighted Fréchet spaces of holomorphic functions are defined by

$$HW(\mathbb{C}) := \mathrm{proj}_n Hw_n(\mathbb{C}) \text{ and } HW_0(\mathbb{C}) := \mathrm{proj}_n H(w_n)_0(\mathbb{C}).$$

For each $n \in \mathbb{N}$, let B_n, resp. $B_{n,0}$, be the closed unit ball of $Hw_n(\mathbb{C})$, resp. $H(w_n)_0(\mathbb{C})$, and $C_n := B_n \cap HW(\mathbb{C})$, resp. $C_{n,0} := B_{n,0} \cap HW_0(\mathbb{C})$. By $\overline{B_n}$, $\overline{B_{n,0}}$, $\overline{C_n}$, $\overline{C_{n,0}}$ we denote the *co*-closures of the corresponding sets. The sequence $\left(\frac{1}{n}C_n\right)_{n\in\mathbb{N}}$, resp. $\left(\frac{1}{n}C_{n,0}\right)_{n\in\mathbb{N}}$, constitutes a 0-neighborhood base of $HW(\mathbb{C})$, resp. $HW_0(\mathbb{C})$. Without loss of generality we may assume that $(C_n)_{n\in\mathbb{N}}$, resp. $(C_{n,0})_{n\in\mathbb{N}}$, is a 0-neighborhood base. Put

$$\overline{W} := \{\overline{w} : \mathbb{C} \to]0,\infty[;\ \overline{w} \text{ continuous on } \mathbb{C},\ w_n\overline{w} \text{ is bounded on } \mathbb{C}\ \forall n \in \mathbb{N}\},$$

and $C_{\overline{w}} := \{f \in HW(\mathbb{C}); |f| \le \overline{w} \text{ on } \mathbb{C}\}$, resp. $C_{\overline{w},0} := C_{\overline{w}} \cap HW_0(\mathbb{C})$, $\overline{w} \in \overline{W}$. We write $\overline{C_{\overline{w}}}$ and $\overline{C_{\overline{w},0}}$ to refer to the *co*-closure. $(C_{\overline{w}})_{\overline{w}\in\overline{W}}$, resp. $(C_{\overline{w},0})_{\overline{w}\in\overline{W}}$, is a fundamental system of bounded subsets of $HW(\mathbb{C})$, resp. $HW_0(\mathbb{C})$.
Let v be a weight on $\mathbb{C}$. Its *associated growth condition* (see [4]) is defined by

$$\tilde{v}(z) := \sup\{|g(z)|; g \in H(\mathbb{C}),\ |g| \le v\},\ z \in \mathbb{C}.$$

A weight v on $\mathbb{C}$ is said to be *radial* if $v(z) = v(|z|)$ holds for every $z \in \mathbb{C}$.

We use standard notation on locally convex spaces (see e.g. Jarchow [13], Köthe [14], Meise-Vogt [17] and Pérez Carreras-Bonet [18]). For a locally convex space E, E' is the topological dual and E'_b the strong dual. If E is a locally convex space, $\mathcal{U}_0(E)$ and $\mathcal{B}(E)$ stand for the families of all absolutely convex 0-neighborhoods and absolutely convex bounded sets in E, respectively.
A locally convex space E is called *quasinormable* if

$$\forall U \in \mathcal{U}_0(E)\ \exists V \in \mathcal{U}_0(E)\ \forall \lambda > 0\ \exists B \in \mathcal{B}(E) :\ V \subset B + \lambda U.$$

Each normed space is quasinormable. By [17, Lemma 26.14] a Fréchet space E with a 0-neighborhood base $(U_n)_{n\in\mathbb{N}}$ is quasinormable if and only if

$$\forall n \in \mathbb{N}\ \exists m > n\ \forall k \geq n\ \forall \varepsilon > 0\ \exists \delta > 0 :\ U_m \subset \delta U_k + \varepsilon U_n.$$

A dense linear subspace F of a quasinormable Fréchet space E need not be quasinormable, as an example due to Bonet and Dierolf shows (see [9]), but Bonet, Dierolf and Aye Aye showed that F is quasinormable if and only if it is large in E, see [10].
The question of the inheritance properties of quasinormability under formation of injective tensor products led to the stronger property *quasinormable by operators* (QNo) (due to Peris, see [19]). Let E be a Fréchet space with a 0-neighborhood base $(U_n)_{n\in\mathbb{N}}$. Then E is said to be (QNo) if

$$\forall n \in \mathbb{N}\ \exists m > n\ \forall \varepsilon > 0\ \exists P \in L(E,E) :\ P(U_m) \in \mathcal{B}(E) \text{ and } (I-P)(U_m) \subset \varepsilon U_n.$$

2 Results

Definition 1 (Bierstedt-Bonet-Taskinen [5, Definition 2.1]) *Given constants $A > 0$ and $a > 0$, we say that a continuous, radial strictly positive weight function $w : \mathbb{C} \to \mathbb{R}_+$ of the form*

$$w(r) := v(r)e^{-ar},\ r \in [0,\infty)$$

belongs to the class $(E)_{A,a}$ if $v : \mathbb{R}_+ \to \mathbb{R}_+$ is differentiable, strictly increasing and has the property

$$\sup_{r\in[0,\infty)} \frac{rv'(r)}{av(r)} \leq A.$$

If $a = 1$, we denote the class by $(E)_A$.

The proof of the following proposition is strongly based on concepts given by Lusky in [15], for details see [5].

Proposition 2 (Bierstedt-Bonet-Taskinen [5, Proposition 2.3]) *Let $A, a > 0$ be fixed. There exists a sequence $(T_n)_{n\in\mathbb{N}}$ of finite rank operators on the space of all polynomials with the following properties:*

(a) The operators T_n satisfy T_nT_m=0 if $|n-m| \geq 2$ and $T_nT_{n+1} = T_{n+1}T_n$.

(b) For every polynomial p we have $\sum_n T_np = p$, and the sum is finite.

(c) There is a constant $D \geq 1$ such that for every $r > 0$ and every polynomial p we have

$$\sup_{|z|=r} |T_np(z)| \leq D \sup_{|z|=r} |p(z)|.$$

(d) *There exists increasing positive sequences* $(\rho_n)_{n\in\mathbb{N}}$ *and* $(\sigma_n)_{n\in\mathbb{N}}$, $\rho_n < \sigma_n$, *such that for each weight* $w \in (E)_{A,a}$,$w(r) = v(r)e^{-ar}$, *there exist a constant* $C(w) > 0$ *such that for every polynomial* p,

$$\frac{1}{D}\sup_{n\in\mathbb{N}}\sup_{\rho_n\le|z|\le\sigma_n} v(\rho_n)e^{-ar}|T_n p(z)| \le \|p\|_w \le$$
$$C(w)\sup_{n\in\mathbb{N}}\sup_{\rho_n\le|z|\le\sigma_n} v(\rho_n)e^{-ar}|T_n p(z)|;$$

here D *is the constant of statement (c), which does not depend on the weight* w.

(e) *There is a constant* $0 < d \le 1$, *independent of the weight, such that (with the notations of item (d))* $v(\rho_n) \ge dv(\rho_{n+1})$ *for all* n.

The following lemma is well-known. We will omit the proof, for details see [25].

Lemma 3 *Let* E *be a locally convex space and* F *a dense subspace of* E. *Assume that* F *is quasinormable. Then* E *is also quasinormable.*

Theorem 4 *Let* $W = (w_n)_{n\in\mathbb{N}}$ *be an increasing sequence of strictly positive continuous radial functions on the complex plane* $\mathbb{C}$ *such that each* w_n *belongs to the class* $(E)_{A,a}$. *The following are equivalent:*

(1) $HW_0(\mathbb{C})$ *is quasinormable.*

(2) $HW(\mathbb{C})$ *is quasinormable.*

(3) *For every* $l \in \mathbb{N}$ *there is* $j > l$ *such that for every* $i \ge l$ *and for every* $\mu > 0$ *we can find* $\xi > 0$ *such that*
$$C_{j,0} \subset \xi C_{i,0} + \mu C_{l,0}.$$

(4) *For every* $l \in \mathbb{N}$ *there is* $j > l$ *such that for every* $i \ge l$ *and for every* $\varepsilon > 0$ *we can find* $\lambda > 0$ *such that*
$$\left(\frac{1}{w_j}\right)^{\sim} \le \frac{\lambda}{w_i} + \frac{\varepsilon}{w_l} \text{ on } \mathbb{C}.$$

(5) *For every* $l \in \mathbb{N}$ *there is* $j > l$ *such that for every* $\alpha > 0$ *there is* $\overline{w} \in \overline{W}$ *with*
$$\left(\frac{1}{w_j}\right)^{\sim} \le \overline{w} + \frac{\alpha}{w_l} \text{ on } \mathbb{C}.$$

Proof. Since $HW_0(\mathbb{C})$ is a Fréchet space, the equivalence of (1) and (3) follows from [17, Lemma 26.14]. The equivalence of (1) and (2) and the fact that (2) implies (5) are particular cases of results in a more general setting given in [25, Propositions 17 and 19]. It is easy to see that (4) follows from (5). It remains to show that (4) yields (3). Our proof was inspired by the proof of [5, Lemma 3.1]. By [3] the polynomials $\mathcal{P}$ are dense in $HW_0(\mathbb{C})$. Hence by Lemma 3 it is enough to consider only polynomials (see also [25]). We denote $C_1 = C(w_i)$, $C_2 = C(w_l)$ in the sense of Proposition 2. We fix $l \in \mathbb{N}$ and select $j > l$. For fixed $i \geq l$ and $\mu > 0$ we put $\varepsilon := \frac{\mu}{6D^2 d^{-1} C_2}$, where D and d are the constants of Proposition 2. For this ε, apply (4) to select $\lambda > 0$. We fix $p \in \mathcal{P} \cap C_{j,0}$. Hence $|p| \leq \frac{1}{w_j}$, or equivalently, $|p| \leq \left(\frac{1}{w_j}\right)^{\sim}$ on $\mathbb{C}$. Condition (4) implies

$$\left(\frac{1}{w_j}\right)^{\sim} \leq \frac{\lambda}{w_i} + \frac{\varepsilon}{w_l} \leq \max\left(\frac{2\lambda}{w_i}, \frac{2\varepsilon}{w_l}\right).$$

We put $u := \min\left(\frac{w_i}{2\lambda}, \frac{w_l}{2\varepsilon}\right)$ and get

$$|p| \leq \left(\frac{1}{w_j}\right)^{\sim} \leq \max\left(\frac{2\lambda}{w_i}, \frac{2\varepsilon}{w_l}\right) = \frac{1}{u} \text{ on } \mathbb{C}.$$

Hence $u|p| \leq 1$ on $\mathbb{C}$. Put $a_1 := \frac{1}{2\lambda}$, $a_2 := \frac{1}{2\varepsilon}$, $u_1 := w_i$, $u_2 := w_l$, i.e. $u = \min(a_1 u_1, a_2 u_2)$. Set $s := \min(a_1 s_1, a_2 s_2)$, where $s_1 := v_i$ and $s_2 := v_l$. For each $n \in \mathbb{N}$ choose $k(n) \in \{1, 2\}$ such that

$$s(\rho_n) = a_{k(n)} s_{k(n)}(\rho_n),$$

where $(\rho_n)_{n \in \mathbb{N}}$ is the sequence of Proposition 2. Denote $N_1 := \{n \in \mathbb{N}; k(n) = 1\}$ and $N_2 := \{n \in \mathbb{N};\ k(n) = 2\}$. We have that $\mathbb{N}$ is the disjoint union of N_1 and N_2. For $k = 1, 2$ we define

$$p_k := \sum_{n \in N_k} T_n p.$$

The sum has in fact only finitely many terms, since p is a polynomial. First notice that by (a) of Proposition 2:

$$\begin{aligned} T_n p_k &= \sum_{m \in N_k} T_n T_m p \\ &= (\chi(k, n-1) T_n T_{n-1} + \chi(k, n) T_n^2 + \chi(k, n+1) T_n T_{n+1}) p \\ &= (\chi(k, n-1) T_{n-1} T_n + \chi(k, n) T_n^2 + \chi(k, n+1) T_{n+1} T_n) p, \end{aligned}$$

where $\chi(k, n) := 1$, if $n \in N_k$ and $\chi(k, n) := 0$ otherwise. Hence, d) and c) of

Proposition 2 imply

$$\begin{aligned}
\|p_1\|_i &\leq C_1 \sup_{n\in\mathbb{N}} \sup_{\rho_n\leq|z|\leq\sigma_n} v_i(\rho_n)e^{-ar}|(\chi(k,n-1)T_{n-1}T_n \\
&+ \chi(k,n)T_n^2 + \chi(k,n+1)T_{n+1}T_n)p(z)| \\
&\leq C_1 \sup_{n\in N_1}(\sup_{\rho_{n+1}\leq r\leq\sigma_{n+1}} v_i(\rho_{n+1})e^{-ar} \sup_{\Theta\in[0,2\pi]} |(T_nT_{n+1}p)(re^{i\Theta})| \\
&+ \sup_{\rho_n\leq r\leq\sigma_n} v_i(\rho_n)e^{-ar} \sup_{\Theta\in[0,2\pi]} |(T_n^2p)(re^{i\Theta})| \\
&+ \sup_{\rho_{n-1}\leq r\leq\sigma_{n-1}} v_i(\rho_{n-1})e^{-ar} \sup_{\Theta\in[0,2\pi]} |(T_nT_{n-1}p)(re^{i\Theta})|) \\
&\leq DC_1 \sup_{n\in N_1}(\sup_{\rho_{n+1}\leq|z|\leq\sigma_{n+1}} v_i(\rho_{n+1})e^{-ar}|(T_{n+1}p)(z)| \\
&+ \sup_{\rho_n\leq|z|\leq\sigma_n} v_i(\rho_n)e^{-ar}|(T_np)(z)| \\
&+ \sup_{\rho_{n-1}\leq|z|\leq\sigma_{n-1}} v_i(\rho_{n-1})e^{-ar}|(T_{n-1}p)(z)|) \qquad (1)
\end{aligned}$$

The choice of N_1 yields $v_i(\rho_n) = a_1^{-1}s(\rho_n)$ and moreover by Proposition 2

$$\begin{aligned}
da_1v_i(\rho_{n+1}) &\leq a_1v_i(\rho_n) \leq s(\rho_n) \leq ds(\rho_{n+1}) \\
a_1v_i(\rho_{n-1}) &\leq a_1v_i(\rho_n) \leq s(\rho_n) \leq \frac{1}{d}s(\rho_{n-1}).
\end{aligned}$$

Hence (1) is bounded by

$$\begin{aligned}
&3Dd^{-1}C_1a_1^{-1} \sup_{j=-1,0,1,\dots,n+j\in N_1} \sup_{\rho_n\leq|z|\leq\sigma_n} s(\rho_n)e^{-ar}|(T_np)(z)| \\
&\qquad \leq 3D^2d^{-1}C_1a_1^{-1}\|p\|_u \leq 3D^2d^{-1}C_1a_1^{-1}.
\end{aligned}$$

Analogously we obtain $\|p_2\|_l \leq 3D^2d^{-1}C_2a_2^{-1}$. Thus,

$$\begin{aligned}
p = p_1 + p_2 &\in 3D^2d^{-1}C_1a_1^{-1}C_{i,0} + 3D^2d^{-1}C_2a_2^{-1}C_{l,0} \\
&= 6D^2d^{-1}C_1\lambda C_{i,0} + 6D^2d^{-1}C_2\varepsilon C_{l,0} \\
&= 6D^2d^{-1}C_1\lambda C_{i,0} + \mu C_{l,0}.
\end{aligned}$$

Put $\xi := 6D^2d^{-1}C_1\lambda$ and obtain the claim. ∎

Now we show that quasinormable spaces $HW_0(\mathbb{C})$ of the type considered in Theorem 4 are even quasinormable in the sense of Peris [19].

At this point I would like to thank José Bonet who helped me to fix the proof of the following lemma.

Lemma 5 *Let E be a Fréchet space with a basis $\mathcal{U}_0(E)$ of 0-neighborhoods. Then E satisfies (QNo) if and only if the following holds:*

$$\begin{aligned}
(qno) \quad &\forall U \in \mathcal{U}_0(E)\exists V \in \mathcal{U}_0(E)\ \forall W \in \mathcal{U}_0(E)\ \forall\delta > 0 \\
&\exists s = s(\delta, W) > 0\ \exists P \in L(E,E) : P(V) \subset sW \text{ and } (I-P)(V) \subset \delta U.
\end{aligned}$$

Proof. Fix $U_0 \in \mathcal{U}_0(E)$. Proceed inductively using (qno) to find a basis $(U_n)_{n\in\mathbb{N}_0}$ of 0-neighborhoods of E with the property: $\forall n \in \mathbb{N}_0\ \forall\delta > 0\ \forall W \in \mathcal{U}_0(E)\ \exists s = s(\delta, W)\ \exists P \in L(E, E)$:

$$P(U_n) \subset sW \text{ and } (I - P)(U_n) \subset \delta U_{n-1}.$$

Next, for $U_0 \supset U_1$ and $\varepsilon > 0$ we apply (qno) for $\delta = \frac{\varepsilon}{2}$ and $W = U_2$ to find $s_2 > 0$ and $P_1 \in L(E, E)$ with

$$P_1(U_1) \subset s_2U_2 \text{ and } (I - P_1)(U_1) \subset \frac{\varepsilon}{2}U_0.$$

Apply now (qno) for $U_1 \supset U_2$, $\varepsilon > 0$, $\delta = \frac{\varepsilon}{2^2 s_2}$ and $W = U_3$ to find $s_3' > 0$ and $P_2 \in L(E, E)$ with

$$P_2(U_2) \subset s_3'U_3 \text{ and } (I - P_2)(U_2) \subset \frac{\varepsilon}{2^2 s_2}U_1.$$

Hence

$$P_2(s_2U_2) \subset s_2s_3'U_3 \text{ and } (I - P_2)(s_2U_2) \subset \frac{\varepsilon}{2^2}U_1.$$

Define $s_3 := s_2s_3'$. Proceeding in this way we get the basis $U_0 \supset U_1 \supset U_2 \supset \dots \supset U_k \supset U_{k+1} \supset \dots$ such that for every $k \in \mathbb{N}$ there are $s_k > 0$ and $P_k \in L(E, E)$:

$$P_k(s_kU_k) \subset s_{k+1}U_{k+1} \text{ and } (I - P_k)(s_kU_k) \subset \frac{\varepsilon}{s^k}U_{k-1}.$$

Observe that, for $x \in U_1$ we get

$$x = P_1x + (I - P_1)x \in s_2U_2 + \frac{\varepsilon}{2}U_0$$

and thus

$$x = P_2P_1x + (I - P_2)P_1x + (I - P_1)x \in s_3U_3 + \frac{\varepsilon}{2^2}U_1 + \frac{\varepsilon}{2}U_0.$$

Finally, in general

$$\begin{aligned} x &= P_k \cdots P_2P_1x + (I - P_k)P_{k-1} \cdots P_2P_1x + \cdots + (I - P_2)P_1x + (I - P_1)x \\ &\in s_{k+1}U_{k+1} + \frac{\varepsilon}{2^k}U_{k-1} + \cdots + \frac{\varepsilon}{2^2}U_1 + \frac{\varepsilon}{2}U_0. \end{aligned} \tag{2}$$

Set $P_0 = I$. We claim that $\sum_{k=1}^{\infty}(I - P_k)P_{k-1} \cdots P_1x$ converges for all $x \in U_1$. Obviously $\sum_{k=l+1}^{l+m}(I - P_k)P_{k-1} \cdots P_1x \in \frac{\varepsilon}{2^l}U_l$. Hence $Qz := \sum_{n=1}^{\infty}(I - P_k)P_{k-1} \cdots P_1z$ converges for every $z \in E$, since U_1 is absorbing. By the theorem of Banach-Steinhaus we know that Q belongs to $L(E, E)$. Moreover $Q(U_1) \subset \varepsilon U_0$. It remains to show that $(I - Q)(U_1)$ is bounded in E. Fix $s \in \mathbb{N}$. Using (2) above, we see, for $y \in U_1$,

$$\begin{aligned} (I - Q)y = P_sP_{s-1} \cdots P_1y + \sum_{k=s+1}^{\infty}(I - P_k)P_{k-1} \cdots P_1y &\in s_{s+1}U_{s+1} + \varepsilon U_s \\ &\subset (s_{s+1} + 1)U_s. \end{aligned}$$

Thus,

$$(I - Q)(U_1) \subset \bigcap_{s\in\mathbb{N}}(s_{s+1} + 1)U_s.$$

Setting $P := I - Q$ we have proved (QNo). The other direction is obvious. ■

Lemma 6 *Let E be a Fréchet space and F a dense subspace with (QNo). Then E also satisfies (QNo).*

Proof. Let $(U_n)_{n\in\mathbb{N}}$ be a fundamental sequence of 0-neighborhoods in E. Then the sets $V_n = F \cap U_n$, $n \in \mathbb{N}$, yield a fundamental sequence of 0-neighborhoods in F. Since F is (QNo) we obtain:
$\forall n \in \mathbb{N}\ \exists m > n\ \forall k > n\ \forall \varepsilon > 0\ \exists \lambda_k > 0\ \exists P_k \in L(F,F) : P_k(V_m) \subset \lambda_k V_k$ and $(I - P_k)(V_m) \subset \varepsilon V_n$. The sets $\overline{V_n} = \overline{U_n \cap F}$, $n \in \mathbb{N}$, give a fundamental sequence of 0-neighborhoods of E. Next we consider the unique extension $\tilde{P}_k \in L(E,E)$ of P_k and obtain

$$\tilde{P}_k(\overline{V_m}) \subset \overline{\lambda_k V_k} = \lambda_k \overline{V_k} \text{ and } (\tilde{I} - \tilde{P}_k)(\overline{V_m}) \subset \overline{\varepsilon V_n} = \varepsilon \overline{V_n}$$

and the claim follows. ∎

Theorem 7 *Let $W = (w_n)_{n\in\mathbb{N}}$ be an increasing sequence of strictly positive continuous radial functions on the complex plane $\mathbb{C}$ such that each w_n belongs to the class $(E)_{A,a}$. Then $HW_0(\mathbb{C})$ is quasinormable if and only if it is (qno) or equivalently (QNo).*

Proof. First, we assume that $HW_0(\mathbb{C})$ is quasinormable. Then, by Theorem 4 we know that for every $n \in \mathbb{N}$ there is $m > n$ such that for every $k > n$ and $\mu > 0$ we can find $\xi_k > 0$ such that

$$\left(\frac{1}{w_m}\right)^{\sim} \leq \frac{\xi_k}{w_k} + \frac{\mu}{w_n} \text{ on } \mathbb{C}. \qquad (*)$$

Since the polynomials are dense in $HW_0(\mathbb{C})$ (see [3]), by Lemma 6 it is enough to consider only the polynomials $\mathcal{P}$. Now, the proof is quite analogous the proof of Theorem 4 that (4) yields (3). We keep the notation we used there. We fix $n \in \mathbb{N}$ and select $m > n$. Moreover, for fixed $k > n$ and $\mu > 0$, there is $\xi_k > 0$ such that $(*)$ is fulfilled. Put now $\varepsilon := \frac{\mu}{6D^2 d^{-1} C_2}$. Following the lines of the proof of Theorem 4 we choose $p \in C_{m,0}$ and set $P_k p := \sum_{n\in N_1} T_n p$. Then $(I - P_k)p = \sum_{n\in N_2} T_n p$, since $\mathbb{N}$ is the disjoint union of N_1 and N_2. We obtain $\|P_k p\|_k \leq 6D^2 d^{-1} C_1 \xi_k$ and $\|(I - P_k)p\|_n \leq 6D^2 d^{-1} C_2 \varepsilon = \mu$. If we put $\xi_k := 6D^2 d^{-1} C_1 \lambda_k$, we obtain the claim. The other direction follows from the definition. ∎

Corollary 8 *Let $W = (w_n)_{n\in\mathbb{N}}$ be an increasing sequence of strictly positive continuous radial functions on the complex plane $\mathbb{C}$ such that each w_n belongs to the class $(E)_{A,a}$. If E is (QNo) and $HW_0(\mathbb{C})$ is quasinormable, then $HW_0(\mathbb{C}, E)$ is also quasinormable.*

Proof. This follows directly from [19, Proposition 3.4] and [2, Corollary 30]. ∎

References

[1] Bastin, F., Ernst, B., A criterion for $CV(X)$ to be quasinormable, *Results Math.* **14**, no. 3-4, (1988) 223-230.

[2] Bierstedt, K.D., Tensor products of weighted spaces, *Function spaces and dense approximation (Proc. Conf., Univ. Bonn, Bonn, 1974), Bonn. Math. Schriften* **81** (1975), 26-58.

[3] Bierstedt, K.D., Bonet, J., Galbis, A., Weighted spaces of holomorphic functions on balanced domains, *Michigan Math. J.* **40**, no. 2, (1993) 271-297.

[4] Bierstedt, K.D., Bonet, J., Taskinen, J., Associated weights and spaces of holomorphic functions, *Studia Math.* **127**, no. 2, (1998) 137-168.

[5] Bierstedt, K.D., Bonet, J., Taskinen, J., Weighted inductive limits of spaces of entire functions, *Monatsh. Math.* **154**, (2008), 103-120.

[6] Bierstedt, K.D., Meise, R., Distinguished echelon spaces and the projective description of weighted inductive limits of type $V_dC(X)$, pp. 169-226 in *Aspects of Mathematics and its Applications*, North-Holland Math. Library **34**, Amsterdam, 1986.

[7] Bierstedt, K.D., Meise, R., Summers, W.H., Köthe sets and Köthe sequence spaces, pp. 27-91 in *Functional Analysis, Holomorphy and Approximation Theory (Rio de Janeiro 1980)*, North-Holland Math. Stud. **71**, Amsterdam, 1982.

[8] Bonet, J., Dierolf, S., On the lifting of bounded sets, *Proc. Edinb. Math. Soc. (2)* **36** (1993), 277-281.

[9] Bonet, J., Dierolf, S., On distinguished Fréchet spaces, *Progress in Functional Analysis, Proc. Int. meet. Occas. 60th Birthd. M. Valdivia, Peñíscola/Spain, North-Holland Math. Stud. 170*, North-Holland, 1992, 201-214.

[10] Bonet, J., Dierolf, S., Aye Aye, K., Dense subspaces of quasinormable spaces, *Math. Nachr.* **279** (2006), no. 7, 699-704.

[11] Bonet, J., Peris, A., On the injective tensor product of quasinormable spaces, *Results Math.* **20** (1991), 431-443.

[12] Grothendieck, A., Sur les espaces (F) et (DF), *Summa Brasil Math.* **3** (1954) 57-123.

[13] Jarchow, H., *Locally Convex Spaces*, Math. Leitfäden, B.G. Teubner, Stuttgart, 1981.

[14] Köthe, G., *Topological Vector Spaces I*, Grundlehren der math. Wiss. **159**, Springer-Verlag, New York-Berlin, 1969.

[15] Lusky, W., On the isomorphism classes of weighted spaces of harmonic and holomorphic functions, *Studia Math.* **175** (2006), 19-45.

[16] Meise, R., Vogt, D., A characterization of quasinormable Fréchet spaces, *Math. Nachr.* **122** (1985) 141-150.

[17] Meise, R., Vogt, D., *Introduction to Functional Analysis*, Oxford Graduate Texts in Math. **2**, The Clarendon Press, Oxford University Press, New York, 1997.

[18] Pérez-Carreras, P., Bonet, J., *Barrelled Locally Convex Spaces*, North-Holland Math. Stud. **131**, Amsterdam, 1987.

[19] Peris, A., Quasinormable spaces and the problem of topologies of Grothendieck, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **19** (1994), no. 1, 167-203.

[20] Valdivia, M., On quasinormable echelon spaces, *Proc. Edinbourgh Math. Soc.(2)* **24** (1981) 73-80.

[21] Valdivia, M., Basic sequences in the dual of a Fréchet space, *Math. Nachr.* **231** (2001), 169-185.

[22] Valdivia, M., *Topics in locally convex spaces*, North-Holland Math. Stud. **67**, Amsterdam, 1982.

[23] Vogt, D., Some results on continuous linear maps between Fréchet spaces, pp. 349-381 in *Functional analysis: Surveys and Recent Results III*, North-Holland Math. Stud. **90**, Amsterdam, 1984.

[24] Vogt, D., On the functor $Ext(E, F)$ for Fréchet spaces, *Studia Math.* **85** (1985), 163-197.

[25] Wolf, E., Weighted Fréchet spaces of holomorphic functions, *Studia Math.* **174** (2006), no. 3, 255-275.

[26] Wolf, E., A note on quasinormable weighted Fréchet spaces of holomorphic functions, *Bull. Belg. Math. Soc. Simon Stevin* **14** (2007), 587-593.

Mathematical Institute, University of Paderborn,
D-33095 Paderborn, Germany;
e-mail: lichte@math.uni-paderborn.de