Parabolic surfaces in hyperbolic space with constant Gaussian curvature

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Abstract

A parabolic surface in hyperbolic space $\mathbb{H}^3$ is a surface invariant by a group of parabolic isometries. In this paper we describe all parabolic surfaces with constant Gaussian curvature. We study the qualitative properties such as completeness and embeddedness.

1 Introduction

The aim of this paper is to provide new examples of surfaces in three-dimensional hyperbolic space $\mathbb{H}^3$ with constant Gaussian curvature K. We abbreviate by saying K-surface. By the Gaussian curvature K of a surface in $\mathbb{H}^3$ we mean $K = K_{ext} - 1$, where K_{ext} is the extrinsic curvature of the surface, i.e., the determinant of the second fundamental form. For this purpose, we restrict to the family of rotation surfaces because the constant Gauss curvature equation reduces into an ordinary differential equation. In hyperbolic space, there exist three types of rotational surfaces depending of the group of isometries by which they are invariant: spherical, hyperbolic and parabolic. A group of isometries leaves fixed a set $\{p, q\}$ of two points of the ideal boundary $\mathbb{S}^2_\infty$ of $\mathbb{H}^3$. The group is of spherical (resp. hyperbolic, parabolic) type if p and q are fixed (resp. p is carried to q and vice-versa, $p = q$).

We are interested in parabolic isometries. A group of parabolic isometries leaves fixed one point of $\mathbb{S}^2_\infty$ and so, it leaves globally fixed the horocycles passing

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through this point. We say that a surface S is a parabolic surface if it is invariant by a group of parabolic isometries. Parabolic surfaces in $\mathbb{H}^3$ were introduced by Do Carmo and Dajczer in [1] for studying surfaces with constant mean curvature (see also the PhD. thesis of Gomes, [2]).

Leite classified complete rotational spherical hypersurfaces in hyperbolic n-space $\mathbb{H}^n$ with constant scalar curvature [3]. Mori studied in [4] complete rotational hypersurfaces in $\mathbb{H}^n$ with constant scalar curvature and of hyperbolic and parabolic type. In this paper, and for the 3-dimensional space $\mathbb{H}^3$, we give a full description of (complete and not complete) parabolic K-surfaces. A first matter of interest is about the completeness, and more general, if such surfaces can be extended to be complete. In this sense, part of our results coincide with the ones in [4]. Furthermore, we will give a special emphasis on the behaviour of the asymptotic boundary of such surfaces. Let us recall that the asymptotic boundary of a surface is the intersection of its closure with the sphere at infinity $\mathbb{S}^2_\infty$ of $\mathbb{H}^3$. Although the parabolic K-surfaces will be described in the next sections, we can state some facts of their qualitative properties. The first one is about embeddedness (see Theorem 2.3).

Any parabolic K-surface immersed in hyperbolic space $\mathbb{H}^3$ is embedded.

In contrast to this, we point out that this does not occur if we replace K by the mean curvature. Gomes proved the existence of a family of complete parabolic surfaces with constant mean curvature that auto-intersect along horocycles [2] (see also [1]). We also show:

Any non-umbilical parabolic K-surface in $\mathbb{H}^3$, with $K < -1$ or $K \geq 0$, is not complete. Moreover, its asymptotic boundary is the point fixed by the group of parabolic isometries.

When the value of the Gaussian curvature lies in the interval $[-1, 0)$, we obtain:

For each number K with $-1 \leq K < 0$, there exists a non-umbilical complete parabolic K-surface in $\mathbb{H}^3$. For such surface, the asymptotic boundary is formed by two circles tangent at the point fixed by the group of parabolic isometries.

The range of real numbers $[-1, 0]$ of values of K is specially interesting. In this interval of numbers for K, there are umbilical K-surfaces: horospheres ($K = 0$), equidistant surfaces ($-1 < K < 0$) and geodesic planes ($K = -1$). On the other hand, it has been proved that for a closed embedded curve Γ in a horosphere Q (resp. in $\mathbb{S}^2_\infty$), and $K \in (-1, 0)$, there exists a K-surface S with boundary $\partial S = \Gamma$ ([5]). For this, the authors search graphs on bounded domains in Q (resp. $\mathbb{S}^2_\infty$). Then the function that defines the graph satisfies an elliptic equation of Monge-Ampère type which can be solved by using the continuity method and the derivation of *a priori* $C^{2+\alpha}$ estimates of the solutions.

This paper is organized as follows. In Section 2 we derive the equation for the Gauss curvature of a parabolic surface and we give some qualitative properties of their solutions. In Section 3 we solve this equation in all its generality whereas in Section 4 we restrict the study to those solutions that are symmetric by reflections with respect to geodesic planes invariant under the group of parabolic isometries.

2 The equation of constant Gauss curvature

In this section we obtain the differential equations that govern the surfaces of our study. In order to have a simple visualization of these surfaces, we consider the half-space model of the hyperbolic $\mathbb{H}^3$, i.e.,

$$\mathbb{H}^3 =: \mathbb{R}^3_+ = \{(x,y,z) \in \mathbb{R}^3; z > 0\}$$

equipped with the hyperbolic metric

$$\langle , \rangle = \frac{dx^2 + dy^2 + dz^2}{z^2}.$$

In what follows, we will use the words "vertical" or "horizontal" in the usual affine sense of $\mathbb{R}^3_+$. The ideal boundary $\mathbb{S}^2_\infty$ of $\mathbb{H}^3$ is identified with the one point compactification of the plane $\Pi \equiv \{z = 0\}$, that is, $\mathbb{S}^2_\infty = \Pi \cup \{\infty\}$ and it corresponds with the asymptotic classes of geodesics rays of $\mathbb{H}^3$. The asymptotic boundary of a set $\Sigma \subset \mathbb{H}^3$ is defined as $\partial_\infty \Sigma = \overline{\Sigma} \cap \mathbb{S}^2_\infty$, where $\overline{\Sigma}$ is the closure of Σ in $\{z \geq 0\} \cup \{\infty\}$.

A parabolic group of isometries G of $\mathbb{H}^3$ is a group of isometries that admits a fixed double point at $\mathbb{S}^2_\infty$. These isometries leave globally fixed each horocycle tangent to the fixed point. In our model, we take the point $\{\infty\}$ of $\mathbb{S}^2_\infty$ as the point that fixes G. Then the group G is defined by the horizontal (Euclidean) translations in the direction of a horizontal vector ξ, $\xi \in \Pi$: $G = \{T_a; a \in \mathbb{R}, T_a(p) = p + a\xi\}$. The orbits are horizontal straight lines parallel to ξ. We can also view this group as the set of reflections with respect to any geodesic plane orthogonal to ξ. Actually, the parabolic group G is generated by all reflections with respect to the geodesic planes orthogonal to ξ. The space of orbits is then represented in any geodesic plane of this family. This will be done in our study.

Let G be a group of parabolic isometries. Without loss of generality, we assume that the horizontal vector ξ that defines the group of is the vector $\xi = (0,1,0)$. Let $P = \{(x,0,z); z > 0\}$, which it is a vertical geodesic plane orthogonal to ξ. Then a surface S invariant by G intersects P in a curve α called the generating curve of S. If S has constant Gaussian curvature, we shall obtain an ordinary differential equation for the curve α (Equation (3) below). Let $L = \{(x,0,0), x \in \mathbb{R}\}$. If we assume that S is a complete surface, the possibilities about its asymptotic boundary $\partial_\infty S$ are: a circle ($\partial_\infty \alpha$ is a point of L or one point of L together ∞), two tangent circles ($\partial_\infty \alpha$ are two different points of L) or it is one point ($\partial_\infty \alpha = \emptyset$ or ∞).

Remark 2.1. *Throughout this work, we assume that a parabolic surface is invariant under the group of parabolic isometries defined by the vector $\xi = (0,1,0)$.*

We derive the equation for the curvature of a parabolic surface. We assume that α is parameterized by arc length with respect to the Euclidean metric and whose domain of definition I is an open interval of real numbers including zero. A parametrization of S is $\mathbf{X}(s,y) = (x(s), y, z(s))$, $s \in I$, $y \in \mathbb{R}$. Let us introduce the function $\theta(s)$, which measures the angle between the tangent vector of α, $\alpha'(s)$, and the x-axis, that is, $x'(s) = \cos(\theta(s))$ and $z'(s) = \sin(\theta(s))$ for a certain differentiable function θ. Exactly, the derivative $\theta'(s)$ is the (Euclidean) curvature of α. Since in this model of $\mathbb{H}^3$, the hyperbolic and Euclidean metrics are conformal, the Gauss maps of S with each one of the induced metric from $\mathbb{H}^3$ and $\mathbb{R}^3_+$ are parallel. In particular, $\partial\mathbf{X}/\partial s$ and $\partial\mathbf{X}/\partial y$ are the principal directions at each point of S. If κ and κ_E denote a principal curvature with the induced hyperbolic and Euclidean metric on S respectively, then κ and κ_E are related as

$$\kappa(s,y) = z(s)\kappa_E(s,y) + z \circ \mathbf{N_E}(s,y),$$

where $\mathbf{N}_E$ is the Gauss map of S for the Euclidean metric. Since $\mathbf{N}_E(s,y) = (-\sin(\theta(s)), 0, \cos(\theta(s)))$ and the two principal (Euclidean) curvatures are $\kappa_1(s,y) = \theta'(s)$ and $\kappa_2(s,y) = 0$, we deduce that the principal curvatures of S are

$$\kappa_1(s,y) = z(s)\theta'(s) + \cos(\theta(s)), \qquad \kappa_2 = \cos(\theta(s)). \tag{1}$$

The term $z(s)\theta'(s)$ coincides with the hyperbolic curvature of α at s. Therefore the Gaussian curvature $K = K_{ext} - 1 = \kappa_1\kappa_2 - 1$ at each point $\mathbf{X}(s,y)$ of S is

$$K(s,y) = z(s)\theta'(s)\cos(\theta(s)) - \sin(\theta(s))^2. \tag{2}$$

Hence, it follows that the coordinates of the generating curve α of a parabolic K-surface S in $\mathbb{H}^3$ are governed by the next differential equations system:

$$\left\{\begin{array}{lcl} x'(s) & = & \cos(\theta(s)) \\ z'(s) & = & \sin(\theta(s)) \\ \theta'(s)z(s)\cos(\theta(s)) & = & K + \sin^2(\theta(s)) \end{array}\right. \tag{3}$$

In order to define the initial conditions, we consider isometries $h : \mathbb{H}^3 \to \mathbb{H}^3$ that keeps unchanged the parabolic group G by conjugations, that is, $h^{-1}Gh = G$. In such case, if S is a parabolic surface invariant by the group G, then $h(S)$ is a surface invariant by the same group G. First, we take a translation in the x-direction. This allows to assume that $x(0) = 0$. On the other hand, we consider the hyperbolic isometries that fix the origin $O = (0,0,0) \in \mathbb{S}^2_\infty$, that is, homotheties from O. The group G is invariant under conjugation by any such homotheties. With an appropriate homothety, we can assume that $z(0) = 1$. As conclusion and without loss of generality, we suppose that the functions $x(s)$, $z(s)$ and $\theta(s)$ satisfy the initial conditions:

$$x(0) = 0, \quad z(0) = 1, \quad \theta(0) = \theta_0. \tag{4}$$

In our family of parabolic K-surfaces, there are umbilical surfaces. If we require that $\kappa_1 = \kappa_2$ at every point of the surface, it means that θ is a constant function θ_0 and that α parameterizes an open set of a straight line. With our notation, and from (2), these umbilical surfaces are the following:

1. A horosphere and $K = 0$, if $\sin(\theta_0) = 0$.
2. An equidistant surface and $-1 < K < 0$, if $0 < |\sin(\theta_0)| < 1$.
3. A (vertical) geodesic plane and $K = -1$, if $\sin(\theta_0) = \pm 1$.

We return to the solutions of (3). We show that if K is constant, then do not exist points on the surface with vertical tangent plane, unless that the surface is a vertical geodesic plane. A vertical tangent plane occurs if $\partial \mathbf{X}/\partial s$ is vertical, that is, $\cos(\theta(s)) = 0$.

Lemma 2.2. *Let α be the generating curve of a parabolic K-surface S in $\mathbb{H}^3$. With the above notation, if there exists $s \in I$ such that* $\cos(\theta(s)) = 0$, *then S is a vertical geodesic plane.*

Proof. It suffices in showing that $A = \{s \in I; \cos(\theta(s)) = 0\}$ is an open set. In such case, the hypothesis of Lemma says that $A \neq \emptyset$ and thus, $A = I$, proving the Lemma. We assume that A is not an open set and we will arrive to a contradiction. Let $s_0 \in I$ such that $\cos(\theta(s_0)) = 0$ and assume that s_0 is a limit point of numbers s with $\cos(\theta(s)) \neq 0$. From (2) and since $\cos(\theta(s_0)) = 0$, the value of the Gauss curvature K is -1. Then $z(s)\theta'(s) = -\cos(\theta(s))$. By differentiation this expression, we conclude that $\theta''(s) = 0$. This means that $\theta(s) = rs + a$ for some constants r, a. If $r = 0$, $\theta' = 0$, which means that $\cos(\theta(s)) = 0$: contradiction. Thus, $r \neq 0$. But then (2) writes as $rz(s) = -\cos(\theta(s))$ and by letting $s \to s_0$, we conclude that $z(s_0) = 0$, a contradiction again. ∎

As a consequence, we have

Theorem 2.3. *Any parabolic K-surface in $\mathbb{H}^3$ is a vertical geodesic plane parallel to ξ or its a graph on a strip of* Π. *In particular, it is embedded.*

Proof. From Lemma 2.2 and excluding the case that S is a geodesic plane, the planar curve α has not points with vertical tangent line, which it shows that α is a graph on some interval of the line L. Thus S is a graph on a domain D of Π and the parabolicity property of S implies that D is a strip. ∎

The above theorem informs us about the completeness of a parabolic K-surface. We point out that the very curve $\alpha \subset S$ is a geodesic of S. Thus S is complete if and only if the length of α is infinite. Consider $\alpha = \alpha(x) = (x, 0, z(x))$, $x \in (x_1, x_2)$, with $-\infty \leq x_1 < x_2 \leq \infty$. Then the length is infinite if $\partial_\infty \alpha = \{\alpha(x_1), \alpha(x_2)\}$.

We now establish a result concerning the Euclidean curvature θ' of α, which says that θ' can not vanish unless that α is a straight line.

Lemma 2.4. *Let α be the generating curve of a parabolic K-surface S. If the curvature of α vanishes at some point, then α is a straight line and S is an umbilical surface.*

Proof. Assume that $\theta'(s_0) = 0$ for a real number $s_0 \in I$. Then the functions

$$\begin{aligned}
\bar{x}(s) &= \cos(\theta(s_0))(s - s_0) + x(s_0) \\
\bar{z}(s) &= \sin(\theta(s_0))(s - s_0) + z(s_0) \\
\varphi(s) &= \theta(s_0)
\end{aligned}$$

are solutions of (3) with the same initial conditions as α at $s = s_0$. Then the uniqueness of solutions implies that $\alpha(s) = (\bar{x}(s), 0, \bar{z}(s))$. This says that α parameterizes a straight line and that S is an umbilical surface. ■

As a consequence of Lemma 2.4, if we view α as the graph of $z = z(x)$, we conclude that either z is a linear function or z is convex (or concave) in all its domain. By analogy, we will say that α is convex (or concave) respectively if z does.

We also obtain a result of symmetry of a parabolic K-surface provided that $z(s)$ has a critical point. If $z'(s) = 0$, the tangent vector $\partial\mathbf{X}/\partial s$ is horizontal, and geometrically, it says that the tangent plane to S at $\alpha(s)$ is horizontal. The surfaces described in the next Lemma will be studied in Section 4.

Lemma 2.5 (Symmetry). *Let α be the generating curve of a parabolic K-surface S. Assume that at $s = s_0$, $z'(s_0) = 0$. Then α is symmetric with respect to the vertical line $x = x(s_0)$ and S is symmetric with respect to the geodesic plane $\{x = x(s_0)\}$.*

Proof. Without loss of generality, we assume that $s_0 = 0$. As $\theta(0) = 0$, then $\sin(\theta(0)) = 0$. Thus the functions

$$\bar{x}(s) = x(-s), \quad \bar{z}(s) = z(-s), \quad \varphi(s) = \theta(-s)$$

satisfy the differential equations system (3) with the same initial conditions as α. By uniqueness, $\alpha(s) = \alpha(-s)$ and this concludes the proof. ■

3 The solutions of the constant Gauss curvature equation

In this section we study the solutions of (3) in all its generality without restrictions on the initial conditions (4). Recall that the assumption $\theta(0) = 0$ on the starting angle implies by Lemma 2.5 some properties of symmetry of the solutions. As we will see in the description below, it is possible that the angle θ does not reach the value $\theta_0 = 0$ and the surface would be not symmetric with respect to vertical geodesic planes parallel to the ξ-direction.

In the above system of equations (3), we have been successful in finding exact solutions of the function $z(s)$, which will be presented below. The third equation in (3) writes as $z(s)z''(s) = K + z'(s)^2$. Consider $z'(s)$ as a function of the next variable $z(s)$. If we put $p = z'$ and $x = z$, we have $xp(x)p'(x) = K + p(x)^2$. Setting $y = p^2$, we write

$$xy'(x) = 2K + 2y(x).$$

The solutions of this equation are $y(x) = \lambda x^2 - K$, that is,

$$z'(s)^2 = \lambda z(s)^2 - K. \tag{5}$$

At $s = 0$, we have $\lambda = K + \sin^2(\theta_0)$. A new differentiation in (5) gives

$$z''(s) = (K + \sin^2(\theta_0))z(s). \tag{6}$$

It is well known that the solutions of (6) depends on the sign of the constant $K + \sin^2(\theta_0)$. We then analyze them according to this sign.

3.1 Case $K+\sin^2(\theta_0)>0$

The solution of (6) with the initial condition (4) is

$$z(s)=\cosh\left(\sqrt{K+\sin^2(\theta_0)}\,s\right)+\frac{\sin(\theta_0)}{\sqrt{K+\sin^2(\theta_0)}}\sinh\left(\sqrt{K+\sin^2(\theta_0)}\,s\right).$$

We can also write the solutions according to the sign of K. Put $R=K+\sin^2(\theta_0)$. Then we have:

$$z(s)=\begin{cases}\sqrt{\dfrac{K}{R}}\cosh\left(\sqrt{R}\,s+\epsilon a\right), \text{ with } \cosh(a)=\sqrt{\dfrac{R}{K}}, & K>0\\ \exp(\sin(\theta_0)\,s), & K=0\\ \sqrt{\dfrac{-K}{R}}\sinh\left(\epsilon\sqrt{R}\,s+a\right), \text{ with } \sinh(a)=\sqrt{\dfrac{R}{-K}}, & K<0\end{cases}$$

Here a is a non-negative number with $\epsilon=1$ if $\sin(\theta_0)>0$ and $\epsilon=-1$ if $\sin(\theta_0)<0$. If $\sin(\theta_0)=0$, the value of a is $a=0$ (only possible if $K>0$).

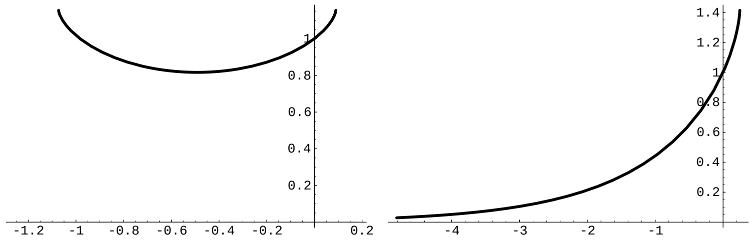
We compute the domains of α. In order to simplify notations, we assume that $\sin(\theta_0)\geq 0$. We have to impose that $z'(s)^2<1$, which imposes conditions on the parameter s. We point out a remark in the case $K<0$. In such setting, we know that $|\sqrt{R}s+a|<\text{arc}\cosh(1/\sqrt{-K})$. However we have an extra condition, since $z(s)$ could vanish. This implies that $s>-a/\sqrt{R}$ and this condition is stronger than $s>-\frac{1}{\sqrt{R}}(\text{arc}\cosh(1/\sqrt{-K})+a)$. Moreover, this proves that one branch of α meets the asymptotic boundary $\mathbb{S}^2_\infty$. Depending on the sign of K, the interval I of definition of the solution α is:

$$\begin{cases}\left(-\dfrac{1}{\sqrt{R}}(\text{arc}\sinh(\dfrac{1}{\sqrt{K}})+a),\dfrac{1}{\sqrt{R}}(\text{arc}\sinh(\dfrac{1}{\sqrt{K}})-a)\right), & K>0\\ \left(-\infty,-\dfrac{\log(\sin(\theta_0))}{\sin(\theta_0)}\right), & K=0\\ \left(-\dfrac{a}{\sqrt{R}},\dfrac{1}{\sqrt{R}}(\text{arc}\cosh(\dfrac{1}{\sqrt{K}})-a)\right), & K<0\end{cases}$$

Finally we study if the curve α has a horizontal tangent line, that is, if there exist values s such that $\sin(\theta(s))=0$. It is immediate that it is not possible if $K\leq 0$ and that when $K>0$ this occurs at $s=-a/\sqrt{R}$. In such point, $z''(s)=\theta'(s)\cos(\theta(s))>0$, which says that it is minimum of the function $z(s)$.

Theorem 3.1. *Let α be the generating curve of a parabolic K-surface S in $\mathbb{H}^3$. Assume that the starting angle θ_0 of α satisfies $K+\sin^2(\theta_0)>0$. Then the surface is not complete. Moreover*

1. *If $K>0$, α is a convex curve with exactly one minimum and $\partial_\infty S=\{\infty\}$. See Fig. 1 (a).*
2. *If $K=0$, α is a convex curve without minimum, one branch of α is asymptotic to the line L and $\partial_\infty S=\{\infty\}$. See Fig. 1 (b).*
3. *If $K<0$, then α is a convex curve without minimum that meets Π at one point and $\partial_\infty S$ is a circle of $\mathbb{S}^2_\infty$. See Fig. 2 (a).*



(a) (b)

Figure 1: The generating curves of parabolic K-surfaces. The initial angle is $\theta(0) = \pi/4$. Case (a): $K = 1$; Case (b): $K = 0$.

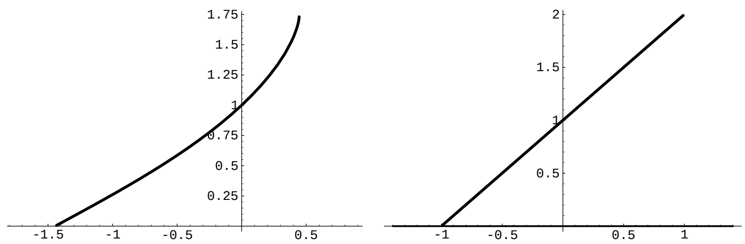
3.2 Case $K + \sin^2(\theta_0) = 0$

The solution is a linear function, namely,

$$z(s) = \sin(\theta_0)\, s + 1.$$

Then $x(s) = \cos(\theta_0)s$ and the parabolic surface generated by α is a Euclidean plane parallel to the ξ vector. Thus the surface is: a geodesic plane, an equidistant surface or a horosphere. See also Lemma 2.4.

Theorem 3.2. *Let α be the generating curve of a parabolic K-surface S in $\mathbb{H}^3$. Assume that the starting angle θ_0 of α satisfies $K + \sin^2(\theta_0) = 0$. Then the surface is umbilical. See Fig. 2 (b).*



(a) (b)

Figure 2: The generating curves of parabolic K-surfaces. The initial angle is $\theta(0) = \pi/4$. Case (a): $K = -1/4$; Case (b): $K = -1/2$.

3.3 Case $K + \sin^2(\theta_0) < 0$

The solution $z(s)$ is given by

$$z(s) = \cos\left(\sqrt{-(K + \sin^2(\theta_0))}\, s\right) + \frac{\sin(\theta_0)}{\sqrt{-(K + \sin^2(\theta_0))}} \sin\left(\sqrt{-(K + \sin^2(\theta_0))}\, s\right).$$

Again, putting $R = K + \sin^2(\theta_0)$, we write $z(s)$ as

$$z(s) = \sqrt{\frac{K}{R}} \sin\left(\epsilon\sqrt{-R}s + a\right), \quad \text{with } \sin(a) = \sqrt{\frac{R}{K}},$$

where, as in the case $R > 0$, a is a non-negative number with $\epsilon = 1$ if $\sin(\theta_0) \geq 0$ and $\epsilon = -1$ if $\sin(\theta_0) < 0$.

We study the domain of the solution α. Without loss of generality, we suppose that $\sin(\theta_0) \geq 0$. Let us recall that the above expression of z can vanish. This implies that $s > -a/\sqrt{-R}$. This condition, together the fact $z'(s)^2 < 1$ determines the interval of definition of α. We distinguish three cases.

1. Case $-1 < K < 0$. Here α intersects the line L at two different points making an angle θ_1 such that $\sin(\theta_1) = \sqrt{-K}$. Then I is
$$\left(-\frac{a}{\sqrt{-R}}, \frac{\pi - a}{\sqrt{-R}}\right).$$
Moreover, in this interval, the function $z'(s)$ vanishes, which says that α presents a maximum at the point $s_1 = (\pi - 2a)/2\sqrt{-R}$ and α is symmetric with respect to the line $x = x(s_1)$.

2. Case $K = -1$. Then α is a halfcircle that intersects orthogonally the line L. This situation will be studied below.

3. Case $K < -1$. Here the domain is
$$\left(\frac{1}{\sqrt{-R}}(\text{arc}\cos(\frac{1}{\sqrt{-K}}) - a), \frac{1}{\sqrt{-R}}(\pi - a - \text{arc}\cos(\frac{1}{\sqrt{-K}}))\right).$$
In particular, α does not meet the line L. Again, $z'(s)$ vanishes at one maximum of the function $z = z(s)$.

Theorem 3.3. *Let α be the generating curve of a parabolic K-surface S in $\mathbb{H}^3$. Assume that the starting angle θ_0 of α satisfies $K + \sin^2(\theta_0) < 0$. Then:*

1. *If $-1 \leq K < 0$, α is a concave curve with exactly one maximum. The surface S is complete and $\partial_\infty S$ is formed by two circles tangent at ∞. See Fig. 3 (a). If $K = -1$, α is a Euclidean halfcircle that meets orthogonally the line L.*

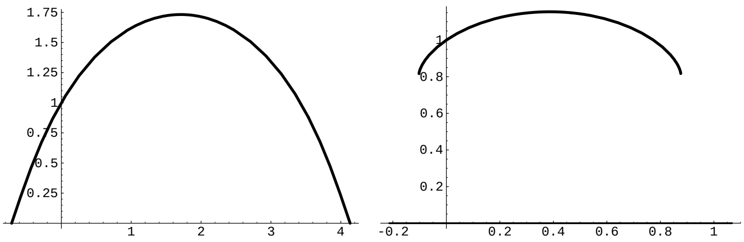
2. *If $K < -1$, then α is a concave curve with one maximum. The surface is not complete and $\partial_\infty S = \{\infty\}$. See Fig. 3 (b).*

This concludes the discussion of the solutions of the differential equation (6) according to the sign of the constant $K + \sin^2(\theta_0)$.

In the results obtained, we were able to give explicitly the function $z(s)$ of the generating curve α. However, and with respect to the function $x(s)$, we can not determine it and we only express $x(s)$ in terms of elliptic integral from the equality

$$x(s) = \int_0^s \sqrt{1 - z'(t)^2}\, dt. \tag{7}$$

Now, we bring up two cases where one can completely solve (3), obtaining explicit parametrizations of the function $x(s)$ and so, of α.



(a) (b)

Figure 3: The generating curves of parabolic K-surfaces. The initial angle is $\theta(0) = \pi/4$. Case (a): $K = -3/4$; Case (b): $K = -2$.

1. Case $K = 0$. Then $K + \sin^2(\theta_0) = \sin^2(\theta_0) \geq 0$. If $\sin(\theta_0) = 0$, we know that the surface S is a horizontal plane, that is, a horosphere. Assume $\sin(\theta_0) \neq 0$. In the next reasoning, we assume that $\sin(\theta_0) > 0$ (analogously if $\sin(\theta_0) < 0$). Then,

$$z(s) = e^{\sin(\theta_0)\, s}, \qquad -\infty < s < -\frac{\log(\sin(\theta_0))}{\sin(\theta_0)}.$$

   Since $z \to 0$ as $s \to -\infty$, the curve α is asymptotic to the line L of the plane Π as $\to -\infty$. See Fig. 1 (b). For the integration of $x(s)$, we denote $m = \sin(\theta_0)$. Then

$$x(s) = \frac{1}{m}\left(\sqrt{1 - m^2 e^{2ms}} - \cos(\theta_0) - \operatorname{arctanh}(\sqrt{1 - m^2 e^{2ms}}) \right. \\ \left. + \operatorname{arctanh}(\cos(\theta_0))\right).$$

   In view of Theorem 2.3, if we consider $x = x(s)$ as independent variable and α as the graph of $z = z(x)$, the domain of z is a non-bounded interval of type $(-\infty, x_1)$, where

$$x_1 = \frac{\operatorname{arc\,tanh}(\cos(\theta_0)) - \cos(\theta_0)}{\sin(\theta_0)}.$$

2. Case $K = -1$. Then $K + \sin^2(\theta_0) \leq 0$. The case of equality means that S is a vertical geodesic plane. We now assume that $K + \sin^2(\theta_0) < 0$. Then

$$z(s) = \frac{1}{\cos(\theta_0)} \sin(\cos(\theta_0)\, s + a), \qquad \sin(a) = \cos(\theta_0)$$

   and

$$x(s) = -\frac{1}{\cos(\theta_0)} \cos(\cos(\theta_0)\, s + a) + \tan(\theta_0).$$

   As $z(s) > 0$, then $-a < \cos(\theta_0)\, s < \pi - a$. Thus α is a halfcircle of radius $1/\cos(\theta_0)$ whose center is the point $(\tan(\theta(_0)), 0)$. The end points of α are in the asymptotic boundary $\mathbb{S}^2_\infty$ and α meets orthogonally the line L. See Figure 4. The function $x(s)$ goes from $(\sin(\theta_0) - 1)/\cos(\theta_0)$ until $(\sin(\theta_0) + 1)/\cos(\theta_0)$

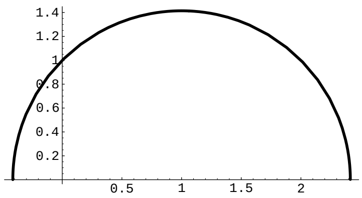


Figure 4: The generating curve of a parabolic K-surfaces with $K = -1$. The initial angle is $\theta(0) = \pi/4$.

Let us return to the description of the generating curve α of S as the graph of the function $z = z(x)$. We focus to the study of the domain of the variable x, $x \in (x_1, x_2)$. From the expression (7) for $x(s)$, $|x(s)| < s$. Hence we deduce that if the interval I of definition of α is bounded, the same occurs for the interval (x_1, x_2).

Theorem 3.4. *Let α be the generating curve of a parabolic K-surface S in $\mathbb{H}^3$. Assume $\theta(0) = \theta_0$.*

1. *If $K + \sin^2(\theta_0) > 0$ with $K \neq 0$, or , $K + \sin^2(\theta_0) < 0$, then x_1 and x_2 are real numbers.*
2. *If $K + \sin^2(\theta_0) > 0$ and $K = 0$, then one of the values x_1 or x_2 is a real number and the other one is ∞.*

Remark 3.5. *If we view α as the graph of $z = z(x)$, the system (3)-(4) is equivalent to the next initial value problem:*

$$z(x)z''(x) - (1 + z'(x)^2)\left((1+K)z'(x)^2 + K\right) = 0$$
$$z(0) = 1, \qquad z'(0) = 0.$$

A first integral concludes that

$$z''(x) = \frac{K}{(1 + K(1 - z(x)^2))^2} z(x).$$

In the cases $K = 0$ and $K = -1$, it follows then

$$\begin{aligned} z(x) &= 1 && \text{if } K = 0 \\ z(x) &= \sqrt{1 - x^2} && \text{if } K = -1 \end{aligned}$$

4 Parabolic surfaces with a horizontal tangent plane

The aim of this section is the study of parabolic K-surfaces whose generating curve α presents a horizontal tangent line at some point. This means that the surface S has points whose tangent plane is parallel to the infinity plane $\mathbb{S}^2_\infty$. Moreover, Lemma 2.4 says that the curve α is symmetric with respect to the vertical line $x = 0$. Following the notation previously used, we assume $\theta_0 = 0$ in (4). The value of R in Section 3 is now simply K, hence that, as in the above section, we distinguish cases depending on the sign of K. Next, we summarize the results obtained.

1. Case $K > 0$. The solution $z(s)$ is
$$z(s) = \cosh\left(\sqrt{K}s\right)$$
whose domain is the bounded interval $(-s_1, s_1)$ with
$$s_1 = \frac{1}{\sqrt{K}}\text{arcsinh}\left(\frac{1}{\sqrt{K}}\right).$$
If we study the behaviour of α near the end points of $(-s_1, s_1)$, we get:
$$\lim_{s\to s_1} z(s_1) = \sqrt{\frac{K+1}{K}} \qquad \lim_{s\to s_1} z'(s_1) = 1.$$
We compute the height of S, that is, the hyperbolic distance between the horospheres at the levels $z(s_1)$ and $z = 1$. It is not difficult to find that this height coincides with
$$\frac{1}{2}\log\left(\frac{K+1}{K}\right).$$
2. Case $K = 0$. The solution is $\alpha(s) = (s, 0, 1)$. Thus, the curve α is a horizontal straight line and the surface is a horosphere.
3. Case $K < 0$. The solution is
$$z(s) = \cos\left(\sqrt{-K}s\right).$$
Depending on the value of K, the generating curve α meets $\mathbb{S}^2_\infty$. If $-1 \leq K < 0$, α intersects $\mathbb{S}^2_\infty$ making an angle θ_1 such that $\sin\theta_1 = \sqrt{-K}$. The domain of α is now $(-\pi/2, \pi/2)$. In the particular case that $K = -1$, α is a halfcircle that orthogonally meets L. If $K < -1$, S is not complete and the curve α is a graph on a bounded domain of L. The parameter s goes in the range $\left(-\frac{1}{\sqrt{-K}}\arcsin\left(\frac{1}{\sqrt{-K}}\right), \frac{1}{\sqrt{-K}}\arcsin\left(\frac{1}{\sqrt{-K}}\right)\right)$.
Analogously as in the case $K > 0$, the height of the surface is
$$\frac{1}{2}\log\left(\frac{K-1}{K}\right).$$

Theorem 4.1. *Let α be the generating curve of a parabolic K-surface S in $\mathbb{H}^3$, where α is the solution of (3). Assume that the initial velocity of α is horizontal. Then we distinguish three possibilities:*

1. *Case $K > 0$. The curve α is convex with exactly one minimum and it is a graph on L defined in some bounded interval $I = (-x_1, x_1)$.*
2. *Case $K = 0$. The curve α is a horizontal straight line and S is a horosphere.*
3. *Case $K < 0$. The curve α is concave with exactly one maximum and it is a graph on L defined in some bounded interval $I = (-x_1, x_1)$. If $K < -1$, α does not intersect the line L and at the end points, the curve is vertical. If $-1 \leq K < 0$, the curve α meets the line L making an angle θ_1 with $\sin\theta_1 = \sqrt{-K}$.*

In the cases 1) and 3), the height of S is $\frac{1}{2}\log\left(\frac{K+1}{K}\right)$ and $\frac{1}{2}\log\left(\frac{K-1}{K}\right)$ respectively.

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