A Kaplansky-Meyer theorem for subalgebras*

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Abstract

In this note we show that, for an arbitrary Hausdorff locally m-convex topology on a subalgebra A of the algebra $C(X)$, the boundedness radius β is nothing but the uniform norm, whenever A is a $C_b(X)$-module and closed under the complex conjugation. We then deduce a Theorem of Kaplansky-Meyer type for subalgebras.

1 Introduction and Preliminaries

A well known result of I. Kaplansky ([4], p. 407) states the following : if X is a locally compact Hausdorff space and $C_0(X)$ is the algebra of all complex or real valued continuous functions on X vanishing at infinity, then every submultiplicative norm on $C_0(X)$ is at least as large as the uniform norm. On the other hand, B. Yood gives in [12] a condition on a topological space T so that the algebra $C(T)$ does not admit any algebra norm. He then conjectured that if $C(T)$ admits a submultiplicative norm, then T must be pseudo compact (i.e. every continuous function on T must be bounded, in other words $C(X) = C_b(X)$). This conjecture was later proved in [5] by M. J. Meyer for an arbitrary topological space T. It is to be noted that Meyer's result fails to hold if, instead of the whole $C(T)$, one takes an arbitrary subalgebra of it. This occurs, for instance, if $T = \mathbb{C}$ and A is the algebra of all polynomial functions endowed with the norm

$$||P|| := \sup\{|P(z)|, |z| \leq 1\},\ P \in A.$$

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This is an algebra norm on A; but T is not pseudo compact. The same algebra A, seen as a subalgebra of $C(T)$ for $T = [0,1]$, shows that an algebra norm on A need not be larger than the uniform norm. Take for example the norm

$$||P|| := \sup\{|P(\frac{1}{n})|,\ 1 \leq n\},\ P \in A.$$

In this note, we show analogs of Kaplansky's and Meyer's Theorems for a large class of subalgebras of $C(T)$ including in particular the Nachbin algebras as in [7]. To this purpose, we first give the expression of the boundedness radius β, as defined in [1], in a subalgebra A of $C(T)$. Namely, we show that, for every Hausdorff locally m-convex topology on A, β is nothing but the uniform norm, whenever A is a $C_b(T)$-module and closed under the complex conjugation.

Henceforth, a topological algebra will be any algebra A on the field $\mathbb{K}$ (= $\mathbb{R}$ or $\mathbb{C}$) endowed with a linear topology τ such that the multiplication of A is separately continuous with respect to τ. We will say that (A,τ) is a locally convex algebra (l. c. a.) if (A,τ) is, in addition, a locally convex space. A l. c. a. (A,τ) will be said to be locally m-convex (l. m. c. in short) if the topology τ can be given by a family $\mathbb{P}$ of submultiplicative seminorms (see [6]); this is to say,

$$\forall P \in \mathbb{P}, P(xy) \leq P(x)P(y),\ \ x,\ y \in A.$$

Following [1], if (A,τ) is a topological algebra and $x \in A$, the boundedness radius of x is the quantity

$$\beta(x) := \inf\left\{\alpha > 0 : \left(\frac{x^n}{\alpha^n}\right)_n \text{ tends to zero as } n \text{ tends to } \infty\right\},\ \ \text{with}\ \ \inf\emptyset = +\infty.$$

It is known [1, 9] that

$$\begin{aligned}\beta(x) &= \inf\left\{\alpha > 0 : \left(\frac{x^n}{\alpha^n}\right)_n \text{ is bounded}\right\}\\ &= \sup_{U\in\mathcal{U}} \limsup_{n\geq 1} (P_U(x^n))^{\frac{1}{n}},\end{aligned}$$

where $\mathcal{U}$ denotes any pseudo base of 0-neighborhoods for τ and P_U the gauge functional of U. The element x is said to be bounded if $\beta(x)$ is finite.

2 Boundedness radius in m-convex subalgebras of $C(X)$

From now on, X will denote a topological space, υX the Hewitt realcompactification of X and βX its Stone-Čech compactification [3]. It is known that υX is a realcompact Hausdorff completely regular space, while βX is even compact. By δ we mean the Dirac transformation. This is a continuous function from X into both υX and βX whose range $\delta(X)$ is dense. Notice that, whenever X is Hausdorff and completely regular, it can be seen as a dense topological subspace of υX as well as of βX.

Let $C(X)$ (resp. $C_b(X)$) denote the algebra of all $\mathbb{K}$-valued continuous (resp. continuous and bounded) functions on X. For each $f \in C(X)$, there is one unique $f^v \in C(vX)$, namely the Gelfand transform of f, such that $f = f^v \circ \delta$. Similarly, if $f \in C_b(X)$, there is one unique, again the Gelfand transform of f, $f^\beta \in C(\beta X)$ such that $f = f^\beta \circ \delta$. Let φ denote both the mappings $f \mapsto f^v$ and $f \mapsto f^\beta$. This is an isomorphism from $C(X)$ onto $C(vX)$ as well as from $C_b(X)$ onto $C(\beta X)$. Since $\delta(X)$ is dense in both vX and βX, we have

$$||f||_u = ||f^v||_u, \ \forall f \in C(X), \text{ and } ||f||_u = ||f^\beta||_u, \ \forall f \in C_b(X).$$

where $|| \ ||_u$ denotes the uniform norm.

Our main result is the following:

Theorem 2.1. *Let X be a Hausdorff completely regular space and A a subalgebra of $C(X)$ which is either a $C_b(X)$-module or closed under the complex conjugation. If τ is a locally m-convex Hausdorff topology on A and β the corresponding boundedness radius. Then*

$$\beta(f) = \|f\|_u, \ \forall f \in A.$$

As a consequence, we get the following Kaplansky-Meyer type theorem :

Theorem 2.2. *Let X be an arbitrary topological space and A a unitary subalgebra of $C(X)$ which is either a $C_b(X)$-module or closed under the complex conjugation. Then there exists an algebra norm on A if and only if $A \subset C_b(X)$. In this case, every algebra norm on A is at least as large as the uniform norm.*

Proof : It is clear that, whenever $A \subset C_b(X)$, there exists an algebra norm on $C(X)$, namely the uniform norm. For the converse, assume that $\| \ \|$ is an algebra norm on A. Then $\varphi(A)$ is a subalgebra of $C(vX)$ satisfying the conditions of Theorem 2.1. Moreover the quantity $\|\varphi(f)\| := \|f\|$ defines an algebra norm on $\varphi(A)$. Hence, since vX is completely regular and Hausdorff, by Theorem 2.1, $\beta(\varphi(f)) = \|\varphi(f)\|_u$, for every $f \in A$. But in any normed algebra, β is less than or equal to the norm. Hence $\beta(\varphi(f)) \leq \|\varphi(f)\|$. Whereby

$$\|f\|_u = \|\varphi(f)\|_u = \beta(\varphi(f)) \leq \|\varphi(f)\| := \|f\|, \ f \in A.$$

Since, for every $f \in A$, $\|f\| < \infty$, it follows that f is bounded on X and then that $A \subset C_b(X)$. The second part of the proof is due to Theorem 2.1 and again to the fact that, in a normed algebra β is less than or equal to the norm. ∎

In order to prove Theorem 2.1, we need some additional results. The following lemma is taken from [7].

Lemma 2.3. *Let X be a Hausdorff completely regular space and A a subalgebra of $C(X)$ which is either a $C_b(X)$-module or closed under the complex conjugation. Then every character on A is an evaluation at some point of βX.*

Lemma 2.4. *Let X be a Hausdorff completely regular space, $A \subset C(X)$ a unitary algebra which is both a $C_b(X)$-module and closed under the complex conjugation. If τ is a locally m-convex Hausdorff topology on A, then every open set $U \subset \beta X$ contains, at least, some x_U the evaluation at which is continuous on A.*

Proof : Under our hypothesis, X is (identified to) a topological subspace of βX. Let $U \subset \beta X$ be an open set and fix $x_0 \in U \cap X$. Since A is a $C_b(X)$-module, we can choose $g \in A$ so that $g(X) \subset [0,1]$, $g(x_0) = 0$ and $g \equiv 1$ identically on the complement U^c of U. Replacing, if necessary, g by $2\max(\frac{1}{2}, g) - 1$, we may assume that g vanishes on an open neighborhood V of x_0. If $\widehat{A}$ denotes the completion of (A, τ), then g cannot be invertible in $\widehat{A}$. Indeed, if g had an inverse f in $\widehat{A}$, then for any non zero $h \in C_b(X)$ vanishing outside of V, we would have

$$h = h(gf) = (hg)f = 0$$

which is a contradiction. Now, since $\widehat{A}$ is a commutative complete locally m-convex algebra with identity, there is some continuous character χ on $\widehat{A}$ such that $\chi(g) = 0$. But the restriction to A of χ is, by Lemma 2.3, the evaluation at some point x_U of βX. From $g(x_U) = 0$ derives $x_U \in U$ and the proof is achieved. ∎

In the following the spectrum $\mathrm{Sp}(x)$ of an element x of a real algebra A is defined as the spectrum of x in the complexification $A_{\mathbb{C}}$ of A, namely:

$$\mathrm{Sp}(x) := \{\lambda \in \mathbb{C} \setminus \{0\} : \frac{x}{\lambda} \text{ is not quasi} - \text{invertible in } A_{\mathbb{C}}\} \cup O$$

O being the empty set or the singleton $\{0\}$ according to whether x is invertible in A or not. The spectral radius of x is then defined as

$$\rho(x) := \sup\{|\lambda|, \lambda \in \mathrm{Sp}(x)\} \text{ with } \sup \emptyset = 0.$$

Lemma 2.5. *Let A be a subalgebra of $C(X)$ which is a $C_b(X)$-module. Then, for every $f \in A$, the spectrum of f is contained in the closure $\overline{f(X)}$ of $f(X)$. In particular*

$$\rho(f) \leq \|f\|_u, \ \forall f \in A,$$

Proof : Assume that λ is a spectral point of f with $\lambda \notin \overline{f(X)}$. Then there is some $\epsilon > 0$ so that

$$|f(x) - \lambda| > \epsilon, \forall x \in X.$$

Since A is a $C_b(X)$-module, the case $\lambda = 0$ cannot occur, for

$$\frac{1}{f} = \frac{1}{f^2} f$$

would belong to A and this contradicts the fact that λ belongs to the spectrum of f. Assume then that $\lambda \neq 0$. Since again A is a $C_b(X)$-module,

$$\frac{f}{f - \lambda} \in A_{\mathbb{C}}.$$

This means that $\frac{f}{\lambda}$ is quasi invertible in $A_{\mathbb{C}}$ which is also a contradiction. Whence the result. ∎

Proof of Theorem 2.1 : Since in a complex locally m-convex algebra $\beta \leq \rho$ (see for example [1] or [9]), we get $\beta \leq || \ ||_u$ by Lemma 2.5. In order to show the equality, it is enough to show that, for every $x_0 \in X$ and every $f \in A$, the inequality $|f(x_0)| \leq \beta(f)$ holds. Fix then $x_0 \in X$ and choose a fundamental system $\mathcal{U}$ of open neighborhoods of x_0 in βX. By Lemma 2.4, for every $U \in \mathcal{U}$, there exists $x_U \in U$ such that the evaluation at x_U is continuous on A. Therefore there exists a continuous submultiplicative seminorm P_U such that

$$|\chi_{x_U}(f)| = |f^\beta(x_U)| \leq P_U(f), \ \ f \in A.$$

Hence, for every $f \in A$ and every $n \in \mathbb{N}$, we have

$$|f^\beta(x_U)| \leq (P_U(f^n))^{\frac{1}{n}}.$$

This leads to

$$|f^\beta(x_U)| \leq \limsup_{n\to\infty}(P_U(f^n))^{\frac{1}{n}} \leq \beta(f).$$

Whereby

$$|f^\beta(x_U)| \leq \beta(f), \ \ f \in A, U \in \mathcal{U}.$$

But the net $(x_U)_{U\in\mathcal{U}}$ converges in βX to x_0. Hence

$$|f(x_0)| \leq \beta(f), \ \ f \in A. \qquad \blacksquare$$

Since a pseudo complete locally A-convex algebra (A, τ) can be equipped with a locally m-convex topology $M(\tau)$ stronger than τ and having the same m-bounded sets as τ (see [8]), we also get:

Corollary 2.6. *If A is as in Theorem 2.1 and τ is a pseudo complete locally A-convex Hausdorff topology on A. Then*

$$\beta(f) = ||f||_u, \ \ \forall f \in A.$$

Remark 2.7. 1. Theorem 2.1 fails to hold if the topology τ is not assumed to be Hausdorff. For such an example, equip $C(X)$ with the topology of uniform convergence on a given compact subset $K \subset X$ with $K \neq X$.

2. Theorem 2.1 fails also to hold if A is not assumed to be a $C_b(X)$-module. Actually, if A consists of all polynomial functions on $[0, 1]$ endowed with the algebra norm

$$||f|| = \sup\{|f(\frac{1}{n})|, n \geq 2\},$$

then, for $f : x \mapsto x$, we have $\beta(f) \leq ||f|| = \frac{1}{2}$, while $||f||_u = 1$. Hence $|| \ ||_u$ does not agree with β.

3. The boundedness radius β is a submultiplicative seminorm in any commutative locally m-convex algebra. However, it need be neither subadditive nor submultiplicative in a general topological algebra. The following proposition gives conditions under which β is submultiplicative or subadditive.

Proposition 2.8. *Let (A, τ) be a topological algebra and $x, y \in A$. If $xy = yx$ and the product of any two idempotent bounded sets is bounded, then*

$$\beta(xy) \leq \beta(x)\beta(y), \text{ here } 0\infty = \infty.$$

If, in addition, the convex hull of an idempotent bounded set is bounded, then

$$\beta(x+y) \leq \beta(x) + \beta(y).$$

Proof : If $\beta(x) = +\infty$ or $\beta(y) = +\infty$, the result is trivial. Assume then that x and y are bounded. The first assertion derives from the fact that, for any positive numbers r and s, we have

$$\left\{ \frac{(xy)^n}{(rs)^n}, n \in \mathbb{N} \right\} \subset \left\{ \frac{x^n}{r^n}, n \in \mathbb{N} \right\} \left\{ \frac{y^n}{s^n}, n \in \mathbb{N} \right\}.$$

The second assertion is a consequence of the following :

$$(x+y)^n = \sum_{p=0}^{n} C_n^p r^p s^{n-p} \frac{x^p}{r^p} \frac{y^{n-p}}{s^{n-p}} \in (r+s)^n B,$$

here B denotes the convex hull of the idempotent bounded set

$$\left\{ \frac{x^n}{r^n}, n \in \mathbb{N} \right\} \left\{ \frac{y^n}{s^n}, n \in \mathbb{N} \right\}. \qquad \blacksquare$$

It is clear that the product of any two idempotent bounded sets is bounded whenever the multiplication of A is sequentially continuous. Actually, this is also the case whenever (A, τ) is a commutative pseudo complete locally convex algebra [1]. By a similar proof as in [1], one can easily show that this remains also true if (A, τ) is pseudo-barrelled. This means that every idempotent bounded set is contained in a barrelling idempotent bounded disc, where a disc B is said to be barrelling if the linear hull A_B of B endowed with the gauge of B is a barrelled space.

Typical algebras satisfying the conditions of Theorem 2.1 are the Nachbin ones. In order to give applications of our results, we recall some notions connected to such algebras. A Nachbin family on a Hausdorff completely regular space X is any collection V of non negative upper semicontinuous functions on X such that:

$$\forall v_1, v_2 \in V,\ x \in X,\ \lambda > 0,\ \exists v \in V :\ v(x) > 0 \text{ and } \lambda v_i \leq v,\ i = 1, 2.$$

With each Nachbin family V on X is associated the so-called weighted locally convex space

$$CV(X) := \{ f \in C(X) : |f|_v := \sup_{x \in X} v(x)|f(x)| < +\infty,\ v \in V \}$$

with its natural topology given by the seminorms $(| \ |_v)_{v \in V}$. In general, this space need not be an algebra, but it always contains many interesting ones. It is shown in [7] that the largest locally convex algebra (with respect to the relative topology induced by $CV(X)$ and the pointwise multiplication) is

$$C_\ell V(X) := \{f \in CV(X) : \forall v \in V,\ \exists u \in V \text{ with } |f(x)|v(x) \leq u(x),\ x \in X\}.$$

Such an algebra and some of its subalgebras are called Nachbin algebras (see [7] for examples). They are $C_b(X)$-modules and closed under complex conjugation so that we can apply Theorem 2.2. We then get:

Proposition 2.9. *If $C_\ell V(X)$ is unitary, the following three conditions are equivalent:*
1. There is an algebra norm on $C_\ell V(X)$.
2. $C_\ell V(X) \subset C_b(X)$.
3. $C_\ell V(X)$ is a uniformly locally A-convex algebra.

Proof : The implication 1. $\Longrightarrow$ 2. is due to Theorem 2.2, while 2. $\Longrightarrow$ 3. is obvious. As to 3. $\Longrightarrow$ 1., it is a consequence of Theorem 4 (1) of [7] and the fact that the uniform norm is an algebra norm on $C_b(X)$.

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