Coefficient characterization for certain classes of univalent functions*

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Abstract

Necessary and sufficient coefficient conditions are established for certain classes of analytic functions to be univalent in the unit disk. In addition, characterizations and convolution results are established for these classes of functions. As a consequence univalency preservation type operators are discussed.

1 Introduction

Let $\mathcal{H}(\mathbb{D})$ denote the space of all functions analytic in the unit disk $\mathbb{D} := \{z : |z| < 1\}$ of the complex plane $\mathbb{C}$. Here we think of $\mathcal{H}(\mathbb{D})$ as a topological vector space endowed with the topology of uniform convergence over compact subsets of $\mathbb{D}$. Further, let $\mathcal{A}$ denote the class of all functions f analytic in $\mathbb{D}$, with the normalization $f(0) = 0$ and $f'(0) = 1$. Denote by $\mathcal{S}$ the family consisting of functions $f \in \mathcal{A}$ that are univalent in $\mathbb{D}$. We observe that mappings in $\mathcal{S}$ can be associated with the mappings in Σ, namely the univalent functions F,

$$F(\zeta) = \zeta + \sum_{n=0}^{\infty} c_n \zeta^{-n}, \quad 1 < |\zeta| \leq \infty,$$

by the correspondence

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n = \frac{1}{F(1/z)}, \quad z \in \mathbb{D}.$$

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If we write $\zeta = 1/z$, then the association $f(z) = 1/F(1/z)$ quickly yields the formula

$$F'(\zeta) = \left(\frac{z}{f(z)}\right)^2 f'(z).$$

An investigation on various subclasses of $\mathcal{S}$ has a long history and continues to occupy a prominent place in function theory. Now, let $\mathcal{U}$ be the subclass of functions $f \in \mathcal{A}$ such that

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right| < 1 \ \text{ for } z \in \mathbb{D}.$$

It is well-known (see [1, 7]) that $\mathcal{U} \subsetneq \mathcal{S}$. Further, for $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ in $\mathcal{U}$, one has

$$\frac{f(z)}{z} \neq 0 \ \text{ and } \ \left(\frac{z}{f(z)}\right)^2 f'(z) = 1 + (a_3 - a_2^2)z^2 + \cdots, \quad z \in \mathbb{D},$$

which may be written as

$$-z\left(\frac{z}{f(z)}\right)' + \frac{z}{f(z)} = \left(\frac{z}{f(z)}\right)^2 f'(z) = 1 + w(z), \quad z \in \mathbb{D}, \tag{1}$$

with $w \in \mathcal{B}_1$, where

$$\mathcal{B}_1 = \{w \in \mathcal{H} : \ w(0) = w'(0) = 0 \ \text{ and } \ |w(z)| < 1, \, z \in \mathbb{D}\}.$$

By (1), we have the representation

$$\frac{z}{f(z)} = 1 - a_2 z - z^2 \int_0^1 \frac{w(tz)}{z^2 t^2}\, dt, \quad z \in \mathbb{D}, \tag{2}$$

where throughout the discussion

$$a_2 = a_2(f) = \frac{f''(0)}{2}.$$

This representation together with many others which follow from this led to a number of recent investigations, see for example [3, 4, 5, 6]. However, because $w \in \mathcal{B}_1$, the Schwarz lemma gives $|w(z)| \leq |z|^2$ in $\mathbb{D}$. Consequently, we have

$$\left|\frac{z}{f(z)} + a_2 z - 1\right| \leq |z|^2, \quad z \in \mathbb{D}. \tag{3}$$

We observe that if z is fixed ($0 < |z| < 1$), then this inequality determines the range of the functional

$$\frac{z}{f(z)} + a_2 z$$

in the class $\mathcal{U}$. In particular, if $a_2 = 0$ then by a computation (3) gives that

$$\left|\frac{f(z)}{z} - \frac{1}{1 - |z|^4}\right| \leq \frac{|z|^2}{1 - |z|^4}, \quad z \in \mathbb{D}$$

so that, for every $f \in \mathcal{U}$ with $f''(0) = 0$, we have

$$\frac{|z|}{1+|z|^2} \leq |f(z)| \leq \frac{|z|}{1-|z|^2}, \quad z \in \mathbb{D}$$

and

$$\operatorname{Re}\left(\frac{f(z)}{z}\right) \geq \frac{1}{1+|z|^2} > \frac{1}{2}, \quad z \in \mathbb{D}. \tag{4}$$

In this paper we are interested in functions $f \in \mathcal{A}$ which have the form

$$\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots \quad \text{with } b_n \geq 0 \text{ for all } n \geq 2 \tag{5}$$

and for all z in a neighborhood of $z = 0$. We remark that, since f is analytic in $\mathbb{D}$, it follows that $\frac{z}{f(z)} \neq 0$ for $z \in \mathbb{D}$, but $\frac{z}{f(z)}$ may have poles in $\mathbb{D}$. Such class of functions has been considered by many authors, see for instance [10].

2 Main results and their proofs

Theorem 6. *Let $f \in \mathcal{A}$ have the form* (5)*. Then we have the following equivalence:*

(a) $f \in \mathcal{S}$

(b) $\dfrac{f(z)f'(z)}{z} \neq 0$ *for* $z \in \mathbb{D}$

(c) $\displaystyle\sum_{n=2}^{\infty}(n-1)b_n \leq 1$

(d) $f \in \mathcal{U}$.

Proof. (a) $\Rightarrow$ (b): Let $f \in \mathcal{S}$ be of the form (5). Then, $f'(z) \neq 0$ and $f(z)/z \neq 0$ in $\mathbb{D}$.

(b) $\Rightarrow$ (c): From the representation of f and (1) we quickly see that for $z \in \mathbb{D}$,

$$\left(\frac{rz}{f(rz)}\right)^2 f'(rz) = 1 - \sum_{n=2}^{\infty}(n-1)b_n r^n z^n$$

from which, as $z/f(z) \neq 0$, it follows that $f'(rz) \neq 0$ is equivalent to

$$1 - \sum_{n=2}^{\infty}(n-1)b_n r^n z^n \neq 0.$$

We claim that $\sum_{n=2}^{\infty}(n-1)b_n \leq 1$. Suppose on the contrary that $\sum_{n=2}^{\infty}(n-1)b_n > 1$. Then, on the one hand, there exists a positive integer m such that

$$\sum_{n=2}^{m}(n-1)b_n > 1$$

and so there exists an r_0 with $0 < r_0 < 1$ and

$$\sum_{n=2}^{m}(n-1)b_n r_0^n > 1.$$

On the other hand, as $b_n \geq 0$ for $n \geq 2$, we have that

$$\left(\frac{r_0}{f(r_0)}\right)^2 f'(r_0) = 1 - \sum_{n=2}^{\infty}(n-1)b_n r_0^n \leq 1 - \sum_{n=2}^{m}(n-1)b_n r_0^n < 0$$

and, since $f'(r)$ is a continuous function of r with $f'(0) = 1$ and $f'(r_0) < 0$, there exists an r_1 $(0 < r_1 < r_0 < 1)$ such that $f'(r_1) = 0$. This is a contradiction. Consequently, we must have

$$\sum_{n=2}^{\infty}(n-1)b_n \leq 1.$$

(c) $\Rightarrow$ (d): Suppose that $\sum_{n=2}^{\infty}(n-1)b_n \leq 1$. Then $\sum_{n=2}^{\infty} b_n \leq 1$ and therefore,

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n \in \mathcal{H}(\mathbb{D})$$

and hence, $f(z)/z \neq 0$ for $z \in \mathbb{D}$. Then, as $b_n \geq 0$ for $n \geq 2$, from (1) (see also [3, 4, 6]) it follows that $f \in \mathcal{U}$.

(d) $\Rightarrow$ (a): $\mathcal{U} \subset \mathcal{S}$ is a well-known fact. ■

Remark 7. For the class $\mathcal{S}$, if $f \in \mathcal{S}$ has the form $\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots$ with $b_n \in \mathbb{C}$, then a necessary condition is that $\sum_{n=2}^{\infty}(n-1)|b_n|^2 \leq 1$. This is a consequence of the Area Theorem [4, (17)]. Observe that, from the condition of Theorem 6, $|b_n| \leq 1/(n-1)$ (≤ 1) for all $n \geq 2$. Thus, a comparison of the last coefficient result with Theorem 6 shows that for the case $b_n \geq 0$ $(n \in \mathbb{N})$ we have $\sum_{n=2}^{\infty}(n-1)b_n \leq 1$, which seems to be a well-suited natural one to apply in special situations. ■

As a motivation, we present specific examples. To do this, we consider the Gauss hypergeometric function $F(a,b;c;z)$, which is defined by (see [9])

$${}_2F_1(a,b;c;z) := F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} z^n, \quad z \in \mathbb{D}.$$

Here a, b, c, in general, are complex numbers such that $c \neq -m$, $m = 0, 1, 2, 3, \ldots$, where $(a)_n$ denotes the Pochhammer symbol

$$(a)_0 = 1, \ (a)_n := a(a+1)\cdots(a+n-1) \ \text{ for } n \in \mathbb{N}.$$

In the exceptional case $c = -m$, $m = 0, 1, 2, 3, \ldots$, $F(a,b;c;z)$ is defined if $a = -j$ or $b = -j$, where $j = 0, 1, 2, \ldots$ and $j \leq m$. Note that $F(a,b;c;z)$ is analytic in $\mathbb{D}$. We have the following well-known result

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}\,(c-a-b) > 0, \ c \notin -\mathbb{N} \cup \{0\}. \tag{8}$$

The following result is then easy to prove.

Lemma 9. *For $a > -1$, $b > -1$ with $ab > 0$ and $c \geq (a+1)(b+1)$, let*

$$T = \sum_{n=2}^{\infty} (n-1)\frac{(a)_n(b)_n}{(c)_n(1)_n}.$$

Then

$$T = 1 - [c-(a+1)(b+1)]\frac{\Gamma(c)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} \leq 1.$$

Proof. First we observe that the hypotheses on a, b, c implies that

$$\begin{cases} c > 0,\ c-a-b-1 \geq ab > 0 \\ c-a > b+1 > 0 \\ c-b > a+1 > 0 \\ c-(a+1)(b+1) \geq 0. \end{cases}$$

Now, we rewrite the sum as follows

$$\begin{aligned} T &= \frac{ab}{c}\sum_{n=2}^{\infty} n\frac{(a+1)_{n-1}(b+1)_{n-1}}{(c+1)_{n-1}(1)_n} - \sum_{n=2}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} \\ &= \frac{ab}{c}[F(a+1,b+1;c+1;1)-1] - \left[F(a,b;c;1)-1-\frac{ab}{c}\right] \end{aligned}$$

where we observe that the first sum is finite if $c > a+b+1$ whereas the second sum is finite if $c > a+b$. By the hypotheses on a, b, c both these conditions hold obviously. In order to determine the sum explicitly, we need to use the formula (8). Consequently, we have

$$\begin{aligned} T &= 1 + \frac{ab}{c}\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} - \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \\ &= 1 - \frac{\Gamma(c)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}[c-(a+1)(b+1)] \leq 1 \end{aligned}$$

and desired result follows. ∎

Now if we let $F(a,b;c;z) = 1 + b_1 z + b_2 z^2 + \cdots$, then by Lemma 9, we see that $b_n \geq 0$ for all $n \geq 2$ and

$$T = \sum_{n=2}^{\infty}(n-1)b_n \leq 1$$

if the hypotheses of Lemma 9 are satisfied. Thus, according to Theorem 6 and Lemma 9, it follows that the function $f(z)$ defined by

$$f(z) = \frac{z}{F(a,b;c;z)}$$

is in $\mathcal{U}$ whenever $F(a,b;c;z)$ is nonvanishing in $\mathbb{D}$ and if the hypotheses of Lemma 9 are satisfied. More precisely, we have

Corollary 10. *Suppose that* $a, b > -1$, $ab > 0$ *and* $c \geq \max\{ab, (a+1)(b+1)\}$. *Then* $\dfrac{z}{F(a,b;c;z)}$ *is in* $\mathcal{U}$.

Proof. To complete the proof, we need to recall a well-known result of Eneström-Kakeya theorem which states that if $1 \geq c_1 \geq c_2 \geq \cdots \geq c_n \geq \cdots \geq 0$ and if $g(z) = 1 + c_1 z + c_2 z^2 + \cdots$, then $g(z) \neq 0$ in the unit disk $\mathbb{D}$. We now apply this result for $g(z) = F(a,b;c;z)$ so that $c_n = A_n^{a,b;c}$, where

$$A_n^{a,b;c} = \frac{(a)_n (b)_n}{(c)_n (1)_n}.$$

Since

$$(c+n)(1+n) - (a+n)(b+n) = c - ab + (c+1-a-b)n,$$

then the hypotheses $a, b > -1$, $ab > 0$ and $c \geq \max\{ab, a+b-1\}$ imply that

$$0 < \frac{(a+n)(b+n)}{(c+n)(1+n)} \leq 1 \ \text{ for } n \geq 1$$

and so

$$0 < c_{n+1} = \frac{(a+n)(b+n)}{(c+n)(1+n)} c_n \leq c_n = \frac{(a)_n (b)_n}{(c)_n (1)_n} \leq 1 \quad \text{for all } n \geq 0.$$

Therefore, according to the Eneström-Kakeya theorem, it follows that $F(a,b;c;z) \neq 0$ for all $z \in \mathbb{D}$. Finally, the result follows from Lemma 9 because $c \geq (a+1)(b+1)$ and $ab > 0$ imply that $c \geq a+b-1$. ∎

This corollary contains that $\dfrac{z}{F(a,b;c;z)}$ is in $\mathcal{U}$ if $a, b > 0$ and $c \geq (a+1)(b+1)$. In particular, we have

1. $\dfrac{z}{F(1,1;c;z)}$ is in $\mathcal{U}$ if $c \geq 4$
2. $\dfrac{z}{F(-1/2,-1/2;c;z)}$ is in $\mathcal{U}$ if $c \geq 1/4$.

We remark that $F(-1/2,-1/2;c;z)$ for $c = 1$ represents the complete elliptic integral of the second kind.

An inspection of the preceding argument also yields the following result.

Corollary 11. *Suppose that* $a \in \mathbb{C}\backslash\{0\}$ *and* $c \geq \max\{|a|^2, |a+1|^2\}$. *Then the function* $\dfrac{z}{F(a,\overline{a};c;z)}$ *belongs to* $\mathcal{U}$.

3 Coefficient Multiplier theorem

For $g(z) = z + \sum_{n=2}^{\infty} g_n z^n$ in $\mathcal{A}$, we have a natural convolution operator defined by

$$zF(a,b;c;z) * g(z) := \sum_{n=1}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} g_n z^n, \quad c \notin -\mathbb{N}, \quad z \in \mathbb{D}. \tag{12}$$

Here $*$ stands for the convolution (Hadamard product) of two analytic functions in $\mathbb{D}$:

$$\psi(z) = \sum_{n=0}^{\infty} \phi_n z^n, \;\; \phi(z) = \sum_{n=0}^{\infty} \phi_n z^n \implies (\psi * \psi)(z) = \sum_{n=0}^{\infty} \psi_n \phi_n z^n, \quad z \in \mathbb{D}.$$

Using the Euler integral representation for the hypergeometric function, one has (see [9])

$$F(a,b;c;z) = \int_0^1 \lambda(t) \frac{1}{(1-tz)^a}\, dt, \quad z \in \mathbb{D}, \qquad \mathrm{Re}\, c > \mathrm{Re}\, b > 0, \tag{13}$$

where $\lambda(t)$ is given by

$$\lambda(t) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} t^{b-1}(1-t)^{c-b-1}. \tag{14}$$

Thus, by (13), we have for $g \in \mathcal{A}$,

$$zF(1,b;c;z) * g(z) = z\left(F(1,b;c;z) * \frac{g(z)}{z}\right), \quad z \in \mathbb{D}, \quad \mathrm{Re}\, c > \mathrm{Re}\, b > 0,$$

which may be rewritten in the following form:

$$zF(1,b;c;z) * g(z) = \int_0^1 g(tz) \frac{\lambda(t)}{t}\, dt, \quad z \in \mathbb{D}, \quad \mathrm{Re}\, c > \mathrm{Re}\, b > 0$$

where $\lambda(t)$ is given by (14). We remark that the integral representation of this form is also known for the general case involving $zF(a,b;c;z)$ in [2]. In particular, we have the classical Bernardi transform of g:

$$zF(1,\gamma;\gamma+1;z) * g(z) = \int_0^1 \gamma t^{\gamma-2} g(tz)\, dt, \quad z \in \mathbb{D}, \quad \mathrm{Re}\, \gamma > 0. \tag{15}$$

Next we consider the following question: *Given a univalent function f, is it possible to generate functions in $\mathcal{S}$?* There exist solutions to this problem (see [4, Theorem 3]). Using Theorem 6, we can provide another method of obtaining functions in $\mathcal{S}$. Indeed, we state and prove our next result which is simple but has interesting consequences.

Theorem 16 (Multiplier theorem). *Let $f \in \mathcal{S}$ have the form* (5)*. Suppose that $g(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ is an analytic function in $\mathbb{D}$ with $0 \le c_n \le 1$ for all $n \ge 2$ and such that $(z/f(z)) * g(z) \ne 0$ on $\mathbb{D}$. Then, H defined by*

$$H(z) = \frac{z}{(z/f(z)) * g(z)}$$

is in the class $\mathcal{U}$.

Proof. By hypotheses, $\frac{z}{H(z)} \neq 0$ for $z \in \mathbb{D}$. Since

$$\sum_{n=2}^{\infty}(n-1)b_n c_n \leq \sum_{n=2}^{\infty}(n-1)b_n \leq 1,$$

by Theorem 6, we conclude that $H \in \mathcal{U}$. ■

Corollary 17. *Let $f \in \mathcal{S}$ have the form* (5). *Suppose that $a, b, c > -2$ with $c \neq 0, -1, -2, \ldots$, and satisfy*

$$0 \leq \frac{ab(a+1)(b+1)}{2c(c+1)} \leq 1, \quad c \geq \max\left\{a+b-1, \frac{2(a+b-1)+ab}{3}\right\} \tag{18}$$

and that $(z/f(z)) * F(a,b;c;z) \neq 0$ for all $z \in \mathbb{D}$. If $H = H_f^{a,b,c}$ is the transform defined by

$$H(z) = \frac{z}{(z/f(z)) * F(a,b;c;z)}, \quad z \in \mathbb{D}, \tag{19}$$

then, H is in the class $\mathcal{U}$.

Proof. In order to apply Theorem 16 we set $g(z) = F(a,b;c;z)$ so that $c_n = A_n^{a,b;c}$. We need to show that

$$0 \leq c_n = \frac{(a)_n(b)_n}{(c)_n(1)_n} \leq 1 \quad \text{for all } n \geq 2.$$

We see that this holds if

$$0 \leq \frac{(a+n)(b+n)}{(c+n)(1+n)} \leq 1 \text{ for } n \geq 2 \text{ and } 0 \leq c_2 = \frac{ab(a+1)(b+1)}{2c(c+1)} \leq 1.$$

The first condition is equivalent to

$$(c+n)(1+n) \geq (a+n)(b+n)$$

and this holds for all $n \geq 2$ if $c - ab + 2(c+1-a-b) \geq 0$. Finally, under the hypotheses, the conclusion follows. ■

An inspection of the argument of Corollary 17 yields the following result

Corollary 20. *Suppose that $a \in \mathbb{C}\backslash\{0\}$, $c \neq 0, -1$, $c > -2$ and satisfy*

$$0 \leq \frac{|a(a+1)|^2}{2c(c+1)} \leq 1, \quad c \geq \max\left\{2\mathrm{Re}\, a - 1, \frac{2(2\mathrm{Re}\, a - 1) + |a|^2}{3}\right\}$$

and that $(z/f(z)) * F(a,\bar{a};c;z) \neq 0$ for all $z \in \mathbb{D}$. If $H = H_f^{a,\bar{a},c}$ is the transform defined by

$$H(z) = \frac{z}{(z/f(z)) * F(a,\bar{a};c;z)}, \quad z \in \mathbb{D},$$

then, H is in the class $\mathcal{U}$.

The case $a = 1$ of Corollary 17 reduces to

Corollary 21. *Let $f \in \mathcal{S}$ have the form* (5), *$b, c > -2$ with $c \neq 0, -1, -2, \ldots$, and satisfy $c \geq b$ and $0 \leq \frac{b(b+1)}{c(c+1)} \leq 1$. Suppose that $(z/f(z)) * F(1, b; c; z) \neq 0$ for all $z \in \mathbb{D}$ and $H = H_f^{b,c}$ is the transform defined by*

$$H(z) = \frac{z}{(z/f(z)) * F(1, b; c; z)}, \quad z \in \mathbb{D}, \tag{22}$$

Then, H is in the class $\mathcal{U}$.

Choosing $b = \gamma$ and $c = b + 1$, we easily have the following: If $f \in \mathcal{S}$ has the form (5) with $f''(0) = 0$, $\gamma > 0$, and $H = H_f^{\gamma}$ is the transform defined by

$$\frac{z}{H(z)} = \gamma \int_0^1 \frac{tz}{f(tz)} t^{\gamma - 1} \, dt, \quad z \in \mathbb{D}, \tag{23}$$

then, H is in the class $\mathcal{U}$.

Remark 24. In [4], it has been observed that $(z/f(z)) * F(1, \gamma; \gamma + 1; z) \neq 0$ for all $z \in \mathbb{D}$, $\operatorname{Re} \gamma \geq 0$, if $f \in \mathcal{U}$ with $f''(0) = 0$. Consequently, this condition has been removed in the above special case. ∎

The conclusion of the following lemma readily follows by using the Herglotz' representation for g.

Lemma 25. *If g is analytic in $\mathbb{D}$, $g(0) = 1$, and $\operatorname{Re} g(z) > 1/2$ in $\mathbb{D}$, then for any function F, analytic in $\mathbb{D}$, then function $g * F$ takes values in the convex hull of the image of $\mathbb{D}$ under F.*

Suppose that $f \in \mathcal{U}$ with $f''(0) = 0$. Then, by (4), we have $\operatorname{Re} (z/f(z)) > 0$ in $\mathbb{D}$. Thus, if g is an analytic function $\mathbb{D}$ with $g(0) = 1$, and $\operatorname{Re} g(z) > 1/2$ in $\mathbb{D}$, then by Lemma 25 we have that

$$\operatorname{Re} \{(z/f(z)) * g(z)\} > 0, \quad z \in \mathbb{D}.$$

There are a large class of functions g satisfying the condition $\operatorname{Re} g(z) > 1/2$ in $\mathbb{D}$. For instance, the class of functions g for which $g(z) = G(z)/z$ where $G \in \mathcal{C}$, the normalized class of convex functions in $\mathbb{D}$. Another example is given by the second author in [8] where conditions on a, b, c are established so that $\operatorname{Re} F(a, b; c; z) > 1/2$ in $\mathbb{D}$.

There are a number of other transformations which fit into the language of Corollary 17. For instance,

(i) for $a, b \in \mathbb{C} \backslash \{-2, -3, \ldots\}$, define

$$G(a, b; z) = \sum_{n=1}^{\infty} \frac{(1 + a)(1 + b)}{(n + a)(n + b)} z^{n-1}.$$

If $\operatorname{Re} a > -1$ and $\operatorname{Re} b > -1$, then we have the integral form

$$G(a, b; z) = \int_0^1 \frac{\varphi(t)}{1 - tz} \, dt,$$

where
$$\varphi(t) = \begin{cases} (a+1)(b+1)\frac{t^a - t^b}{b-a} & \text{for } b \neq a \\ (a+1)^2 t^a \log(1/t) & \text{for } b = a. \end{cases} \tag{26}$$

(ii) For $p \geq 0$ and $a > -1$, consider
$$\Phi_p(a;z) = \sum_{n=1}^{\infty} \frac{(1+a)^p}{(n+a)^p} z^{n-1} = \int_0^1 \frac{\varphi(t)}{1-tz}\, dt$$
where
$$\varphi(t) = \frac{(1+a)^p}{\Gamma(p)} (\log 1/t)^{p-1} t^a. \tag{27}$$

In these two cases, an analog of Corollary 17 takes the following forms which we state without proof.

Corollary 28. *Let $f \in \mathcal{S}$ have the form* (5)*. Suppose that a, b satisfy either*
$$(a+1)(b+1) > 0 \ \text{ with } \ a+b+4 \geq 0,$$
or
$$a \in \mathbb{C} \setminus \{-2, -3, \ldots\}, \ \mathrm{Re}\, a \geq -2, \ \text{ and } \ b = \overline{a}.$$
Let $H = H_f^{a,b}$ be the transform defined by
$$H(z) = \frac{z}{(z/f(z)) * G(a,b;z)}, \quad z \in \mathbb{D},$$
or equivalently, if moreover $\mathrm{Re}\, a \geq -1$, $\mathrm{Re}\, b \geq -1$, *by*
$$\frac{z}{H(z)} = \int_0^1 \varphi(t) \frac{tz}{f(tz)}\, dt,$$
where $\varphi(t)$ is given by (26)*. Then, H is in the class $\mathcal{U}$ whenever $(z/f(z)) * G(a,b;z) \neq 0$ in $\mathbb{D}$.*

Corollary 29. *Let $f \in \mathcal{S}$ have the form* (5)*. Suppose that $a > -1$, $p \geq 1$ and $H = H_f^{a,p}$ be the transform defined by*
$$H(z) = \frac{z}{(z/f(z)) * \Phi_p(a;z)}, \quad z \in \mathbb{D}$$
or equivalently,
$$\frac{z}{H(z)} = \int_0^1 \varphi(t) \frac{tz}{f(tz)}\, dt,$$
where $\varphi(t)$ is given by (27)*. Then, H is in the class $\mathcal{U}$ whenever $(z/f(z)) * \Phi_p(a;z) \neq 0$ in $\mathbb{D}$.*

Remark 30. Corollary 17 continues to hold if $p \geq 1$ is an even integer and $-2 < a < -1$, except for the integral representation. ■

Now, we consider some sort of a converse of Corollary 17.

Theorem 31. *Suppose that $a,b,c > -2$ satisfy* (18). *If $H \in \mathcal{S}$ and has the representation*

$$\frac{z}{H(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (c_n \geq 0 \text{ for all } n \in \mathbb{N}),$$

then the analytic function f, with $f(0) = 0 = f'(0) - 1$ and $\frac{z}{f(z)} \neq 0$ in $|z| < r^\star$, defined by the relation (19) *is univalent in $|z| < r^\star$, where*

$$r^\star = r^\star_{a,b,c} := \inf\left\{ \left(A_n^{a,b;c}\right)^{\frac{1}{n}} \,:\, n = 2,3,\ldots \right\} \quad \left(A_n^{a,b;c} = \frac{(a)_n(b)_n}{(c)_n(1)_n}\right).$$

Proof. From the representation (19), we have that

$$\frac{z}{f(z)} * F(a,b;c;z) = \frac{z}{H(z)}$$

and the series that represents $z/H(z)$ converges for $|z| < 1$, since $H \in \mathcal{S}$. Moreover, using for example the ratio test, we see that the series

$$1 + \sum_{n=1}^{\infty} \frac{1}{A_n^{a,b;c}} z^n$$

converges for $|z| < 1$. Thus,

$$\frac{rz}{f(rz)} = 1 + \sum_{n=1}^{\infty} \frac{c_n}{A_n^{a,b;c}} r^n z^n$$

converges for $|z| < 1/r$. Now, we set $g(z) = r^{-1} f(rz)$. By Theorem 6, g is univalent in $\mathbb{D}$ if $g \in \mathcal{A}$, $rz/f(rz) \neq 0$ for all $z \in \mathbb{D}$ and if

$$\sum_{n=2}^{\infty} (n-1) \frac{c_n}{A_n^{a,b;c}} r^n \leq 1. \tag{32}$$

On the other hand, since $H \in \mathcal{S}$, Theorem 6 shows that $\sum_{n=2}^{\infty}(n-1)c_n \leq 1$ and so the condition (32) will be satisfied if $r \leq \left(A_n^{a,b;c}\right)^{\frac{1}{n}}$, for each $n = 2,3,\ldots$. Consequently, $f(z)$ is univalent in $|z| < r^\star_{a,b,c}$. ∎

Remark 33. Analogous results in the above formulation may also be stated for Corollaries 28 and 29. ∎

Corollary 34. *Let γ be real number with $\gamma > 0$. If $H \in \mathcal{S}$ and has the representation as in Theorem* 31, *then the analytic function f, with $f(0) = 0 = f'(0) - 1$ and $\frac{z}{f(z)} \neq 0$ in $|z| < r^\star$, defined by the relation* (23) *is univalent in $|z| < r^\star$, where*

$$r^\star = \inf_{n \geq 2} \left(\frac{\gamma}{\gamma + n}\right)^{\frac{1}{n}}.$$

4 Characterization of certain class of univalent functions

Denote by $\mathcal{S}_+$ the subclass of functions $f \in \mathcal{S}$ which have the form (5). Now, we state a necessary and sufficient condition for functions to be in $\mathcal{S}_+$.

Theorem 35. *Let $f \in \mathcal{A}$. We have $f \in \mathcal{S}_+$ if and only if f has the form*

$$\frac{z}{f(z)} = b_1 z + \sum_{n=1}^{\infty} \lambda_n \frac{z}{f_n(z)}, \tag{36}$$

for some sequence $\{\lambda_n\}_{n=1}^{\infty}$ of nonnegative real numbers with $\sum_{n=1}^{\infty} \lambda_n = 1$, and

$$\frac{z}{f_n(z)} = \begin{cases} 1 & \text{if } n = 1 \\ 1 + \dfrac{1}{n-1} z^n & \text{if } n = 2, 3, \ldots. \end{cases}$$

Proof. Suppose that a function $f \in \mathcal{A}$ has the form (36) for some sequence $\{\lambda_n\}$ and $\{f_n\}$ with the stated condition. We need to show that such an f belongs to $\mathcal{S}_+$. To do this, we rewrite the expression on the right of (36) conveniently as

$$\begin{aligned} \frac{z}{f(z)} &= b_1 z + \lambda_1 + \sum_{n=2}^{\infty} \lambda_n \frac{z}{f_n(z)} \\ &= 1 + b_1 z + \sum_{n=2}^{\infty} \frac{\lambda_n}{n-1} z^n, \quad \text{by the definition of } f_n. \end{aligned}$$

Because

$$\sum_{n=2}^{\infty} (n-1) \frac{\lambda_n}{n-1} = \sum_{n=2}^{\infty} \lambda_n = 1 - \lambda_1 \leq 1,$$

the condition for univalence in Theorem 6 is satisfied and so, $f \in \mathcal{S}_+$.

Conversely, let $f \in \mathcal{S}_+$. Then $z/f(z)$ is a nonvanishing analytic function in $\mathbb{D}$ and therefore, we may set

$$\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots, \quad z \in \mathbb{D}, \text{ with } b_n \geq 0, n \geq 2.$$

But then, by Theorem 6, we have

$$\sum_{n=2}^{\infty} (n-1) b_n \leq 1$$

and so, $b_n \leq 1/(n-1)$ for all $n = 2, 3, \ldots$. Consequently, we can set

$$\lambda_n = (n-1) b_n \text{ for } n = 2, 3, \ldots \textit{ and } \lambda_1 = 1 - \textstyle\sum_{n=2}^{\infty} \lambda_n.$$

Thus, $0 \leq \lambda_n \leq 1$ for each $n \in \mathbb{N}$ with $\sum_{n=1}^{\infty} \lambda_n = 1$, and $z/f(z)$ has the form

$$\begin{aligned} \frac{z}{f(z)} &= 1 + b_1 z + \sum_{n=2}^{\infty} \frac{\lambda_n}{n-1} z^n \quad \text{(by the definition of } \lambda_n'\text{s)} \\ &= b_1 z + \lambda_1 + \sum_{n=2}^{\infty} \lambda_n \frac{z}{f_n(z)} \quad \text{(by the definition of } f_n\text{'s)} \\ &= b_1 z + \sum_{n=1}^{\infty} \lambda_n \frac{z}{f_n(z)} \end{aligned}$$

and the result follows. ∎

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