On the orthogonal polynomials with weight having singularities on the boundary of regions in the complex plane

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Abstract

The order of the weight of orthogonal polynomials is analyzed, when this weight function shows singularities on the boundary of a region in the complex plane.

1 Introduction

Let $G \subset \mathbb{C}$ be a finite region, with $0 \in G$, bounded by a Jordan curve $L := \partial G$, let σ be the two-dimensional Lebesgue measure, and let $h(z) \in L^1(G, d\sigma)$ be a weight function defined in G.

A system of polynomials $\{K_n(z)\}_{n=0}^{\infty}$, $\deg K_n = n$, satisfying the condition

$$\iint_G h(z)\, K_n(z)\, \overline{K_m(z)} d\sigma_z = \delta_{n,m},$$

where $\delta_{n,m}$ is the Kronecker delta, is called a system of orthonormal polynomials for the pair (G, h). It is determined uniquely if the coefficient of the highest degree term is positive.

Let $\{z_j\}_{j=1}^{m}$ be a fixed system of points on L and the weight function $h(z)$ defined as the follows:

$$h(z) = h_0(z) \prod_{j=1}^{m} \left|z - z_j\right|^{\gamma_j}, \tag{1.1}$$

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where $\gamma_j > -2$ for $j = \overline{1,m}$ and $h_0(z)$ is uniformly separated from zero in G:

$$h_0(z) \geq c > 0, \ \forall z \in G.$$

In this paper we continue the study of the estimation problem of the maximum norm

$$\|K_n\|_{C(\overline{G})} := \max\left\{|K_n(z)|, \ z \in \overline{G}\right\}$$

of orthogonal polynomials over a region with respect to a weight. The polynomials are defined by the pair (G, h). Therefore, the variation of the norm of these polynomials depends on the properties of the region G and of the weight $h(z)$. Similar problems have been studied in [1],[2],[3], in case of orthogonality along a curve and in [4]-[10],[11], in case of orthogonality over a region. In addition, we also generalize this problem for arbitrary algebraic polynomials $P_n(z)$ of degree at most n.

2 Main results

Throughout this paper $c, c_1, c_2, ...$ are positive, and $\varepsilon, \varepsilon_1, \varepsilon_2, ...$ sufficiently small positive constants (mostly different in different relations), which, in general, depend on G.

For $\delta > 0$ and $z \in \mathbb{C}$ let us put : $B(z,\delta) := \{\zeta : |\zeta - z| < \delta\}$, $B := B(0,1)$, $\Delta(z,\delta) := ext\ \overline{B(z,\delta)}$ (with respect to $\overline{\mathbb{C}}$), $\Delta := extB$, $\Omega := extG$, $\Omega(z,\delta) := \Omega \cap B(z,\delta)$; $w = \varphi(z)$ $(w = \Phi(z))$ the univalent conformal mapping of G (Ω) onto the $B(\Delta)$ normalized by $\varphi(0) = 0, \varphi'(0) > 0$ $(\Phi(\infty) = \infty, \Phi'(\infty) > 0)$, $\psi := \varphi^{-1}$ $(\Psi := \Phi^{-1})$.

Definition 2.1. *A bounded Jordan region G is called a k-quasidisk, $0 \leq k < 1$, if the conformal mapping ψ can be extended to a K-quasiconformal, $K = \frac{1+k}{1-k}$, homeomorphism of the plane $\overline{\mathbb{C}}$ onto $\overline{\mathbb{C}}$. In that case the curve $L := \partial G$ is called a k-quasicircle. The region G (resp. curve L) is called a quasidisk (resp. quasicircle), if it is a k-quasidisk (k-quasicircle) for some $0 \leq k < 1$.*

Theorem A[9]. *Let G be a k-quasidisk for some $0 \leq k < 1$, and let the weight function $h(z)$ be defined by (1.1) with $\gamma_j = 0, \ j = \overline{1,m}$. Then, for every $n = 1, 2, ...$*

$$\|K_n\|_{C(\overline{G})} \leq c_1 n^{1+k}.$$

Definition 2.2. *We say that $G \in Q_\alpha$, $0 < \alpha \leq 1$, if*

a) L is a quasicircle,

b) Φ satisfies the Lipschitz condition of order α on $\overline{\Omega}$: $\Phi \in Lip_\alpha(\overline{\Omega})$.

Theorem B[9]. *Let $G \in Q_\alpha$, for some $0 < \alpha \leq 1$ and let the weight function $h(z)$ be defined by (1.1) with $\gamma_j = 0, \ j = \overline{1,m}$,*

1) if $\alpha \geq \frac{1}{2}$, then for every $n = 1, 2, ...$

$$\|K_n\|_{C(\overline{G})} \leq c_2 n^{\frac{1}{\alpha}},$$

2) if $\alpha < \frac{1}{2}$, *then there exist a number* $\delta = \delta(\alpha, G)$, $\delta \in [1,2]$, *such that for every* $n = 1, 2, ...$

$$\|K_n\|_{C(\overline{G})} \leq c_2' n^{\delta}. \tag{2.1}$$

Now we assume that the weight function $h(z)$ defined as (1.1) where $\gamma_j \neq 0$, for some $j \geq 1$. Note that throughout this paper we use the same sequence of singular points $\{z_j\}_{j=1}^{m}$ defined by (1.1).

We now state two theorems the proof of which is given in the next section.

Theorem 2.1. *Let G be a k -quasidisk for some* $0 \leq k < 1$, *and let the weight function* $h(z)$ *be defined by (1.1). Then, for each point* z_j, $j = \overline{1, m}$, *and for every* $n = 1, 2, ...$

$$|K_n(z_j)| \leq c_3 n^{(1+\frac{\gamma_j}{2})(1+k)}.$$

Corollary 2.2. *Under the same conditions as in Theorem 2. 1, one has*

$$\|K_n\|_{C(\overline{G})} \leq c_4 n^{(1+\frac{\gamma}{2})(1+k)},$$

$$\gamma := \max\left\{0;\ \gamma_j,\ j = \overline{1, m}\right\},\ n = 1, 2, ...$$

Theorem 2.3. *Let* $G \in Q_\alpha$, *for some* $0 < \alpha \leq 1$, *and let the weight function* $h(z)$ *be defined by (1.1). Then, for each point* z_j, $j = \overline{1, m}$, *and for every* $n = 1, 2, ...$

$$|K_n(z_j)| \leq c_5 n^{(1+\frac{\gamma_j}{2})\mu},$$

where

$$\mu = \begin{cases} \frac{1}{\alpha}, & if\ \alpha \geq \frac{1}{2}, \\ \delta, & if\ \alpha < \frac{1}{2} \end{cases}$$

and δ *is defined as in (2.1).*

Corollary 2.4. *Under the same conditions as in Theorem 2. 3, one has*

$$\|K_n\|_{C(\overline{G})} \leq c_4 n^{(1+\frac{\gamma}{2})\mu},$$

$$\gamma := \max\left\{0;\ \gamma_j,\ j = \overline{1, m}\right\},\ n = 1, 2, ...$$

In our previous work [7, Prop. 1-3], we discussed the sharpness of results similar to those contained in Theorems 2.1, 2.3. Therefore, using a similar reasoning we can also determine the sharpness in the Theorems 2.1, 2.3.

Definition 2.3. *Let* $z \in L$ *and* $\nu \in (0, 1)$ *be fixed. We say that* $\Omega \in Q(z; \nu)$, *if L is a quasicircle and there exists* $r > 0$ *such that a closed circular sector* $S(z; r, \nu) := \left\{\zeta : \zeta = z + re^{i\theta}, 0 \leq \theta_0 < \theta < \theta_0 + \nu\right\}$ *of radius r and opening* $\nu\pi$ *lies in* $\overline{G}$ *with vertex at z.*

Definition 2.4. *Let* $\nu_1, ..., \nu_m$ *and* α, *with* $0 < \nu_1, ..., \nu_m < \alpha \leq 1$, *be fixed. We say that* $\Omega \in Q_\alpha(\zeta_1, \zeta_2, ..., \zeta_m; \nu_1, ..., \nu_m)$, *if for every j,* $\Omega \in Q\left(\zeta_j; \nu_j\right)$ *and* $\Phi \in Lip_\alpha(\overline{\Omega} \setminus \left\{\zeta_j\right\})$.

Let $\Omega \in Q_\alpha(\zeta_1, \zeta_2, ..., \zeta_m; \nu_1, ..., \nu_m)$, $0 < \nu_1, ..., \nu_m < \alpha \leq 1$. Assume that the system of points $\{z_j\}$, $j = \overline{1,m}$ and $\left\{\zeta_j\right\}$, $j = \overline{1,m}$ mentioned in (1.1) and in Definition 2.4 respectively, are identically ordered on L, i. e. $z_j \equiv \zeta_j$, $j = \overline{1,m}$. In [9], we showed that if the interference condition

$$1 + \frac{\gamma_j}{2} = \frac{1}{\alpha(2 - \nu_j)}$$

is satisfied for each singular point $\{z_j\}$, $j = \overline{1,m}$, of the weight function and the boundary contour, then the growth rate of the polynomials $K_n(z)$ in $\overline{G}$ does not depend on whether or not the weight function $h(z)$ and the boundary contour L show singularities. In [10], one of the authors investigated this problem in the case where

$$1 + \frac{\gamma_j}{2} < \frac{1}{\alpha(2 - \nu_j)}. \tag{2.2}$$

In the present paper we also investigate the case when the opposite of (2.2) holds.

Theorem 2.5. *Let* $\Omega \in Q_\alpha(z_1, z_2, ..., z_m; \nu_1, ..., \nu_m)$, *for some* $0 < \nu_j < 1$ *and* $\alpha(2 - \nu_j) \geq 1$, $j = \overline{1,m}$, *and let* $h(z)$ *be defined by (1.1). If*

$$1 + \frac{\gamma_j}{2} > \frac{1}{\alpha(2 - \nu_j)} \tag{2.3}$$

holds for each point z_j, $j = \overline{1,m}$, *then for each point* z_j, $j = \overline{1,m}$, *and for every* $n = 1, 2, ...$

$$\max_{z \in \overline{G}} \left(\prod_{j=1}^{m} \left|z - z_j\right|^{\widetilde{\mu}_j} |K_n(z)| \right) \leq c_4 n^{1/\alpha},$$

$$\left|K_n(z_j)\right| \leq c_5 n^{\widetilde{s}_j},$$

where

$$\widetilde{\mu}_j := 1 + \frac{\gamma_j}{2} - \frac{1}{\alpha(2 - \nu_j)},$$

$$\widetilde{s}_j := \left(1 + \frac{\gamma_j}{2}\right)(2 - \nu_j),\ j = \overline{1,m}.$$

The conditions (2.3) might be satisfied when $\gamma_j > 0$, $j = \overline{1,m}$. For that reason we will call (2.3) the algebraic zero conditions of order $\mu_j = \alpha(2 - \nu_j)\left(1 + \frac{\gamma_j}{2}\right) - 1$.

3 Some auxiliary results

For $a > 0$ and $b > 0$ we shall use the notations "$a \prec b$" (order inequality) if $a \leq cb$, and "$a \asymp b$" if $c_1 a \leq b \leq c_2 a$ for some constants c, c_1, c_2 respectively.

Let G be a quasidisk. Then there exists a quasiconformal reflection $y(.)$ across L such that $y(G) = \Omega, y(\Omega) = G$ and y(.) fixes the points of L. The quasiconformal reflection $y(.)$ is such that it satisfies the following condition [12, p. 26], [13]:

$$\begin{aligned} |y(\zeta) - z| &\asymp |\zeta - z|\,,\ z \in L,\ \varepsilon < |\zeta| < \frac{1}{\varepsilon}, \\ \left|y_{\overline{\zeta}}\right| &\asymp |y_\zeta| \asymp 1,\ \varepsilon < |\zeta| < \frac{1}{\varepsilon}, \\ \left|y_{\overline{\zeta}}\right| &\asymp |y(\zeta)|^2,\ \ |\zeta| < \varepsilon,\ \left|y_{\overline{\zeta}}\right| \asymp |\zeta|^{-2},\ |\zeta| > \frac{1}{\varepsilon}. \end{aligned} \tag{3.1}$$

For $t > 0$, let $L_t := \{z : |\varphi(z)| = t, if\ t < 1,\ |\Phi(z)| = t, if\ t > 1\}, G_t := intL_t$, $\Omega_t := extL_t$. For $R > 1$ let $L^* := y(L_R)$, $G^* := intL^*, \Omega^* := extL^*$; $w = \Phi_R(z)$ be the conformal mapping of Ω^* onto the Δ normalized by $\Phi_R(\infty) = \infty, \Phi_R'(\infty) > 0$;$\Psi_R := \Phi_R^{-1}$. For $t > 1$, let $L_t^* := \{z : |\Phi_R(z)| = t\}$, $G_t^* := intL_t^*$, $\Omega_t^* := extL_t^*$; $d(z, L) := dist(z, L)$.

According to [14], for all $z \in L^*$ and $t \in L$ such that $|z - t| = d(z, L)$ we have

$$d(z, L) \asymp d(t, L_R) \asymp d(z, L_R). \tag{3.2}$$

Lemma 3.1. *[4]. Let G be a quasidisk,* $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z : |z - z_1| \prec d(z_1, L_{r_0})\}$; $w_j = \Phi(z_j)$, $j = 1, 2, 3$. *Then*

a) The statements $|z_1 - z_2| \prec |z_1 - z_3|$ *and* $|w_1 - w_2| \prec |w_1 - w_3|$ *are equivalent. So are* $|z_1 - z_2| \asymp |z_1 - z_3|$ *and* $|w_1 - w_2| \asymp |w_1 - w_3|$.

b) If $|z_1 - z_2| \prec |z_1 - z_3|$, *then*

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{\varepsilon} \prec \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \prec \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{c},$$

where $0 < r_0 < 1$ *a constant, depending on G and k.*

Lemma 3.2. *Let G be a k -quasidisk for some* $0 \leq k < 1$. *Then*

$$|\Psi(w_1) - \Psi(w_2)| \succ |w_1 - w_2|^{1+k},$$

for all $w_1, w_2 \in \overline{\Omega}'$.

This fact follows from of an appropriate result for the mapping $f \in \sum(k)$ [15, p. 287] and the estimate for the functions Ψ [12, Th. 2. 8].

Let $A_p(h, G), p > 0$ denote the class of functions f which are analytic in G and satisfy the condition

$$\|f\|_{A_p} := \|f\|_{A_p(h,G)} := \left(\iint_G h(z)\,|f(z)|^p\,d\sigma_z\right)^{1/p} < \infty.$$

Lemma 3.3. *[8]. Let G be a quasidisk and let $P_n(z)$, $\deg P_n \leq n, n = 1,2,...$, be an arbitrary polynomial and let the weight function $h(z)$ satisfy the condition (1.1). Then, for any $R > 1$, $p > 0$ and $n = 1,2,...$ one has*

$$\|P_n\|_{A_p(h,G_{1+c(R-1)})} \leq c_6 R^{n+\frac{1}{p}} \|P_n\|_{A_p(h,G)}, \tag{3.3}$$

where c, c_1 are independent of n and R.

Lemma 3.4. *Let G be a quasidisk; $z_1 \in L$, and let $z \in L^* := L^*\left(1+\frac{1}{n}\right)$, $n = 1,2,...$, such that $d(z_1, L^*) = |z_1 - z|$. Then, the relation*

$$\{\zeta : |\zeta - z| < c_1 |z_1 - z|\} \subset G$$

holds for some constant $c_1 = c_1(G, D, K)$, $0 < c_1 < 1$.

Proof. Let $d(z, L) = |z_2 - z| \leq |z - z_1|$, $z_2 \in L$. Let $\Gamma = \Gamma(z, z_2; z^*, z_1, G_{R_0})$ be a family of locally rectifiable curves and separating z and z_2 from z_1 and $z^* \in L_{R_0}$ in G_{R_0}, where $R_0 = R_0(G, \varphi, y) > 1$ is a fixed constant. Using the quasiconformal reflection $y(.)$ we can extend the function φ to a quasiconformal homeomorphism $\widetilde{\varphi} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, $\widetilde{\varphi}(0) = 0$, $\widetilde{\varphi}(\infty) = \infty$. Let $\Gamma' = \tilde{\varphi}(\Gamma)$. Then, it is easily shown that the module $m(\Gamma)$ and $m(\Gamma')$ satisfy

$$m(\Gamma) \geq \frac{1}{2\pi} \ln c_2 \left| \frac{z_1 - z}{z_2 - z} \right|, \tag{3.4}$$

$$m(\Gamma') \leq \frac{1}{2\pi} \ln c_3 \left| \frac{\varphi(z_1) - \varphi(z)}{\varphi(z_2) - \varphi(z)} \right|, \tag{3.5}$$

where $c_j = c_j(G, R_0)$, $j = 2,3$, are independent of z, z_1, z_2. As $|z_2 - z| \leq |z_1 - z|$, according to Lemma 3.1, we get $|\varphi(z_2) - \varphi(z)| \prec |\varphi(z_1) - \varphi(z)|$. On the other hand, let $d(\varphi(z_1), \varphi(L^*)) = |\varphi(z_1) - t_1|$, $t_1 \in \varphi(L^*)$ and $d(\varphi(z), \partial B) = |\varphi(z) - t_2|$, $t_2 \in \partial B$. We then obtain $|\varphi(z_1) - t_1| \asymp |\varphi(z_1) - \varphi(z)| \asymp |\varphi(z) - t_2| \asymp |\varphi(z_2) - \varphi(z)|$, and consequently, $|\varphi(z_1) - \varphi(z)| \leq c_4 |\varphi(z_2) - \varphi(z)|$. Hence

$$m(\Gamma') \leq \frac{1}{2\pi} \ln c_3.c_4 = c_5.$$

So, considering the modules to be quasi-invariant [13, p. 14], it follows from (3.4) and (3.5) that

$$c_5 \geq m(\Gamma') \geq C^{-2}(K) m(\Gamma) \geq \frac{1}{C^2(K) 2\pi} \ln c_2 \left| \frac{z_1 - z}{z_2 - z} \right|,$$

where $C(K)$ is the quasiconformality coefficient of the reflection $y(.)$. Then,

$$|z_2 - z| \geq c_2 e^{-2\pi C^2(K) c_5} |z_1 - z|.$$

Taking c_1 as

$$c_1 := \frac{1}{2} c_2 e^{-2\pi C^2(K) c_5}$$

completes the proof. ■

4 Proof of Theorems 2.1 and 2.3

Proof. We first give the proof of Theorem 2. 1. Without loss of generality we may take $j=1$. As L is a quasicircle, we also have that each L_R, $R=1+\frac{1}{n}$, is also a quasicircle. Therefore, we can construct a $c(K)$-quasiconformal reflection $y_R(z)$, $y_R(0)=\infty$, across L_R such that $y_R(G_R)=\Omega_R$, $y_R(\Omega_R)=G_R$ and $y_R(.)$ fixes the points of L that satisfy conditions (3.1) rewritten for $y_R(z)$. By using $y_R(z)$ constructed in this way, we can write the following integral representation for $K_n(z)$ [12, p. 105]

$$K_n(z)=-\frac{1}{\pi}\iint_{G_R}\frac{K_n(\zeta)\,y_{R,\zeta}}{(y_R(\zeta)-z)^2}d\sigma_\zeta\,,\quad z\in G_R. \tag{4.1}$$

We put $U_\varepsilon(z):=\{\zeta:|\zeta-z|<\varepsilon\}$, $\varepsilon>0$; without loss of generality we may take $U_\varepsilon:=U_\varepsilon(0)\subset G^*$. For $z_1\in L$ we have

$$|K_n(z_1)|\le\frac{1}{\pi}\iint_{U_\varepsilon}\frac{|K_n(\zeta)|\left|y_{R,\bar\zeta}\right|}{|y_R(\zeta)-z_1|^2}d\sigma_\zeta+\frac{1}{\pi}\iint_{G_R}\frac{|K_n(\zeta)|\left|y_{R,\bar\zeta}\right|}{|y_R(\zeta)-z_1|^2}d\sigma_\zeta=:J_1+J_2. \tag{4.2}$$

To estimate the integral J_1, we multiply the numerator and denominator of the integrand by $\sqrt{h(\zeta)}$, and applying the Holder inequality we obtain

$$\begin{aligned}J_1^2 &\le \iint_{U_\varepsilon}h(\zeta)\,|K_n(\zeta)|^2\,d\sigma_\zeta\cdot\iint_{U_\varepsilon}\frac{\left|y_{R,\bar\zeta}\right|^2}{h(\zeta)\,|y_R(\zeta)-z_1|^4}d\sigma_\zeta\\ &\prec \iint_{U_\varepsilon}\frac{\left|y_{R,\bar\zeta}\right|^2}{|\zeta-z_1|^{\gamma_1}\,|y_R(\zeta)-z_1|^4}d\sigma_\zeta\asymp\iint_{U_\varepsilon}\frac{\left|y_{R,\bar\zeta}\right|^2}{|y_R(\zeta)-z_1|^4}d\sigma_\zeta.\end{aligned}$$

According to (3.1) $\left|y_{R,\bar\zeta}\right|\asymp|y_R(\zeta)|^2$, for all $\zeta\in U_\varepsilon$, because of $|\zeta-z_1|\ge\varepsilon$, $|y_R(\zeta)-z|\asymp|y_R(\zeta)|$ for $z\in L$ and $\zeta\in U_\varepsilon$. On the other hand, if $J_{y,R}:=\left|y_{R,\zeta}\right|^2-\left|y_{R,\bar\zeta}\right|^2$ is the Jacobian of the reflection $y_R(\zeta)$, we obtain

$$\left|J_{y,R}\right|\succ\left|y_{R,\bar\zeta}\right|^2$$

as in [9]. Then, we can find

$$J_1^2\prec\iint_{U_\varepsilon}\frac{\left|y_{R,\bar\zeta}\right|^2}{\left|J_{y,R}\right|\,|\zeta-z_1|^4}d\sigma_\zeta\prec\iint_{|\zeta-z_1|\ge c_1}\frac{d\sigma_\zeta}{|\zeta-z_1|^4}\prec 1. \tag{4.3}$$

For the integral J_2 we have

$$J_2^2=\iint_{G_R\setminus U_\varepsilon}\frac{\left|y_{R,\bar\zeta}\right|^2d\sigma_\zeta}{|y_R(\zeta)-z_1|^{4+\gamma_1}}\cdot\iint_{G_R\setminus U_\varepsilon}|y_R(\zeta)-z_1|^{\gamma_1}\,|K_n(\zeta)|^2\,d\sigma_\zeta=:J_{21}\cdot J_{22}. \tag{4.4}$$

First we establish that

$$|\zeta - z_1| \prec |y_R(\zeta) - z_1| \prec |\zeta - z_1| + d(z_1, L_R) \tag{4.5}$$

for all $\zeta \in G_R \backslash U_\varepsilon$ and $z_1 \in L$.

Let $|z_1 - t| = d(z_1, L_R)$, $t \in L_R$. According to (3.1) we have $c_1 \leq \left| y_{R,\bar{\zeta}} \right| \leq c_2$ and $c_3 |\zeta - z| \leq |y_R(\zeta) - z| \leq c_4 |\zeta - z|$, for all $\zeta \in G_R \backslash U_\varepsilon$ and $z \in L_R$. Then

$$\begin{aligned} |\zeta - z_1| &\leq |\zeta - t| + |y_R(\zeta) - t| + |y_R(\zeta) - z_1| \\ &\leq \left(c_3^{-1} + 1\right) |y_R(\zeta) - t| + |y_R(\zeta) - z_1| \\ &\prec |y_R(\zeta) - z_1|. \end{aligned}$$

On the other hand

$$\begin{aligned} |y_R(\zeta) - z_1| &\leq |y_R(\zeta) - t| + |t - \zeta| + |\zeta - z_1| \\ &\leq (c_4 + 1)|t - \zeta| + |\zeta - z_1| \prec |t - \zeta| + |\zeta - z_1|. \end{aligned}$$

Using (4.5), we obtain for the integral J_{21}

$$\begin{aligned} J_{21} &\prec \iint\limits_{y(G_R \backslash U_\varepsilon)} \frac{d\sigma_\zeta}{|\zeta - z_1|^{4+\gamma_1}} \\ &\leq \iint\limits_{|\zeta - z_1| \geq d(z_1, L_R)} \frac{d\sigma_\zeta}{|\zeta - z_1|^{4+\gamma_1}} \prec d^{-(\gamma_1+2)}(z_1, L_R). \end{aligned} \tag{4.6}$$

Let $\gamma > 0$. If $\zeta \in U(z_1) =: \{\xi : |\xi - z_1| \leq d(z_1, L_R)\}$, then using (4.5), we have $|y_R(\zeta) - z_1| \leq |\zeta - z_1|$. Therefore, according to Lemma 3.3, we obtain

$$\begin{aligned} J_{22} &= \iint\limits_{G_R \backslash (U_\varepsilon \cup U(z_1))} |y_R(\zeta) - z_1|^{\gamma_1} |K_n(\zeta)|^2 d\sigma_\zeta \\ &\quad + \iint\limits_{U(z_1)} |y_R(\zeta) - z_1|^{\gamma_1} |K_n(\zeta)|^2 d\sigma_\zeta \\ &\prec \iint\limits_{G_R \backslash (U_\varepsilon \cup U(z_1))} |\zeta - z_1|^{\gamma_1} |K_n(\zeta)|^2 d\sigma_\zeta \\ &\quad + d^{\gamma_1}(z_1, L_R) \iint\limits_{U(z_1)} |K_n(z_1)|^2 d\sigma_\zeta \\ &\prec \iint\limits_{G_R} h(\zeta) |K_n(\zeta)|^2 d\sigma_\zeta \\ &\quad + d^{\gamma_1}(z_1, L_R) . \max_{\zeta \in \overline{U(z_1)}} |K_n(\zeta)|^2 . mes U(z_1) \\ &\prec 1 + \max_{\zeta \in \overline{U(z_1)}} |K_n(\zeta)|^2 . d^{2+\gamma_1}(z_1, L_R). \end{aligned} \tag{4.7}$$

Using the lemma of Bernstein-Walsh [16] and Lemma 3.4 we obtain

$$\max_{\zeta\in\overline{U(z_1)}}|K_n(\zeta)| \leq \max_{\zeta\in\overline{G}_R}|K_n(\zeta)| \prec \max_{\zeta\in\overline{G}}|K_n(\zeta)| \prec \max_{\zeta\in\overline{G^*}}|K_n(\zeta)|. \tag{4.8}$$

Let $\zeta \in L^*$ be any point. Applying the Mean Value Theorem to the polynomial $K_n(z)$ in the disc $|z-\zeta| < c_1 d(z_1, L^*)$, with the constant $c_1 < 1$ taken from Lemma 3.4, we have

$$\begin{aligned}|K_n(\zeta)|^2 &\leq \frac{1}{\pi c_1^2 d^2(z_1,L^*)} \iint\limits_{|z-\zeta|<c_1 d(z_1,L^*)} |K_n(z)|^2 d\sigma_z \\ &\prec \frac{1}{d^2(z_1,L^*)} \iint\limits_{|z-\zeta|<c_1 d(z_1,L^*)} \frac{|z-z_1|^{\gamma_1}|K_n(z)|^2}{|z-z_1|^{\gamma_1}} d\sigma_z \\ &\prec \frac{1}{d^2(z_1,L^*)} \cdot \left[\frac{1}{(1-c_1)\,d(z_1,L^*)}\right]^{\gamma_1} \iint\limits_{|z-\zeta|<c_1 d(z_1,L^*)} |z-z_1|^{\gamma_1}|K_n(z)|^2 d\sigma_z.\end{aligned}$$

Thus,

$$|K_n(\zeta)|^2 \prec d^{-(2+\gamma_1)}(z_1, L^*),$$

by Lemma 3.3. From (4.7), (4.8) and (3.2) for all $p > 0$ we get

$$J_{22} \prec 1 + d^{2+\gamma_1}(z_1, L_R)\,.d^{-(2+\gamma_1)}(z_1, L^*) \prec 1 \tag{4.9}$$

If $-2 < \gamma_1 \leq 0$, then $|y_R(\zeta) - z_1|^{\gamma_1} \prec |\zeta - z_1|^{\gamma_1}$, and, consequently, according to Lemma 3.3, we have

$$J_{22} \prec 1. \tag{4.10}$$

Relations (4.2), (4.3), (4.4), (4.6)-(4.10) yield

$$|K_n(z_1)| \prec d^{-\left(1+\frac{\gamma_1}{2}\right)}(z_1, L_R), \tag{4.11}$$

By Lemma 3.2 the proof of Theorem 2.1 is completed.

The proof of Theorem 2.3 is obtained if we combine the following estimate with (4.11):

$$d(z_1, L_R) \succ (R-1)^{\mu},$$

where $\mu = \frac{1}{\alpha}$, if $\alpha \geq \frac{1}{2}$ and $\mu = \delta$, if $\alpha < \frac{1}{2}$ with $\delta = \delta(\alpha, G), 1 \leq \delta \leq 2$, a certain number. ■

5 Case of arbitrary polynomials

Theorems 2.1- 2.5 can be generalized to arbitrary algebraic polynomials. Let $P_n(z)$ be an arbitrary polynomial of degree at most n and let $M_{n,p} := \|P_n\|_{A_p(h,G)}$.

Theorem 5.1. *Let G be a k-quasidisk for some $0 \leq k < 1$, and let the weight function $h(z)$ be defined by (1.1). Then, for each point $z_j \in L$, $j = \overline{1,m}$, and for every $n = 1, 2, ...$*

$$|P_n(z_j)| \leq c_7 n^{\frac{(2+\gamma_j)(1+k)}{p}} M_{n,p}.$$

Theorem 5.2. *Let $G \in Q_\alpha$, for some $0 < \alpha \le 1$, and let the weight function $h(z)$ be defined by (1.1). Then, for each point $z_j \in L$, $j = \overline{1,m}$, and for every $n = 1, 2, ...$*

$$\left|P_n(z_j)\right| \le c_8 n^{\frac{(2+\gamma_j)\mu}{p}} M_{n,p},$$

where μ defined as in (2.1).

Theorem 5.3. *Let $p > 1, \Omega \in Q_\alpha(z_1, z_2, ..., z_m; \nu_1, ..., \nu_m)$ for some $0 < \nu_j < 1$ and $\alpha(2 - \nu_j) \ge 1$, and let $h(z)$ be defined by (1.1). If*

$$1 + \frac{\gamma_j}{2} > \frac{1}{\alpha(2 - \nu_j)}$$

holds for each point z_j, $j = \overline{1,m}$, then, for every $n = 1, 2, ...$

$$\max_{z \in \overline{G}} \left(\prod_{j=1}^{m} \left|z - z_j\right|^{\mu_j^*} |P_n(z)| \right) \le c_9 n^{2/\alpha p} M_{n,p}, \tag{5.1}$$

$$\left|P_n(z_j)\right| \le c_{10} n^{s_j^*} M_{n,p}, \tag{5.2}$$

where

$$\mu_j^* := \frac{2 + \gamma_j}{p} - \frac{2}{p\alpha(2 - \nu_j)}, \; s_j^* := \frac{(2 + \gamma_j)(2 - \nu_j)}{p}, \; j = \overline{1,m}.$$

The proofs of the Theorems 5.1-5.2 are completely similar to the proofs of the Theorems 2.1-2.3.

Proof of Theorem 5. 3. Let us introduce the Blaschke functions with respect to the singular points of the weight functions $h(z)$:

$$B_R(z) = \prod_{j=1}^{m} B_R^j(z) := \prod_{j=1}^{m} \frac{\Phi_R(z) - \Phi_R(z_j)}{1 - \overline{\Phi_R(z_j)}\Phi_R(z)}, \; z \in \Omega^*.$$

It is easily seen that $B_R(z_j) = 0$ and $|B_R(z)| \equiv 1$ at $z \in L^*$. As the system of points $\{z_j\}_{j=1}^m$ on L is finite, we may assume without loss of generality that $j = 1$, $\mu^* := \mu_1^*$; $s^* := s_1^*$ For $R > 1$ we put $R_1 := 1 + \frac{R-1}{2}$, $\widetilde{L}^* := y(L_R)$, $w = \Phi_R(z)$, $w_1 = \Phi_R(z_1)$, and

$$h_R(w) := \left[\frac{\Psi_R(w) - \Psi_R(w_1)}{wB_R(\Psi_R(w))}\right]^{\mu^*} \frac{P_n(\Psi_R(w))}{w^{n+1}}.$$

Let $z \in L$. The Cauchy integral representation for an unbounded region yields

$$h_R(w) = -\frac{1}{2\pi i} \int\limits_{|t|=R_1} h_R(t) \frac{dt}{t - w}.$$

As for all $|t| = R_1 > 1$, $|B_R(\Psi_R(t))| \geq 1$, $|t|^{n+1} = R_1^{n+1} > 1$, we obtain

$$\begin{aligned} A_n &:= |\Psi_R(w) - \Psi_R(w_1)|^{\mu^*} |P_n(\Psi_R(w))| \\ &\leq |wB_R(\Psi_R(w))|^{\mu^*} |w|^{n+1} \frac{1}{2\pi} \int\limits_{|t|=R_1} |\Psi_R(t) - \Psi_R(w_1)|^{\mu^*} |P_n(\Psi_R(t))| \frac{|dt|}{|t-w|}. \end{aligned} \tag{5.3}$$

As

$$\begin{aligned} \left|wB_R^1(z)\right| &= \left| w \cdot \frac{w - \Phi_R(z_1)}{\frac{1}{\overline{\Phi_R(z_1)}} - w} \cdot \frac{1}{\overline{\Phi_R(z_1)}} \right| \\ &= \left| \frac{w}{\overline{\Phi_R(z_1)}} \right| \cdot \left| \frac{w - \Phi_R(z_1)}{\overline{\Phi_R(z_1)} - w} \right| = \left| \frac{w}{\overline{\Phi_R(z_1)}} \right|, \end{aligned}$$

we obtain from (3.2) that

$$|wB_R(\Psi_R(w))|^{\mu^*} \prec 1, \quad |w|^{n+1} \prec 1.$$

So, from (5.3) it follows that

$$A_n \prec \int\limits_{|t|=R_1} |\Psi_R(t) - \Psi_R(w_1)|^{\mu^*} |P_n(\Psi_R(t))| \frac{|dt|}{|t-w|}.$$

To estimate the integral of at the right hand side, we multiply the numerator and denominator of integrand by $|\Psi_R(t) - \Psi_R(w_1)|^{\frac{\gamma}{p}} |\Psi_R'(t)|^{\frac{2}{p}}$; then applying the Holder inequality we obtain

$$\begin{aligned} A_n &\prec \left(\int\limits_{|t|=R_1} |\Psi_R(t) - \Psi_R(w_1)|^{\gamma} |P_n(\Psi_R(t))|^p |\Psi_R'(t)|^{2p} |dt| \right)^{\frac{1}{p}} \\ &\quad \times \left(\int\limits_{|t|=R_1} \frac{|\Psi_R(t) - \Psi_R(w_1)|^{\mu^* q - \gamma(q-1)}}{|\Psi_R'(t)|^{2(q-1)} |t-w|^{2q}} |dt| \right)^{\frac{1}{q}} \\ &=: A_n^1 \cdot B_n^1. \end{aligned} \tag{5.4}$$

Let

$$f_n(t) := (\Psi_R(t) - \Psi_R(w_1))^{\frac{\gamma}{p}} P_n(\Psi_R(t)) \left(\Psi_R'(t)\right)^{\frac{2}{p}}.$$

Now we partition the circle $|t| = R_1$ into n equal parts δ_n with $mes(\delta_n) = \frac{2\pi R_1}{n}$; applying the Mean Value Theorem to the integral A_n^1 we get

$$A_n^1 = \sum_{k=1}^{n} \int\limits_{\delta_k} |f_n(t)|^p |dt| = \sum_{k=1}^{n} \left| f_n\left(t_k'\right) \right|^p mes(\delta_k), \ \ t_k' \in \delta_k.$$

On the other hand, applying the mean value estimate

$$\left|f_n\left(t_k'\right)\right|^p \leq \frac{1}{\pi\left(1-\left|t_k'\right|\right)^2} \iint\limits_{\left|\xi-t_k'\right|<1-\left|t_k'\right|} \left|f_n\left(\xi\right)\right|^p d\sigma_\xi,$$

we obtain

$$A_n^1 \prec \sum_{k=1}^{n} \frac{mes(\delta_k)}{\pi\left(1-\left|t_k'\right|\right)^2} \iint\limits_{\left|\xi-t_k'\right|<1-\left|t_k'\right|} \left|f_n\left(\xi\right)\right|^p d\sigma_\xi, \quad t_k' \in \delta_k.$$

Taking into account that at most two of the discs with origin at the points t_k' are intersecting, we obtain

$$A_n^1 \prec \frac{mes\delta_1}{\left(1-\left|t_1'\right|\right)^2} \iint\limits_{1<|\xi|<R_1} \left|f_n\left(\xi\right)\right|^p d\sigma_\xi \prec n \iint\limits_{1<|\xi|<R_1} \left|f_n\left(\xi\right)\right|^p d\sigma_\xi.$$

According to (3.3) we obtain for A_n^1:

$$A_n^1 \prec n \iint\limits_{G_{R_1}^* \setminus G^*} |z-z_1|^\gamma \left|P_n\left(z\right)\right|^p d\sigma_z \prec n \cdot M_{n,p}^p. \tag{5.5}$$

In order to estimate the integral B_n^1, we take into account the estimate for the functions Ψ_R (see e. g. [12, Th. 2. 8]). We put

$$\{t : |t| = R_1\} = \bigcup_{j=1}^{3} K_j,$$

where

$$\begin{aligned} K_1 &:= \{t : |t| = R_1, |t-w| < \varepsilon_1\}, \\ K_2 &:= \{t : |t| = R_1, |t-w_1| < \varepsilon_2\}, \\ K_3 &:= \{t : |t| = R_1, |t-w| \geq \varepsilon_1, |t-w_1| \geq \varepsilon_2\}, \\ w &= \Phi\left(z\right), \ w_1 = \Phi\left(z_1\right). \end{aligned}$$

Then we have

$$\begin{aligned} B_n^1 &\prec \int\limits_{|t|=R_1} \frac{\left|\Psi_R\left(t\right)-\Psi_R\left(w_1\right)\right|^{\mu^* q-\gamma(q-1)}\left(|t|-1\right)^{2(q-1)}}{\left(\left|\Psi_R\left(t\right)\right|-1\right)^{2(q-1)}} \frac{|dt|}{|t-w|^q} \\ &= \left(\int\limits_{K_1} + \int\limits_{K_2} + \int\limits_{K_3}\right)[idem] =: B_n^{11} + B_n^{12} + B_n^{13}. \end{aligned} \tag{5.6}$$

We estimate each integral separately

$$\begin{aligned} B_n^{11} &\prec \int\limits_{K_1} \frac{\left|\Psi_R\left(t\right)-\Psi_R\left(w_1\right)\right|^{\mu^* q-\gamma(q-1)}}{\left(|t|-1\right)^{\left(\frac{1}{\alpha}-1\right)2(q-1)}} \frac{|dt|}{|t-w|^q} \\ &\prec \left(\frac{1}{n}\right)^{-2(q-1)\left(\frac{1}{\alpha}-1\right)} \int\limits_{K_1} \frac{|dt|}{|t-w|^q} \prec n^{\frac{2(q-1)}{\alpha}-(q-1)}. \end{aligned} \tag{5.7}$$

Similarly

$$B_n^{13} \prec \int\limits_{K_3} \frac{|dt|}{(|t|-1)^{\left(\frac{1}{\alpha}-1\right)2(q-1)}\,|t-w|^q} \prec n^{\frac{2(q-1)}{\alpha}-(q-1)}. \tag{5.8}$$

For the estimate of B_n^{12} we consider two cases.

a) $|\Psi_R(t)-\Psi_R(w_1)| \preceq |\Psi_R(t)|-1$. In this case, according to [17], we get

$$B_n^{12} \prec \int\limits_{K_2} \frac{(|t|-1)^{2(q-1)}}{(|\Psi_R(t)|-1)^{(2+\gamma)(q-1)-\mu^* q}} \frac{|dt|}{|t-w|^q}$$

$$\prec \int\limits_{K_2} \frac{(|t|-1)^{2(q-1)}}{(|t|-1)^{(2-\nu_j)[(2+\gamma)(q-1)-\mu^* q]}} \frac{|dt|}{|t-w|^q}$$

$$\prec n^{(2-\nu_j)[(2+\gamma)(q-1)-\mu^* q]} \left(\frac{1}{n}\right)^{2(q-1)} \int\limits_{K_2} \frac{|dt|}{|t-w|^q} \prec n^{\frac{2(q-1)}{\alpha}-(q-1)}.$$

b) $|\Psi_R(t)|-1 < |\Psi_R(t)-\Psi_R(w_1)| < c$. In this case, according to the (3.1), we have $|t|-1 < |t-w_1| < c_1$. Let us put $\varepsilon_0 := |t|-1$. According to (3.1), for all points $\xi \in L^*$ satisfying $d(\xi, L_R^*) \asymp d(\xi, L)$ holds $|t|-1 \asymp |w_1|-R_1$. We then take the discs centered at the point w_1, with radius $2^s\varepsilon_0$, $s=1,2,...N$, where we choose a number N such that the circles $Q_N = \left\{\tau : |\tau-w_1| = 2^N\varepsilon_0\right\}$ satisfy the conditions $Q_N \cap \{t : |t| = R_1\} \neq \varnothing$, and $Q_{N+1} \cap \{t : |t| = R_1\} = \varnothing$. Then, putting $K_2^s := K_2 \cap \left\{t : 2^{s-1}\varepsilon_o \le |t-w_1| \le 2^s\varepsilon_o\right\}$, we have consecutively

$$\begin{aligned}
B_n^{12} &\prec \int\limits_{K_2} \frac{|\Psi_R(t)-\Psi_R(w_1)|^{2(q-1)}(|t|-1)^{2(q-1)}}{(|\Psi_R(t)|-1)^{2(q-1)}|\Psi_R(t)-\Psi_R(w_1)|^{\frac{2}{\alpha(2-\nu_j)}(q-1)}} \frac{|dt|}{|t-w|^q} \\
&\prec \sum_{s=1}^{\infty} \int\limits_{K_2^s} \left[\frac{|\Psi_R(t)-\Psi_R(w_1)|}{|\Psi_R(t)|-1}\right]^{2(q-1)} \frac{(|t|-1)^{2(q-1)}}{|t-w_1|^{\frac{2(q-1)}{\alpha}}} \frac{|dt|}{|t-w|^q} \\
&\prec \sum_{s=1}^{\infty} \int\limits_{K_2^s} \left[\frac{|t-w_1|}{|t|-1}\right]^{2\varepsilon(q-1)} \frac{(|t|-1)^{2(q-1)}}{|t-w|^q} |dt| \\
&\prec \sum_{s=1}^{\infty} \frac{(2^s\varepsilon_o)^{2\varepsilon(q-1)}(\varepsilon_o)^{2(1-\varepsilon)(q-1)}}{(2^{s-1}\varepsilon_o)^{\frac{2(q-1)}{\alpha}}} \int\limits_{K_2^s} \frac{|dt|}{|t-w|^q} \\
&= \sum_{s=1}^{\infty} \frac{2^{s2\varepsilon(q-1)}(\varepsilon_o)^{2(1-\varepsilon)(q-1)+2\varepsilon(q-1)}}{2^{\frac{2(q-1)(s-1)}{\alpha}}(\varepsilon_o)^{\frac{2(q-1)}{\alpha}}} \int\limits_{K_2^s} \frac{|dt|}{|t-w|^q} \\
&\prec 2^{\frac{2(q-1)}{\alpha}} n^{\frac{2(q-1)}{\alpha}} \left(\frac{1}{n}\right)^{2(q-1)} \sum_{s=1}^{\infty} \left(\frac{2^\varepsilon}{2^{\frac{1}{\alpha}}}\right)^{2s(q-1)} \int\limits_{K_2^s} \frac{|dt|}{|t-w|^q} \\
&\prec n^{\frac{2(q-1)}{\alpha}-2(q-1)} \int\limits_{K_2^s} \frac{|dt|}{|t-w|^q} \sum_{s=1}^{\infty} \left(\frac{2^\varepsilon}{2^{\frac{1}{\alpha}}}\right)^{2s(q-1)} \prec n^{\frac{2(q-1)}{\alpha}-(q-1)},
\end{aligned}$$

where $\varepsilon = \varepsilon(L) < 1$. Therefore

$$B_n^{12} \prec n^{\frac{2(q-1)}{\alpha} - (q-1)}, \tag{5.9}$$

and using (5.6), (5.7), (5.8), and (5.9) we obtain

$$B_n^{1} \prec n^{\frac{2(q-1)}{\alpha} - (q-1)}. \tag{5.10}$$

Relations (5.4), (5.5), and (5.10) yield

$$A_n \prec n^{\frac{2}{p\alpha}} M_{n,p}.$$

As the system of points $\{z_j\}_{j=1}^{m}$ is isolated, we get (5.1).

For the proof of (5.2) we can write the integral representations for $P_n(z)$ by analogy to (4.1):

$$P_n(z_1) = -\frac{1}{\pi} \iint\limits_{G_R} \frac{P_n(\zeta)\, y_{R,\zeta}}{(y_R(\zeta) - z_1)^2} d\sigma_\zeta\,, \quad z_1 \in L.$$

With similar arguments as those used for proving the relations (4.2-4.11), it is easily shown that

$$\begin{aligned} |P_n(z_1)| &\prec M_{n,p} \left(\iint\limits_{y(G_R \setminus U_\varepsilon)} \frac{d\sigma_\zeta}{|\zeta - z_1|^{\gamma_1(q-1)+2q}} \right)^{\frac{1}{q}} \\ &\prec M_{n,p} \left(\iint\limits_{|\zeta - z_1| \geq d(z_1, L_R)} \frac{d\sigma_\zeta}{|\zeta - z_1|^{\gamma_1(q-1)+2q}} \right)^{\frac{1}{q}} \\ &\prec M_{n,p} d^{-\frac{(\gamma_1+2)}{p}}(z_1, L_R). \end{aligned}$$

and so, according to [17], we obtain (5.2). ∎

Note that the Theorems 5.1, 5.2 are sharp. This is easily seen by the example $G = B$, $h(z) \equiv 1$, $P_n(z) = \sum_{j=1}^{n} (j+1) z^j$.

Remark.
As for $K_n(z)$, $M_{n,2} \equiv 1$, the proof of Theorem 2.3 also follows from Theorem 5.2.

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