Generalized almost f-algebras

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Abstract

We introduce and study a new class of lattice ordered algebras.

1 Introduction

The importance of f-algebras in the theory of Riesz spaces has steadily grown since their introduction in the fifties by Birkhoff and Pierce [6]. It is only recently that other various lattice-ordered algebraic structures have been getting more attention. We are thinking here about almost f-algebras introduced by Birkhoff [5]. Recently, Henriksen [9] expressed his wish to see more papers dealing with ℓ-algebras rather than f-algebras. In this prospect, we introduce and give a systematic study a class of ℓ-algebras that is much larger than the class of (almost) f-algebras. Indeed, we define an ℓ-algebra A to be a *generalized almost f-algebra* if $x \wedge y = 0$ in A implies that the product xy is an annihilator element in A. The surprise is that, unlike to (almost) f-algebras, a generalized almost f-algebra need not be commutative and need not have positive squares. In spite of that, our main objective in this paper is to extend various classical facts on (almost) f-algebras to the more general setting of generalized almost f-algebras as introduced above. Furthermore, we show that $a_1..a_p = a_{\sigma(1)}..a_{\sigma(p)}$ for all permutation $\sigma \in S(p)$ such that $p \geq 3$ and $a_i \in A$. This is applied to give a description of the set of nilpotent elements in an Archimedean generalized almost f-algebras. We also present in this paper a generalization of multiplicator operator on an algebra A. An order bounded operator T is said to be a *generalized multiplicator* if $xT(y) - T(x)y$ is an annihilator element in A for all x, y in A. In this regard, it will be focused

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on the relationship between generalized multiplicators and orthomorphisms on generalized almost f-algebras.

We assume that the reader is familiar with the notion of Riesz spaces (also called vector lattices) as presented in [1] by Aliprantis and Burkinshaw and [11] by Luxemburg and Zaanen. For terminology and properties of Riesz spaces and order bounded operators not explained or proved in this paper, we refer to [1,11,14]. We refer to [1,14] for f-algebra and orthomorphism theories and to [11] for the relatively uniform topology..

2 Preliminaries

A vector lattice (also called a Riesz space) L is said to be *Archimedean* if for each non zero $x \in L$ the set $\{nx : n = \pm 1, \pm 2, ...\}$ has no upper bound in L. *In order to avoid unnecessary repetition we will assume throughout that all vector lattices and lattice ordered algebras under consideration are Archimedean.* The vector lattice A is said to be a *lattice ordered algebra* (briefly, an ℓ-algebra) if there exists an associative multiplication in A with the usual algebra properties such that $xy \in A^+$ for all $x, y \in A^+$. For an ℓ-algebra A, we denote

$$N(A) = \{x \in A : x^n = 0 \text{ for some } n = 1, 2, ..\},$$

that is, $N(A)$ is the set of all nilpotent elements in A. Also, for a fixed nonnegative integer n, we put

$$N_n(A) = \{x \in A : x^n = 0\}.$$

The ℓ-algebra A is said to be *semiprime (or reduced)* if $N(A) = \{0\}$. The ℓ-algebra A is called an f-algebra if A has the property that $x \wedge y = 0$ in A implies $xz \wedge y = zx \wedge y = 0$ for all $z \in A^+$. In a (not necessarily Archimedean) f-algebra A, the equality $|xy| = |x|\,|y|$ holds for all $x, y \in A$ and then squares are positive. The (Archimedean) f-algebra A is automatically commutative and satisfies

$$N(A) = N_2(A) = \{x \in A : xy = 0 \text{ for all } y \in A\}.$$

An *almost f-algebra* is an ℓ-algebra A such that $x \wedge y = 0$ in A implies $xy = 0$. An ℓ-algebra is an almost f-algebra if and only if $|x^2| = |x|^2$ for all $x \in A$. As for f-algebras, any (Archimedean) almost f-algebra is commutative and satisfies

$$N_2(A) = \{x \in A : xy = 0 \text{ for all } y \in A\}$$

and

$$N(A) = N_3(A) = \{x \in A : xyz = 0 \text{ for all } y, z \in A\}.$$

For an (almost) f-algebras A, both $N(A)$ and $N_2(A)$ are ℓ-ideals, that is, order and ring ideals. For more information about almost f-algebras, the reader is referred to [2].

Next, we discuss linear operators on Riesz spaces. Let L and M be Riesz spaces with positive cones L^+ and M^+, respectively, and let T be a linear operator from L into M. One says that T is *order bounded* if for each $x \in L^+$ there exists $y \in M^+$ such that $|T(z)| \leq y$ in M whenever $|z| \leq x$ in L. The linear operator T is said to be *positive* if $T(L^+) \subset M^+$. The linear operator T is called a *Riesz homomorphism* (or *lattice homomorphism)* whenever $x \wedge y = 0$ implies $T(x) \wedge T(y) = 0$. Obviously, every Riesz homomorphism is positive and then order bounded. The set $\mathcal{L}_b(L)$ of all order bounded linear operators on L is an ordered vector space with respect to pointwise operations and order. The positive cone of $\mathcal{L}_b(L)$ is the subset of all positive linear operators. An element T in $\mathcal{L}_b(L)$ is referred to as an *orthomorphism* if, for all $x, y \in L$, $|T(x)| \wedge |y| = 0$ whenever $|x| \wedge |y| = 0$. Under the ordering and operations inherited from $\mathcal{L}_b(L)$, the set $\mathrm{Orth}(L)$ of all orthomorphisms on L is an Archimedean Riesz space. The absolute value in $\mathrm{Orth}(L)$ is given by $|T|(x) = |T(x)|$ for all $x \in L^+$. With respect to the composition as multiplication, $\mathrm{Orth}(L)$ is an Archimedean f-algebra with the identity mapping I_L on L as a unit element (for details on this see, e.g., [1, Theorem 8.24]).

Let L and M be Riesz spaces, $p \in \{2, 3, ...\}$. A map $\Psi : L^p \to M$ is called a *p-linear* map whenever the operator

$$\begin{array}{rccc} \Psi_i : & L & \to & A \\ & x & \to & \Psi(a_1, .., \underset{i}{x}, .., a_p) \end{array}$$

is linear for all $1 \leq i \leq p$ and $a_1, a_2, .., a_p \in L^+$ (where L^p is the cartesian product of L with itself p times). A p-linear map $\Psi : L^p \to M$ is said to be positive if $\Psi(a_1, a_2, .., a_p) \in M^+$ for all $a_1, a_2, .., a_p \in L^+$. A positive p-linear map $\Psi : L^p \to M$ is said to be an *orthosymmetric* map if $a_i \wedge a_j = 0$ implies $\Psi(a_1, a_2, .., a_p) = 0$. The proof of commutativity of Archimedean almost f-algebras, given by Bernau and Huijsmans in [2, Theorem 2.15], doesn't make use of associativity. In fact, Bernau and Huijsmans proved that every orthosymmetric bilinear map is symmetric.

At this point, we shall introduce and give the first properties of the class of ℓ-algebras that will be surveyed in this paper.

Let A be a lattice ordered algebra. Consider the left annihilator $lan(A) = \{x \in A : xA = \{0\}\}$, the right annihilator $ran(A) = \{x \in A : Ax = \{0\}\}$ and the annihilator $ann(A) = lan(A) \cap ran(A)$.

Definition 1. *The ℓ-algebra A is said to be a generalized almost f-algebra if*

$$x \wedge y = 0 \text{ implies } xy \in ann(A).$$

It is not hard to prove that f-algebras and almost f-algebras are generalized almost f-algebras. Now we give an example to show that the classes of almost f-algebras and generalized almost f-algebras are in general distinct.

Example 1. *Take $A = C([-1, 1])$ with the usual operation, order and multiplication $*$ defined by*

$$\begin{cases} (f * g)(x) = \int_0^1 f(t)g(1-t)dt & \text{if } x \in \left[-1, -\frac{1}{4}\right] \\ (f * g)(x) = (-4\int_0^1 f(t)g(1-t)dt)x & \text{if } x \in \left[-\frac{1}{4}, 0\right] \\ (f * g)(x) = 0 & \text{if } x \in [0, 1] \end{cases}$$

it is not hard to show that A is a generalized almost f-algebra under the multiplication $*$. However, A is not an almost f-algebra. Indeed, let f, g, defined by

$$f(x) = \begin{cases} 1 - 4x & \text{if } x \in \left[-1, \frac{1}{4}\right] \\ 0 & \text{if } x \in \left[\frac{1}{4}, 1\right] \end{cases}$$

$$g(x) = \begin{cases} 0 & \text{if } x \in \left[-1, \frac{3}{4}\right] \\ -\frac{4}{3}x + 1 & \text{if } x \in \left[\frac{3}{4}, 1\right] \end{cases}.$$

It is not hard to prove that $f \wedge g = 0$ and $f * g \neq 0$.

We observe from preceding example that a generalized almost f-algebra need not have positive squares.

It seems natural therefore to ask what is missing for a generalized almost f-algebra to be an f-algebra. If some conditions are imposed, then we have the following relationship. Any semiprime generalized almost f-algebra is an f-algebra and any generalized almost f-algebra with unit element e is an f-algebra.

3 Theoretical properties of generalized almost f-algebras

In this section we prove some properties of generalized almost f-algebras which will be used later. We start with our basic Lemma.

Lemma 1. *Let A be a generalized almost f-algebra and $p \in \{3, 4, ..\}$. If $\sigma \in S(p)$ is a permutation then*

$$a_1..a_p = a_{\sigma(1)}..a_{\sigma(p)} \text{ for all } a_1, .., a_p \in A.$$

Proof. Let Ψ the p-linear map from A^p into A defined by

$$\Psi(a_1, .., a_p) = a_1..a_p$$

for all $a_1, .., a_p \in A$. It is not hard to prove that Ψ is orthosymmetric.

Let $i \neq j \in \{1, ..., p\}$ and $0 \leq a_1, ..., a_p \in A$ and define

$$\begin{array}{cccc} \Phi : & A^2 & \rightarrow & A \\ & (x, y) & \rightarrow & \Psi(a_1, .., \underset{i}{x}, .., \underset{j}{y}, .., a_p) \end{array}.$$

Φ is a orthosymmetric bilinear map then symmetric. So $\Psi(a_1, .., a_i, .., a_j, .., a_p) = \Psi(a_1, .., a_j, .., a_i, .., a_p)$.Since $S(p)$ is generated by transpositions, we obtain

$$\Psi(a_1, .., a_p) = \Psi(a_{\sigma(1)}, .., a_{\sigma(p)})$$

for all permutation $\sigma \in S(p)$. This give the desired result. ■

As an application we give the following Proposition.

Proposition 1. *Let A be a generalized almost f-algebra. then*

(i) $a^4 \geq 0$ *for all $a \in A$ (and hence $|a^4| = |a|^4$ for all $a \in A$).*

(ii) $a(a^+)^2 \geq 0$, $(a^+)^2 a \geq 0$, $a^2 b \geq 0$ *for all $a \in A$ and $b \in A^+$.*

(iii) $|a^{2n}| = |a|^{2n}$ *for all $a \in A$ and $n \in \{2, 3, ...\}$.*

Proof. (i) If $a \in A$ then

$$\begin{aligned} a^4 &= (a^+ - a^-)^4 \\ &= (a^+)^4 - 4(a^+)^3(a^-) + 6(a^+)^2(a^-)^2 - 4(a^+)(a^-)^3 + (a^-)^4 \\ &= (a^+)^4 + (a^-)^4. \end{aligned}$$

(ii) Let $a \in A$ and observe that $a(a^+)^2 = (a^+ - a^-)(a^+)^2 = (a^+)^3 \geq 0$. Similarly $(a^+)^2 a = (a^+)^3 \geq 0$.

(iii) All mixed terms in the expansion of $a^{2n} = (a^+ - a^-)^{2n}$ vanish for $n \in \{2, 3, ...\}$, so

$$a^{2n} = (a^+)^{2n} + (a^-)^{2n} \geq 0$$

i.e. $a^{2n} = \left|a^{2n}\right|$ for $n \in \{2, 3, ...\}$. Likewise $|a|^{2n} = (a^+ + a^-)^{2n} = (a^+)^{2n} + (a^-)^{2n} = \left|a^{2n}\right|$ for $n \in \{2, 3, ...\}$. ■

Next, we describe the set of nilpotent elements in a generalized almost f-algebra.

Proposition 2. *If A is a generalized almost f-algebra, then $N(A)$ is an ℓ-ideal in A.*

Proof. Obviously, $N(A)$ is a vector subspace of A. Moreover, Lemma 1 implies directly that $N(A)$ is a ring ideal. Now, we prove that $N(A)$ is an order ideal. It is clear that for $0 \leq b \leq a$, $a \in N(A)$ implies $b \in N(A)$. It remains to show, that $N(A)$ is a sublattice of A. To this end we use Proposition 1(iii). If $a \in N(A)$, then $a^n = 0$ for some $n \in \{2, 3, ...\}$, so $a^{2n} = 0$, so

$$|a|^{2n} = \left|a^{2n}\right| = 0.$$

This give the desired result. ■

Theorem 1. *Let A be an Archimedean generalized almost f-algebra. Then*

$$\begin{aligned} N(A) = N_4(A) &= \left\{a \in A : a^4 = 0\right\} \\ &= \left\{a \in A : a^2bc = 0 \textit{ for all } b, c \in A\right\} \\ &= \left\{a \in A : abcd = 0 \textit{ for all } b, c, d \in A\right\}. \end{aligned}$$

Proof. By Proposition 2 we can assume, without loss of generality that $a \in A^+$. Let $k \in \{5, 6, ...\}$ such that $a^k = 0$. It follows from Lemma 1 and Proposition 1(ii) that

$$(a - na^{k-3})^2 a = a^3 + n^2 a^{2k-5} - 2na^{k-1} \geq 0 \text{ for all } n \in \{1, 2, ...\}.$$

Observe now that for $k \in \{5, 6, ...\}$, we have $2k - 5 \geq k$. Therefore, $a^{2k-5} = 0$. Consequently, $2na^{k-1} \leq a^3$ for all $n \in \{1, 3, ...\}$. The Archimedean property gives $a^{k-1} = 0$. Repeating this argument another $k - 5$ times, we finally find $a^4 = 0$. So,

$$N(A) = \left\{a \in A : a^4 = 0\right\}.$$

The rest of proof is an easy exercise. ■

Now, let A be an Archimedean generalized almost f-algebra it is easily followed from the descriptions of $N(A)$ that $N(A)$ is a uniformly closed ℓ-ideal of A. Hence, the quotient $A/N(A)$ is Archimedean.

Theorem 2. *If A is an Archimedean generalized almost f-algebra, then $A/N(A)$ is an Archimedean semiprime f-algebra.*

We observe from the preceding Theorem that if A is an Archimedean generalized almost f-algebra then A is four-commutative (i.e $(ab-ba)^4 = 0$ for all $a, b \in A$). This result can be strengthened in the following way, if A is an Archimedean generalized almost f-algebra then A is square-commutative (i.e $(ab-ba)^2 = 0$ for all $a, b \in A$).

4 Product in generalized almost f-algebra

Let A be a uniformly complete generalized almost f-algebra and $p \in \{3, 4, ...\}$.

In this section, we take the following notations:

$$\begin{array}{ll} (i) & \Pi_p(A) = \left\{a_1 a_2 .. a_p : \ a_k \in A, k = 1, .., p\right\}. \\ (ii) & \Sigma_p(A) = \left\{a^p : \ 0 \le a \in A\right\}. \end{array}$$

In this section, we will study the set $\Pi_p(A)$.

First let us prove a useful Proposition.

Proposition 3. *Let A, B be Archimedean vector lattices, $p \in \{2, 3, ...\}$, Ψ an orthosymmetric map from A^p into B and $\pi_1, .., \pi_p \in \mathrm{Orth}(A)$. Then*

$$\Psi(\pi_1(a_1), \pi_2(a_2), .., \pi_p(a_p)) = \Psi(a_1, a_2, .., (\pi_1 .. \pi_p)(a_p))$$

for all $a_1, a_2, .., a_p \in A$.

Proof. It sufficient to prove that if $i \neq j \in \{1, .., p\}$ then

$$\Psi(a_1, .., \pi(a_i), .., a_p) = \Psi(a_1, .., \pi(a_j), .., a_p)$$

for all $a_1, .., a_p \in A$ and $\pi \in \mathrm{Orth}(A)$.

Let $i \neq j \in \{1, .., p\}$, $0 \le \pi \in \mathrm{Orth}(A)$ and define

$$\begin{array}{rccc} \Phi : & A^2 & \to & A \\ & (x, y) & \to & \Psi(a_1, .., \underset{i}{\pi(x)}, .., \underset{j}{y}, .., a_p) \end{array}.$$

It is straightforward to show that $\Phi(x, y) = \Phi(y, x)$. Therefore

$$\Psi(a_1, .., \underset{i}{\pi(x)}, .., \underset{j}{y}, .., a_p) = \Psi(a_1, .., \underset{i}{\pi(y)}, .., \underset{j}{x}, .., a_p).$$

On the other hand $\Psi(a_1, .., \underset{i}{\pi(y)}, .., \underset{j}{x}, .., a_p) = \Psi(a_1, .., \underset{i}{x}, .., \underset{j}{\pi(y)}, .., a_p)$ (because the bilinear map $(x, y) \to \Psi(a_1, .., \underset{i}{x}, .., \underset{j}{y}, .., a_p)$ is symmetric). Consequently,

$$\Psi(a_1, .., \underset{i}{\pi(x)}, .., \underset{j}{y}, .., a_p) = \Psi(a_1, .., \underset{i}{x}, .., \underset{j}{\pi(y)}, .., .., a_p)$$

for all $x, y \in A$. In particular

$$\Psi(a_1, .., \pi(a_i), .., a_j, .., a_p) = \Psi(a_1, .., a_i, .., \pi(a_j), .., .., a_p)$$

and the proof of the Proposition is finished. ∎

Proposition 4. *Let A be an uniformly complete generalized almost f-algebra and $p \in \{3, 4, ...\}$. Then:*

(i) For every $0 \leq a_1, .., a_p \in A$, there exists $0 \leq u \in A$ such that

$$u^p = a_1..a_p.$$

(ii) For every $0 \leq a, b \in A$, there exists $0 \leq u \in A$ such that

$$u^p = a^p + b^p.$$

Proof. (i) Let $0 \leq a_1, .., a_p \in A$ and put $e = a_1 + .. + a_p$. Consider A_e the principal order ideal of A generated by e and Ψ the p-linear map from A^p into A defined by

$$\Psi(a_1, .., a_p) = a_1..a_p.$$

Obviously, Ψ is an orthosymmetric map. Moreover, for every $k \in \{1, .., p\}$, there exists $0 \leq \pi_k \in \mathrm{Orth}(A_e)$ such that $a_k = \pi_k(e)$. So

$$a_1..a_p = \Psi(a_1, .., a_p) = \Psi(\pi_1(e), .., \pi_p(e)) = \Psi(e, .., \pi_1..\pi_p(e))$$

(by the preceding Proposition).

Since $\mathrm{Orth}(A_e)$ is an uniformly complete f-algebra with unit, there exists $0 \leq \pi \in \mathrm{Orth}(A_e)$ such that

$$\pi^p = \pi_1..\pi_p$$

(see [3; Theorem 5]). Consequently

$$a_1..a_p = \Psi(e, .., \pi^p(e)) = \Psi(\pi(e), .., \pi(e)) = \pi(e)^p$$

which is the asked result.

(ii) We use the same argument of the proof of (i). ∎

Now, we arrive at to the main result of this section.

Theorem 3. *Let A be a uniformly complete generalized almost f-algebra and a natural number $p \in \{4, 5, ..\}$. Then, $\Pi_p(A)$ is a semiprime f-algebra under the ordering and multiplication inherited from A with $\Sigma_p(A)$ as positive cone. In particular*

$$a^p \wedge_p b^p = (a \wedge b)^p \text{ and } a^p \vee_p b^p = (a \vee b)^p \text{ for all } 0 \leq a, b \in A.$$

(Where $a^p \wedge_p b^p$ and $a^p \vee_p b^p$ indicate $\inf(a^p, b^p)$ and $\sup(a^p, b^p)$ in $\Pi_p(A)$).

Proof. At first, we prove that $\Pi_p(A)$ is an order vector subspace of A with $(\Pi_p(A))^+ = \Sigma_p(A)$, $a^p \wedge_p b^p = (a \wedge b)^p$ and $a^p \vee_p b^p = (a \vee b)^p$. From the preceding Theorem, it is clear that $\Sigma_p(A)$ is a positive cone in A. Hence $\Sigma_p(A) - \Sigma_p(A)$ is an order vector subspace of A with

$$(\Sigma_p(A) - \Sigma_p(A))^+ = \Sigma_p(A).$$

Let $a, b \in A^+$, we have

$$a^p - b^p = (a - b) \sum_{0 \leq k \leq p-1} a^k b^{p-1-k}.$$

Now, by preceding Proposition there exists $u \in A^+$ such that

$$u^{p-1} = \sum_{0 \le k \le p-1} a^k b^{p-1-k}$$

Therefore

$$a^p - b^p = u^{p-1}(a-b).$$

So $\Sigma_p(A) - \Sigma_p(A) \subset \Pi_p(A)$. Conversely, let $a_1, a_2, ..., a_p \in A$. Observe now that, by using the fact that $a_i = a_i^+ - a_i^-$ and the preceding Proposition, there exists $u, v \in A^+$ such that

$$a_1..a_p = u^p - v^p.$$

So $\Pi_p(A) \subset \Sigma_p(A) - \Sigma_p(A)$. This implies that

$$\Sigma_p(A) - \Sigma_p(A) = \Pi_p(A).$$

Now, let $a, b, c \in A^+$. Obviously, $(a \vee b)^p \ge a^p$ and $(a \vee b)^p \ge b^p$. Assume that

$$c^p \ge a^p, c^p \ge b^p$$

so, $[c]^p \ge [a]^p$, $[c]^p \ge [b]^p$ in $A/N(A)$. Since $A/N(A)$ is an Archimedean semiprime f-algebra we get by [3; Proposition 2]

$$[c] \ge [a], \ [c] \ge [b]$$

so

$$[c-a] \ge 0, [c-b] \ge 0.$$

This implies that

$$(c-a)^-, (c-a)^- \in N(A).$$

Consequently,

$$(c - (a \vee b))^- = (c-a)^- \vee (c-a)^- \in N(A).$$

Observe now that by using Theorem 1, we get

$$\begin{aligned} c^p - (a \vee b)^p &= (c - (a \vee b))\Big(\sum_{0 \le k \le p-1} c^k (a \vee b)^{p-1-k}\Big) \\ &= (c - (a \vee b))^+\Big(\sum_{0 \le k \le p-1} c^k (a \vee b)^{p-1-k}\Big) \ge 0. \end{aligned}$$

Hence $a^p \vee_p b^p$ exists in $\Pi_p(A)$ and satisfies

$$a^p \vee_p b^p = (a \vee b)^p.$$

We conclude that $\Pi_p(A)$ is a vector lattice.

Prove now that $\Pi_p(A)$ is a semiprime f-algebra. Let $a, b \in A$ such that $a^p \wedge_p b^p = (a \wedge b)^p = 0$, so

$$a \wedge b \in N(A).$$

So

$$(a \wedge b)(ab)c = 0$$

for all $c \in A$. Since A is a generalized almost f-algebra, so

$$(a - a \wedge b)(b - a \wedge b)(ab) = 0$$

so

$$(ab)^2 = 0.$$

We get

$$a^p b^p = (ab)^p = 0.$$

This implies that $\Pi_p(A)$ is an almost f-algebra. It is easy to prove that $\Pi_p(A)$ is semiprime. So $\Pi_p(A)$ is a semiprime f-algebra, and the proof is complete. ∎

Now, let A be a uniformly complete generalized almost f-algebra and B be a semiprime f-algebra. If $T : A \to B$ is an algebra homomorphism, then T is disjointness preserving, that is, if $|a| \wedge |b| = 0$ in A then $|Ta| \wedge |Tb| = 0$. Indeed, $|a| \wedge |b| = 0$ implies $abc = 0$ for all $c \in A$, so $(ab)^2 = 0$, so $(T(ab))^2 = T(ab)^2 = 0$. Since B is semiprime, we get $T(ab) = T(a)T(b) = 0$. So, $|Ta| \wedge |Tb| = 0$ and thus the algebra homomorphism $T : A \to B$ is a lattice homomorphism if and only if T is positive.

Theorem 4. *Let A be a uniformly complete generalized almost f-algebra and B be an Archimedean semiprime f-algebra. Then any algebra homomorphism $T : A \to B$ is a lattice homomorphism.*

Proof. By the above remark, it suffices to prove that $T \geq 0$. Let $a \in A^+$, Hence $a^5 \in A^+$ then there exists $b \in A^+$ such that $a^5 = b^4$ (see Theorem 3). Consequently $(Ta)^5 = (Tb)^4 \geq 0$. However, $(Ta)^5 = ((Ta)^+)^5 - ((Ta)^-)^5 \geq 0$, where we use $(Ta)^+(Ta)^- = 0$. It follows from $((Ta)^+)^5 \wedge ((Ta)^-)^5 = 0$ that the above decomposition of $(Ta)^5$ into positive elements is minimal and hence $((Ta)^-)^5 = ((Ta)^5)^- = 0$. Since B is semiprime, we obtain $(Ta)^- = 0$, that is, $Ta \geq 0$. The proof is complete. ∎

5 Generalized multiplicators in generalized almost f-algebras

We call a linear operator T on an algebra A a *multiplicator* after Scheffold [12] if

$$xT(y) = T(x)y \text{ for all } x, y \in A.$$

Using his representation Theorem of almost f-algebras on $C(\Omega)$-spaces, Scheffold proved in [12] that any orthomorphism on a Banach almost f-algebra automatically is an order bounded multiplicator. In the present section we intend to make some contributions to this area. In spite of that, we start our study with a generalization of multiplicator operators. A linear operator T on an algebra A is said to be a *generalized multiplicator* if

$$xT(y) - T(x)y \in ann(A) \text{ for all } x, y \in A.$$

First, let $\mathrm{Mult}_b(A)$ denote the set of all order bounded multiplicators on an ℓ-algebra A and $\mathrm{GMult}_b(A)$ denote the set of all order bounded generalized multiplicator on an ℓ-algebra A.

The main topic of the following result is to establish a relationship between the set of orthomorphisms and the set of order bounded generalized multiplicators.

We plunge into the matter with the following Proposition.

Proposition 5. *Let A be an Archimedean generalized almost f-algebra. Then any orthomorphism on A is an order bounded generalized multiplicator on A.*

Proof. It is clear that we may prove the result only for positive orthomorphisms. Let $0 \leq T \in \mathrm{Orth}(A)$, $c \in A^+$ and define the bilinear map Ψ from $A \times A$ into A by

$$\Psi(x, y) = xT(y)c \text{ for all } x, y \in A.$$

Obviously, Ψ is positive and if $x \wedge y = 0$ then $x \wedge Ty = 0$ and therefore

$$\Psi(x, y) = xT(y)c = 0.$$

This implies that Ψ is symmetric. Hence

$$xT(y)c = \Psi(x, y) = \Psi(y, x) = yT(x)c = T(x)yc$$

and then

$$(xT(y) - T(x)y)c = 0$$

for all $x, y, c \in A$. We derive that T is a generalized multiplicator on A and the proof of the Proposition is complete. ■

Next, we give a characterization of positive generalized multiplicators on a generalized almost f-algebra, which turns out to be useful for later purposes.

Proposition 6. *Let A be an Archimedean generalized almost f-algebra, T a positive operator on A. Then the following are equivalent.*

(i) $T \in \mathrm{GMult}_b(A)$,

(ii) If $x \wedge y = 0$ then $T(x)y \in ann(A)$.

Proof. Assume that $T \in \mathrm{GMult}_b(A)$ and $x \wedge y = 0$. Then $xyT(z) = 0$ for all $z \in A$ (because A is an Archimedean generalized almost f-algebra). Since $T \in \mathrm{GMult}_b(A)$, we obtain

$$T(x)yz = T(x)zy = xT(z)y = 0 \text{ for all } z \in A.$$

Consequently, $T(x)y \in ann(A)$. Therefore, if $0 \leq T$ and $x \wedge y = 0$, then $T(x)y \in ann(A)$. Let $z \in A^+$ and define the bilinear map Φ from $A \times A$ into A by $\Phi(x, y) = T(x)yz$. It is straightforward to show that Φ is a symmetric. Hence $T(x)yz = xT(y)z$ for all $x, y \in A$ and $z \in A^+$. So

$$T(x)y - xT(y) \in ann(A).$$

We derive that $T \in \mathrm{GMult}_b(A)$. ■

As an immediate application we obtain the following result, the proof of which is left to the reader.

Corollary 1. *If A is an Archimedean semiprime f-algebra then*

$$\mathrm{Orth}(A) = \mathrm{Mult}_b(A).$$

Assume now that A is an Archimedean generalized almost f-algebra and $T \in \mathrm{GMult}_b(A)$. Let $s : A \to A/N(A)$ be the canonical surjection. By Proposition 14, s is a lattice and algebra homomorphism. Now let $x \in N(A)$ then $xT(y) \in N(A)$ for all $y \in A$ (because $N(A)$ is a ring ideal). On the other hand, $xT(y) - T(x)y \in N(A)$ and thus $T(x)y \in N(A)$ for all $y \in A$. If one takes $y = T(x)$ then $T(x) \in N(A)$. So, we can define the operators $\widetilde{T}$ from $A/N(A)$ into itself by putting $\widetilde{T}(s(x)) = s(T(x))$. Obviously, $\widetilde{T} \in \mathrm{Mult}_b(A/N(A))$. Since $A/N(A)$ is a semiprime f-algebra, the preceding Corollary, yields that $\widetilde{T}$ is an orthomorphism of $A/N(A)$. This leads to the following result.

Corollary 2. *If $T \in GMult_b(A)$ then $\widetilde{T} \in \mathrm{Orth}(A/N(A))$.*

As an immediate application of preceding result, we prove that $\mathrm{GMult}_b(A)$ is an algebra whenever A is a generalized almost f-algebra and $ann(A) = N(A)$.

Theorem 5. *Let A be an Archimedean generalized almost f-algebra such that $ann(A) = N(A)$. Then $\mathrm{GMult}_b(A)$ is an ordered subalgebra of $\mathcal{L}_b(A)$.*

Proof. We only prove that $\mathrm{GMult}_b(A)$ is closed under composition of operators. Let $R, T \in \mathrm{GMult}_b(A)$, then $\widetilde{R} = s \circ R$, $\widetilde{T} = s \circ T \in \mathrm{Orth}(A/N(A))$. Hence, $\widetilde{R} \circ \widetilde{T} \in \mathrm{Orth}(A/N(A))$=$\mathrm{Mult}_b(A/N(A))$. We derive that $R \circ T(x)y - xR \circ T(y) \in N(A)$ for all $x, y \in A$. Consequently, $R \circ T \in \mathrm{GMult}_b(A)$, which is the desired result. ∎

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