A note on admissibility for unbounded bilinear control systems

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Dedicated to Abdelhaq El Jai on the occasion of his 60th birthday

Abstract

This paper studies infinite-dimensional bilinear control systems described by $y'(t) = Ay(t) + u(t)By(t)$ where A generates a semigroup $(e^{tA})_{t\geq 0}$ on a Banach space Y (state space), $B : D(B)(\subset Y) \to Y$ is an unbounded linear operator and $u \in L^p_{loc}(0, \infty)$ is a scalar control. Sufficient conditions are given for B to be admissible, i.e for any t, the integral $\int_0^t u(s)e^{(t-s)A}By(s)ds$ should be in Y and depends continuously on $u \in L^p(0, \infty)$, $y \in L^q(0, \infty; Y)$ for some appropriate positive numbers p, q. This approach enables us to obtain, through an integrated form, a unique solution for the bilinear system. The results are applied to a heat equation.

1 Introduction

In this paper we deal with abstract infinite-dimensional bilinear control systems of the form

$$\begin{cases} y'(t) = Ay(t) + u(t)By(t), \\ y(0) = y_0, \end{cases} \tag{1.1}$$

where A generates a strongly continuous semigroup $(e^{tA})_{t\geq 0}$ on an infinite dimensional Banach space Y (state space) whose norm will be denoted by $\|.\|$, $B : D(B)(\subset Y) \to Y$ is a possibly unbounded linear operator and the control function $u(.)$ denotes the scalar control. We note that (1.1) is bilinear in the pair

Received by the editors March 2007.

Communicated by P. Godin.

2000 *Mathematics Subject Classification* : 93C25, 93C20.

Key words and phrases : Infinite-dimensional systems, unbounded bilinear control systems, admissibility.

(u, y) and in this case, its possible solution is a nonlinear function of u. Recall that when the operator B is bounded, it has been established in [1] that if $u \in L^1(0, T)$ $(T > 0)$, then (1.1) admits a unique solution $y \in C([0, T]; Y)$. Technically, this result has been obtained by applying the contraction mapping principle to the integrated form of (1.1) given by

$$y(t) = e^{tA}y_0 + \int_0^t u(s)e^{(t-s)A}By(s)ds. \tag{1.2}$$

Here we shall suppose that B is unbounded in the sense that it is bounded from Y to some larger Banach space $X \supset Y$. This fact often occurs naturally when the control is exercised through the boundary or a point for systems governed by partial differential equations (PDEs). Furthermore, we shall suppose that A admits an extension, still denoted by A, which generates a strongly continuous semigroup, still denoted by $(e^{tA})_{t \geq 0}$, on X. Hence for any $y \in Y$

$$\left\| e^{tA}By \right\|_X \leq M \|B\|_{\mathcal{L}(Y,X)} e^{\omega t} \|y\|_X \tag{1.3}$$

for some constants $M \geq 1$, $\omega \geq 0$ so that given $t > 0$, $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$, we have

$$\int_0^t u(s)e^{(t-s)A}By(s)ds \in X, \text{ for all } u \in L^p(0, \infty),\ y \in L^q(0, \infty; Y) \tag{1.4}$$

provided that

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{1.5}$$

This enables us to introduce the bilinear operator

$$\Phi_t : L^p_{loc}(0, \infty) \times L^q_{loc}(0, \infty; Y) \to X$$

given by

$$\Phi_t(u, y) = \int_0^t u(s)e^{(t-s)A}By(s)ds. \tag{1.6}$$

Inspired by the concept of admissibility developed in [10] for unbounded linear control systems, Idrissi has introduced in [6] the following notion of admissibility: the operator B is said to be p-admissible if $\Phi_t(u, y) \in Y$ for all $u \in L^p_{loc}(0, \infty)$, $y \in L^q_{loc}(0, \infty; Y)$ where p and q satisfy (1.5). Unfortunately, it appears along the results obtained in [6] that this definition is too restrictive and it seems hard to find examples of admissible operators enabling us to treat significant examples. In this paper, we shall introduce a new definition of admissibility for the operator B by avoiding the constraint (1.5). Indeed, we can easily deduce by using the generalized Hölder inequality that $\Phi_t(u, y) \in X$ provided that

$$\frac{1}{p} + \frac{1}{q} \leq 1. \tag{1.7}$$

It follows that the constraint (1.5) can be dispensed with. This enables us to introduce a new definition which will give advances in the following directions: (i)

The system (1.1) will admit a unique solution $y \in C([0,\infty);Y)$. (ii) The resulting approach gives an alternative way to study systems (such as the PDE considered in the application below) which are not covered by the treatises concerned with abstract evolution equations (see for instance [9]). (iii) Challenging open problems relative to the control aspect for such systems will be interesting to investigate. Let us mention for instance the problem of stabilization which consists of choosing an appropriate feedback

$$u(t) = \mathcal{F}(y(t))$$

such that the solution of the resulting feedback system satisfies in some sense $y(t) \to 0$ as $t \to \infty$. If we formally compute the time rate of change of the "energy" :

$$\frac{1}{2}\frac{d}{dt}\|y(t)\|^2 = \langle Ay(t), y(t)\rangle + u(t)\langle By(t), y(t)\rangle \tag{1.8}$$

and assuming that the semigroup is of contraction so that $\langle A\varphi, \varphi\rangle \le 0$ for all $\varphi \in D(A)$, we get

$$\frac{1}{2}\frac{d}{dt}\|y(t)\|^2 \le u(t)\langle By(t), y(t)\rangle. \tag{1.9}$$

In order to make the energy nonincreasing, a natural choice for u is

$$u(t) = -\langle By(t), y(t)\rangle. \tag{1.10}$$

Hence the resulting closed-loop system is

$$\begin{cases} y'(t) = Ay(t) - \langle By(t), y(t)\rangle By(t), \\ y(0) = y_0. \end{cases} \tag{1.11}$$

When B is bounded, this feedback system has been studied in [2], [3] where weak and strong stability results were obtained. Moreover, in [5] a decay estimate of the energy has been established. When B is unbounded, a first difficulty arises concerning the meaning of the scalar product $\langle By(t), y(t)\rangle$ and the well-posedness of the system (1.11). Also, we mention the problem of controllability which can be formulated as follows. The system (1.1) is said to be exactly controllable in Y in time $T > 0$ if for any y_0, $y_d \in Y$ there is a control u in some space to be specified such that the solution of (1.1) satisfies $y(T) = y_d$ (desired state). The requirement $y(T) = y_d$ can be relaxed to various notions and degrees of controllability: approximate controllability, null exact controllability (see [4, Vol. 2]). This problem has been studied in [1] when B is bounded.

The plan of the paper is as follows. In section 2, we introduce a new notion of admissibility and state the main results. In section 3, we treat a system governed by the heat equation. Throughout the paper, the norm of any other Banach space, say W, will be specified by $\|.\|_W$. Moreover, we shall denote by C a generic positive constant which may be different at different occurrences. Whenever such a constant depends on some parameter, say r, this fact will be pointed out by C_r. Furthermore, given a function f defined on $[0,\infty)$ and $\tau > 0$, $f(\tau + .)$ denotes the translated function such that $f(\tau + .)(s) = f(\tau + s)$.

2 Main results

In the sequel, p and q are real numbers satisfying (1.7). Then we introduce
Definition 2.1. *The operator B is (p,q)-admissible with respect to the semigroup $(e^{tA})_{t\geq 0}$ if for any $t > 0$, the operator Φ_t defined by (1.6) is bilinear bounded from $L^p(0,\infty) \times L^q(0,\infty;Y)$ to Y.*

Since the semigroup $(e^{tA})_{t\geq 0}$ is clear from the context, we shall simply say that B is (p,q)-admissible. Moreover, the inequality (1.3) combined with (1.7) ensures that for any $t > 0$, the operator Φ_t given by (1.6) is bilinear bounded from $L^p(0,\infty) \times L^q(0,\infty;Y)$ to X. Then the admissibility can be characterized as follows:
Proposition 2.2. *The following conditions are equivalent:*
(i) B is (p,q)-admissible.
(ii) $\Phi_t(u,y) \in Y$ for any $t > 0$ and any $(u,y) \in L^p_{loc}(0,\infty) \times L^q_{loc}(0,\infty;Y)$.
(iii) $\Phi_t(u,y) \in Y$ for any $t > 0$ and any $(u,y) \in L^p(0,\infty) \times L^q(0,\infty;Y)$.

Proof. (i)⇒(ii) Let $t > 0$ and consider $(u,y) \in L^p_{loc}(0,\infty) \times L^q_{loc}(0,\infty;Y)$. We introduce the projection operator given by

$$P_t : L^p_{loc}(0,\infty) \times L^q_{loc}(0,\infty;Y) \to L^p(0,\infty) \times L^q(0,\infty;Y),$$

$$P_t(u,y)(s) = \begin{cases} (u(s),y(s)) \text{ if } 0 \leq s \leq t \\ (0,0) \text{ otherwise} \end{cases}. \tag{2.1}$$

The boundedness of the operator Φ_t from $L^p(0,\infty) \times L^q(0,\infty;Y)$ to X gives

$$\Phi_t(u,y) = \Phi_t[P_t(u,y)] \text{ in } X.$$

On the other hand, since $P_t(u,y) \in L^p(0,\infty) \times L^q(0,\infty;Y)$ and B is admissible, we have $\Phi_t[P_t(u,y)] \in Y$ so that $\Phi_t(u,y) \in Y$.
(ii)⇒(iii) The deduction is trivial from the inclusion

$$L^p(0,\infty) \times L^q(0,\infty;Y) \subset L^p_{loc}(0,\infty) \times L^q_{loc}(0,\infty;Y).$$

(iii)⇒(i) For any $t > 0$, the boundedness of the operator Φ_t from $L^p(0,\infty) \times L^q(0,\infty;Y)$ to X combined with (iii) and the closed graph theorem implies that Φ_t is bilinear bounded from $L^p(0,\infty) \times L^q(0,\infty;Y)$ to Y. This completes the proof of the proposition. ∎

Remark 2.1. From above, it is easy to see that B is (p,q)-admissible if, and only if, for any $t > 0$, there exists some positive constant C_t such that for all $u \in L^p_{loc}(0,\infty)$, $y \in L^q_{loc}(0,\infty;Y)$

$$\|\Phi_t(u,y)\| \leq C_t \|u\|_{L^p(0,t)} \|y\|_{L^q(0,t;Y)}. \tag{2.2}$$

Furthermore, the following proposition shows that the constants C_t can be chosen so as to satisfy $C_{t'} \leq C_t$ if $t' \leq t$.

Proposition 2.3. *Suppose that B is (p,q) admissible. Denote by $\|\Phi_t\|$ the norm of the continuous bilinear operator $\Phi_t : L^p(0,\infty)\times L^q(0,\infty;Y) \longrightarrow Y$ obtained by taking the supremum of $\|\Phi_t(u,y)\|$ for*

$$\|u\|_{L^p(0,\infty)} \le 1,\ \|y\|_{L^q(0,\infty;Y)} \le 1.$$

Then $\|\Phi_t\|$ is nondecreasing as a function of t.

Proof. Suppose that $t > t'$, $t' = t-\tau$, then

$$\Phi_{t'}(u,y) = \int_0^{t'} u(s)e^{(t'-s)A}By(s)ds = \int_0^{t-\tau} u(s)e^{(t-\tau-s)A}By(s)ds.$$

The change of variable $s+\tau=\sigma$ gives

$$\Phi_{t'}(u,y) = \int_\tau^t u(\sigma-\tau)e^{(t-\sigma)A}By(\sigma-\tau)d\sigma$$

$$= \int_0^t \widetilde{u}(\sigma)e^{(t-\sigma)A}B\widetilde{y}(\sigma)d\sigma$$

with

$$\widetilde{u}(\sigma) = \begin{cases} 0 \text{ if } 0\le\sigma<\tau \\ u(\sigma-\tau) \text{ otherwise} \end{cases},$$

$$\widetilde{y}(\sigma) = \begin{cases} 0 \text{ if } 0\le\sigma<\tau \\ y(\sigma-\tau) \text{ otherwise} \end{cases}.$$

Hence, we get

$$\|\Phi_{t'}(u,y)\| = \|\Phi_t(\widetilde{u},\widetilde{y})\|.$$

On the other hand, from $\|u\|_{L^p(0,\infty)} = \|\widetilde{u}\|_{L^p(0,\infty)}$, $\|y\|_{L^q(0,\infty;Y)} = \|\widetilde{y}\|_{L^q(0,\infty;Y)}$, we obtain

$$\|\Phi_{t'}(u,y)\| = \|\Phi_t(\widetilde{u},\widetilde{y})\| \le \|\Phi_t\|\,\|u\|\,\|y\|$$

so that $\|\Phi_{t'}\| \le \|\Phi_t\|$. This completes the proof of the proposition. ∎

In order to make precise the solution for (1.1), we introduce
Definition 2.4. *A function $y \in C([0,\infty);Y)$ is called a mild solution of (1.1) if for any $t>0$, y satisfies (1.2) in Y.*
Then we have:
Theorem 2.5. *Suppose that B is (p,q)-admissible with p, q satisfying*

$$1<p<\infty,\ 1<q<\infty.$$

Then for any $y_0 \in Y$, $u \in L^p_{loc}(0,\infty)$, the system (1.1) admits a unique mild solution $y \in C([0,\infty);Y)$.

Proof. Let $0 < T < 1$, $R > 0$ and consider the set

$$F = \{y \in C([0,T];Y) \mid \|y(t) - y_0\| \le R,\ 0 \le t \le T\}. \tag{2.3}$$

We consider for $u \in L^p_{loc}(0,\infty)$ the mapping $f_u : F \to C([0,T];Y)$ defined by

$$f_u(y)(t) = e^{tA}y_0 + \int_0^t u(s)e^{(t-s)A}By(s)ds. \tag{2.4}$$

Then f_u is well defined. Indeed, we have for $t' > t$, $t' = t + \tau$

$$\|f_u(y)(t') - f_u(y)(t)\| \le \left\|e^{t'A}y_0 - e^{tA}y_0\right\| + \|\Phi_{t'}(u,y) - \Phi_t(u,y)\| \tag{2.5}$$

Obviously $\left\|e^{t'A}y_0 - e^{tA}y_0\right\| \to 0$ as $\tau \to 0$. Furthermore, we get the following decomposition

$$\Phi_{t'}(u,y) - \Phi_t(u,y) = \int_0^{t+\tau} u(s)e^{(t+\tau-s)A}By(s)ds - \int_0^t u(s)e^{(t-s)A}By(s)ds$$

$$= e^{tA}\int_0^\tau u(s)e^{(\tau-s)A}By(s)ds + \int_0^t u(s+\tau)e^{(t-s)A}By(s+\tau)ds - \int_0^t u(s)e^{(t-s)A}By(s)ds$$

$$= e^{tA}\int_0^\tau u(s)e^{(\tau-s)A}By(s)ds + \int_0^t (u(s+\tau) - u(s))e^{(t-s)A}By(s+\tau)ds$$

$$+ \int_0^t u(s)e^{(t-s)A}B(y(s+\tau) - y(s))ds.$$

From the admissibility of B it follows that for τ small enough so as $T + \tau < 1$, we obtain by virtue of Remark 2.1

$$\|\Phi_{t'}(u,y) - \Phi_t(u,y)\| \le C_1\{\|u\|_{L^p(0,\tau)}\|y\|_{L^q(0,\tau;Y)}$$

$$+ \|u(.+\tau) - u\|_{L^p(0,1)}\|y(.+\tau)\|_{L^q(0,1;Y)} + \|u\|_{L^p(0,1)}\|y(.+\tau) - y\|_{L^q(0,1;Y)}\}$$

for some positive constant C_1. It is easy to see that $\|u\|_{L^p(0,\tau)} \to 0$ and $\|y\|_{L^q(0,\tau;Y)} \to 0$ as $\tau \to 0$. On the other hand, by using the dominated convergence theorem and a density argument we get $\|u(.+\tau) - u\|_{L^p(0,1)} \to 0$ and $\|y(.+\tau) - y\|_{L^q(0,1;Y)} \to 0$ as $\tau \to 0$. This yields $f_u(y) \in C([0,T];Y)$. Furthermore, f_u maps F to F provided that

$$\left\|e^{tA}y_0 - y_0\right\| + C_1\|u\|_{L^p(0,T)}\|y\|_{L^q(0,T;Y)} \le R \text{ for all } 0 \le t \le T. \tag{2.6}$$

From

$$\|y\|_{L^q(0,T;Y)} \le (\|y_0\| + R)T^{\frac{1}{q}},$$

inequality (2.6) is verified if

$$\left\|e^{tA}y_0 - y_0\right\| + C_1\|u\|_{L^p(0,T)}(\|y_0\| + R)T^{\frac{1}{q}} < R \text{ for all } 0 \le t \le T. \tag{2.7}$$

This condition is satisfied for T small enough. Moreover, f_u is a contraction map from F to F if there exists some constant $0 < k < 1$ such that for any $y, \widetilde{y} \in F$

$$\|f_u(y)(t) - f_u\widetilde{y}(t)\| < k\,\|y - \widetilde{y}\|_{C(0,T;Y)} \text{ for all } 0 \leq t \leq T. \tag{2.8}$$

From

$$\|f_u(y)(t) - f_u\widetilde{y}(t)\| = \left\| \int_0^t u(s)e^{(t-s)A}B(y(s) - \widetilde{y}(s))ds \right\|$$
$$\leq C_1\,\|u\|_{L^p(0,T)}\,T^{\frac{1}{q}}\,\|y - \widetilde{y}\|_{C(0,T;Y)}$$

we deduce that (2.8) holds again for T sufficiently small. By applying the contraction mapping principle, we conclude that for T small enough, the system (1.1) admits a unique solution $y \in C([0,T]\,;Y)$. Let $[0, T_{max})$ be the maximal interval where the solution exists. We shall see that $T_{max} = \infty$. Indeed, suppose that $T_{max} < \infty$ and let $(t_n)_n$ be an increasing sequence such that $t_n \to T_{max}$ as $n \to \infty$. By proceeding as for (2.5), we obtain for $m < n$

$$\|y(t_n) - y(t_m)\| \leq \left\| e^{t_nA}y_0 - e^{t_mA}y_0 \right\| + \|\Phi_{t_n}(u,y) - \Phi_{t_m}(u,y)\|$$

and consequently $\|y(t_n) - y(t_m)\| \to 0$ as $m, n \to \infty$. Thus, $\lim y(t)$ as $t \to T_{max}$ exists and $y(t)$ can be extended continuously beyond $t = T_{max}$ in a standard way. Hence, $T_{max} = \infty$ and this completes the proof of the theorem. ∎

3 Application

Let Ω be an open bounded domain in $\mathbb{R}^N$ with sufficiently smooth boundary Γ. We consider the bilinear system given by the following heat equation

$$\frac{\partial y}{\partial t} = \Delta y + u(t)(-\Delta)^{\frac{1}{2}}y \text{ on } (0,\infty) \times \Omega, \tag{3.1}$$

$$y = 0 \text{ on } (0,\infty) \times \Gamma, \tag{3.2}$$

$$y(0,x) = y_0(x) \text{ on } \Omega. \tag{3.3}$$

This system has the form (1.1) if we set $Y = L^2(\Omega)$, $B = (-\Delta)^{\frac{1}{2}}$ and

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega),\ Ay = \Delta y. \tag{3.4}$$

Let $0 < \beta_1 < \beta_2 < ... < \beta_n < ...$ be the set of eigenvalues of $-A$ and denote by $\{\varphi_n\}_n$ the corresponding orthonormal basis in $L^2(\Omega)$. Then the semigroup e^{tA} is given explicitly by

$$e^{tA}y = \sum_n \langle y, \varphi_n\rangle\, e^{-\beta_n t}\varphi_n \text{ for all } y \in L^2(\Omega), \tag{3.5}$$

where $\langle .,. \rangle$ denotes the scalar product in $L^2(\Omega)$. Moreover, the operator B can be expressed by

$$By = \sum_n \langle y, \varphi_n\rangle\, \sqrt{\beta_n}\varphi_n. \tag{3.6}$$

Clearly B is unbounded on $L^2(\Omega)$ and it is bounded from $L^2(\Omega)$ onto the space X defined as the completion of $L^2(\Omega)$ for the norm defined by

$$\|y\|_X^2 = \sum_n \frac{(\langle y, \varphi_n \rangle)^2}{\beta_n}. \tag{3.7}$$

Recall that the space $D((-A)^{\frac{1}{2}})$ is normed by

$$\|y\|^2_{D((-A)^{\frac{1}{2}})} = \sum_n \beta_n (\langle y, \varphi_n \rangle)^2. \tag{3.8}$$

It is easy to see that the space X can be interpreted as the dual space of $D((-A)^{\frac{1}{2}})$ with respect to the $L^2(\Omega)$-topology, the space $L^2(\Omega)$ being the pivot space. Furthermore, the restriction (respectively the extension) of the operator A to $D((-A)^{\frac{1}{2}})$ (respectively X) generates a strongly continuous semigroup still denoted by $(e^{tA})_{t\geq 0}$ (see [4, Vol. 1, p. 111]).

The system (3.1)-(3.3) is an example of fractional equation of diffusion type. Such systems are useful models for the description of transport processes in complex systems, slower than the Brownian diffusion. As systems displaying such anomalous behaviour, let us mention the charge carrier transport in amorphous semiconductors, the nuclear magnetic resonance diffusometry in percolative and porous media etc (see [7], [8]). Then the following admissibility result holds.

Theorem 3.1. *Let p, q be real numbers satisfying* $1 < p < \infty$, $1 < q < \infty$ *and*

$$\frac{1}{p} + \frac{1}{q} < \frac{1}{2}. \tag{3.9}$$

Then the operator $(-\Delta)^{\frac{1}{2}}$ is (p, q)-admissible.

Proof. Let $u \in L^p_{loc}(0, \infty)$, $y \in L^q_{loc}(0, \infty; L^2(\Omega))$ and $t > 0$. We set

$$y_n(s) = \langle y(s), \varphi_n \rangle\,, \tag{3.10}$$

so that by adopting the same notations as in (1.6) we get

$$\begin{aligned}\Phi_t(u, y) &= \int_0^t u(s) e^{(t-s)A} (-\Delta)^{\frac{1}{2}} y(s) ds \\ &= \int_0^t u(s) \sum_n \sqrt{\beta_n} y_n(s) e^{-\beta_n (t-s)} \varphi_n ds.\end{aligned}$$

This yields

$$\|\Phi_t(u, y)\|^2 \leq (\int_0^t |u(s)| \left\| \sum_n \sqrt{\beta_n} y_n(s) e^{-\beta_n (t-s)} \varphi_n \right\| ds)^2. \tag{3.11}$$

On the other hand, we have for all $0 < s < t$

$$\left\| \sum_n \sqrt{\beta_n} y_n(s) e^{-\beta_n (t-s)} \varphi_n \right\|^2 = \sum_n \beta_n \left| y_n(s) \right|^2 e^{-2\beta_n (t-s)}$$

$$= \frac{1}{2(t-s)} \sum_n 2\beta_n (t-s) \left| y_n(s) \right|^2 e^{-2\beta_n (t-s)}.$$

Let M denote a positive constant such that

$$se^{-s} \leq M \text{ for all } s > 0.$$

Then it is easy to see that

$$\left\| \sum_n \sqrt{\beta_n} y_n(s) e^{-\beta_n (t-s)} \varphi_n \right\| \leq \frac{\sqrt{M}}{\sqrt{(t-s)}} \left\| y(s) \right\|.$$

This implies

$$\left\| \Phi_t(u,y) \right\| \leq \sqrt{M} \int_0^t \frac{|u(s)|}{\sqrt{(t-s)}} \left\| y(s) \right\| ds. \tag{3.12}$$

Let us consider the real $r > 0$ defined by

$$\frac{1}{r} = 1 - \frac{1}{p} - \frac{1}{q}. \tag{3.13}$$

Then we have

$$1 < r < 2$$

and this ensures the convergence of the integral

$$\int_0^t \frac{ds}{\sqrt{(t-s)}^r} = \frac{2}{2-r} t^{1-\frac{r}{2}}.$$

Moreover, by Hölder inequality we get

$$\left\| \Phi_t(u,y) \right\| \leq \sqrt{M} \left(\frac{2}{2-r} \right)^{\frac{1}{r}} t^{\frac{1}{r} - \frac{1}{2}} \left\| u \right\|_{L^p(0,t)} \left\| y \right\|_{L^q(0,t;Y)}. \tag{3.14}$$

Thus, taking into account Remark 2.1, we deduce that $(-\Delta)^{\frac{1}{2}}$ is (p,q)-admissible. This completes the proof of the theorem. ■

From this theorem, we obtain the following existence result for the solution of the system (3.1)-(3.3).

Corollary 3.2. *Let p be a real number such that $2 < p < \infty$. Then for any $y_0 \in L^2(\Omega)$, $u \in L^p_{loc}(0,\infty)$, there exists a unique solution $y \in C([0,\infty); L^2(\Omega))$ to the system (3.1)-(3.3).*

Proof. Let us consider a positive real number q such that (3.9) holds. From Theorem 3.1 we deduce that $(-\Delta)^{\frac{1}{2}}$ is (p,q)-admissible. Then we can easily conclude by using Theorem 2.5. ■

In order to show that the conjugacy of the parameters p,q is too restrictive, we consider in detail the case of the one-dimensional heat equation ($N = 1$).

Proposition 3.3. *Suppose that $\Omega = (0,1)$. Let p,q be positive real conjugate numbers. Then the operator $(-\Delta)^{\frac{1}{2}}$ is not (p,q)-admissible.*

Proof. The eigenvalues $\{\beta_n\}_n$ and the corresponding orthonormal basis of eigenfunctions $\{\varphi_n\}_n$ are given by

$$\beta_n = n^2\pi^2;\ n = 1, 2, ..., \tag{3.15}$$

$$\varphi_n(x) = \sqrt{2} sin(n\pi x);\ n = 1, 2, ... \tag{3.16}$$

Let z be the vector in $L^2(\Omega)$ defined by

$$z = \sum_n \frac{\varphi_n}{n\pi} \tag{3.17}$$

and consider for $t > 0, \epsilon > 0$ the following functions

$$u(s) = \begin{cases} \frac{1}{(t-s)^{\frac{1}{p}-\epsilon}} \text{ if } 0 \leq s < t \\ 0 \text{ otherwise} \end{cases}, \tag{3.18}$$

$$v(s) = \begin{cases} \frac{1}{(t-s)^{\frac{1}{q}-\epsilon}} \text{ if } 0 \leq s < t \\ 0 \text{ otherwise} \end{cases}, \tag{3.19}$$

$$y(s) = v(s)z. \tag{3.20}$$

Clearly, $u \in L^p(0, \infty)$, $y \in L^q(0, \infty; L^2(\Omega))$ for any $\epsilon > 0$ and

$$\Phi_t(u, y) = \int_0^t u(s)v(s)e^{(t-s)A}(-\Delta)^{\frac{1}{2}} z ds$$

$$= \int_0^t u(s)v(s) \left\{ \sum_n e^{-n^2\pi^2(t-s)} \varphi_n \right\} ds.$$

Since for all $0 \leq s \leq t$

$$\left\| \sum_n e^{-n^2\pi^2(t-s)} \varphi_n \right\|^2_{D(A^{\frac{1}{2}})'} \leq \sum_n \frac{1}{n^2\pi^2},$$

the dominated convergence theorem implies the following equality in $D((-A)^{\frac{1}{2}})'$

$$\Phi_t(u, y) = \sum_n \left\{ \int_0^t u(s)v(s)e^{-n^2\pi^2(t-s)} ds \right\} \varphi_n. \tag{3.21}$$

Let us consider the sequence

$$y_n(t) = \int_0^t u(s)v(s)e^{-n^2\pi^2(t-s)} ds. \tag{3.22}$$

The fact that p and q are conjugate yields

$$y_n(t) = \int_0^t \frac{1}{(t-s)^{1-2\epsilon}} e^{-n^2\pi^2(t-s)} ds. \tag{3.23}$$

The change of variable defined by $t-\sigma = n^2\pi^2(t-s)$ gives

$$y_n(t) = \frac{1}{n^{4\epsilon}\pi^{4\epsilon}} \int_{t-n^2\pi^2 t}^{t} \frac{1}{(t-\sigma)^{1-2\epsilon}} e^{-(t-\sigma)} d\sigma. \tag{3.24}$$

Thus we get for some constant $C > 0$

$$|y_n(t)|^2 \geq \frac{C}{n^{8\epsilon}};\ n = 1, 2, ... \tag{3.25}$$

Hence $\|\Phi_t(u,y)\|^2_{L^2(\Omega)} = \infty$ provided that $\epsilon \leq \frac{1}{8}$. This completes the proof of the proposition. ■

Acknowledgments

The author is grateful to the anonymous referee for his valuable comments and suggestions in revising the paper. This work is partially supported by Académie Hassan II des Sciences et Techniques; Collège Sciences de la Modélisation et de l'Information; Réseau Théorie des Systèmes.

References

[1] BALL, J.M, MARSDEN, J.E and SLEMROD, M., Controllability for distributed bilinear control systems. *SIAM J. Control Optim.* 20, pp. 575-597, 1982.

[2] BALL, J.M and SLEMROD, M., Feedback stabilization of semilinear control systems. *Appl. Math. Optim.* 5, pp. 169-179, 1979.

[3] BALL, J.M and SLEMROD, M., Nonharmonic Fourier series and stabilization of distributed semilinear control systems. *Comm. Pure Appl. Math.* 32, pp. 555-587, 1979.

[4] BENSOUSSAN, A., DA PRATO, G., DELFOUR, M.C and MITTER, S.K, *Representation and control of infinite dimensional systems*, Vol. 1, 2, Birkhauser, 1992.

[5] BERRAHMOUNE, L., Stabilization and decay estimate for distributed bilinear systems. *Systems and Control Letters* 36, pp. 167-171, 1999.

[6] IDRISSI, A., On the unboundedness of control operators for bilinear systems. *Question. Math.* 26, pp. 105-123, 2003.

[7] KILBAS, A.A, SRIVASTAVA, H.M and TRUJILLO, J.J, *Fractional differential equations: an emergent field in applied and mathematical science*, in Factorisation, Singular operators and relative problems (FSORP'02), S. Samka, A. Lebre and A.F dos Santos (Eds), Kluwer Acad. Pub., Boston, pp. 151-173, 2002.

[8] METZLER, R. and KLAFTER, J., The random walk's guide to anomalous diffusion: a fractional dynamic approach, *Physics Reports* 339, pp. 1-77, 2000.

[9] PAZY, A., *Semigroups of linear operators and applications to partial differential equations*, Springer Verlag, 1983.

[10] WEISS, G., Admissibility of unbounded control operators. *SIAM J. Control Optim.* 27, pp. 527-545, 1989.

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