# On nondiscreteness of a higher topological homotopy group and its cardinality

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**Abstract**

Here, we are going to extend Mycielski's conjecture to higher homotopy groups. Also, for an $(n-1)$-connected locally $(n-1)$-connected compact metric space $X$, we assert that $\pi_n^{top}(X)$ is discrete if and only if $\pi_n(X)$ is finitely generated. Moreover, $\pi_n^{top}(X)$ is not discrete if and only if it has the power of the continuum.

## 1 Introduction

In 1998, J. Pawlicowski [5] presented a forcing free proof of a conjecture of Mycielski [4] that the fundamental group of a connected locally connected compact metric space is either finitely generated or has the power of the continuum. In [1], Biss equipped the loop space of $X$ with the compact open topology. Then he put a canonical topology on the fundamental group of $X$ as a quotient of $Hom((S^1, 1), (X, x))$ which is invariant under the homotopy type of $X$ and denoted it by $\pi_1^{top}(X, x)$. He proved among the other things that $\pi_1^{top}(X, x)$ is a topological group which is independent of the base point. Recently, P. Fabel [2] using the Mycielski's conjecture, showed that if $X$ is a peano continuum, then either $X$ has a finitely generated discrete topological fundamental group, or it has a non-discrete topological fundamental group, having the power of the continuum. In [3], we et al. introduced a topology on higher homotopy groups of a pointed space $(X, x)$ as a quotient of $Hom((I^n, \dot{I}^n), (X, x))$ equipped with the

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compact-open topology and denoted it by $\pi_n^{top}(X, x)$. We proved that $\pi_n^{top}(X, x)$ is a topological group. Also, we found necessary and sufficient conditions for which the topology is discrete. In this note, we are going to extend Mycielski's conjecture to higher homotopy groups. At the end, we generalize Fabel's results for topological higher homotopy groups: Suppose $X$ is an $(n-1)$-connected locally $(n-1)$-connected compact metric space. Then $\pi_n^{top}(X)$ is discrete if and only if $\pi_n(X)$ is finitely generated. Moreover, $\pi_n^{top}(X)$ is not discrete if and only if $\pi_n(X)$ has the power of the continuum.

## 2 Main results

We recall a topological space $X$ is called *n-semilocally simply connected at a point x* if there exists an open neighborhood $U$ of $x$ for which any $n$-loop in $U$ is nullhomotopic in $X$. Moreover, $X$ is said to be *n-semilocally simply connected* if it is $n$-semilocally simply connected at each point (see [3]). A space $X$ is called *n-connected* for $n \geq 0$ if it is path connected and $\pi_k(X, x)$ is trivial for every base point $x \in X$ and $1 \leq k \leq n$. $X$ is called *locally n-connected* if for each $x \in X$ and each neighborhood $U$ of $x$, there is a neighborhood $V \subseteq U \subseteq X$ containing $x$ so that $\pi_k(V) \longrightarrow \pi_k(U)$ is zero map for all $0 \leq k \leq n$ and for all basepoint in $V$ (see [6]).

**Lemma 2.1.** *Suppose $X$ is an $(n-1)$-connected, locally $(n-1)$-connected compact metric space and $\pi_n(X)$ is not finitely generated. Then there exists $x \in X$ such that for each positive integer $m$, there exists an $n$-loop $f_m$ at $x$ with diameter $< 2^{-m}$ which is not nullhomotopic. In particular, $X$ is not $n$-semilocally simply connected at $x$.*

*Proof.* For simplicity, we prove the assertion for $n = 2$. Similar argument gives the result in general case. Suppose otherwise. Then for each $x \in X$, there exists $m(x) \in \mathbb{N}$ such that every 2-loop at $x$ which has diameter less than $2^{-m(x)}$ is homotopic to the constant loop at $x$. Let $U_x$ be an open ball containing $x$. By local 1-connectivity of $X$, for each $x \in X$ there is a path connected neighborhood $V_x$ containing $x$, so that $V_x \subseteq U_x$ and $\pi_1(V_x) \longrightarrow \pi_1(U_x)$ is zero map. Suppose $m(x)$ is sufficiently big and by shrinking $V_x$, if it is necessary, we may assume that $V_x$ has diameter $2^{-(m(x)+1)}$. Therefore each $k$-loop, $k = 1, 2$, contained in $V_x$ is nullhomotopic in $U_x$. Suppose that $W_x$ is a path connected neighborhood of $x$ with diameter less than $2^{-(m(x)+3)}$. Now by compactness of $X$, there is a finite cover of $X$ by subsets $W_i$'s, containing $x_i$, $i \leq N_0$, with the above property. If it is necessary, we can omit some $W_i$'s such that the number of remainders is at least to cover $X$. For each $i, j$ with $W_i \bigcap W_j \neq \phi$, fix a path $h_{ij}$ in $W_i \bigcup W_j$ going from $x_i$ to $x_j$. Note that any path $g$ from $x_i$ to $x_j$ which is contained in $W_i \bigcup W_j$ is homotopic to $h_{ij}$ in X. Indeed, suppose $m(x_i) \leq m(x_j)$. Then $g * (h_{ij})^{-1}$ is a 1-loop at $x_i$ with diameter $< 2^{-(m(x_i)+1)}$ contained in $V_i$, so $g * (h_{ij})^{-1}$ is homotopic to constant loop at $x_i$ in $U_i$. This homotopy gives a homotopy from $g$ to $h_{ij}$ in $U_i$. If $m(x_i) > m(x_j)$, consider $(h_{ij})^{-1} * g$, an 1-loop at $x_j$ (similar argument used in [5] for $n = 1$). Each four of these paths $\{h_{ij}, h_{jk}, h_{kl}, h_{li}\}$ with common vertices $\{x_i, x_j, x_k, x_l\}$ induce an inessential 1-loop $f : S^1 \longrightarrow X$. Indeed, suppose that

$m(x_i) \leq min\{m(x_j), m(x_k), m(x_l)\}$ (the argument for other cases is similar). We show that the diam($f$) $< 2^{-(m(x_i)+1)}$, where diam($f$) is the diameter of $f(S^1)$. For, we have

$$\begin{aligned} diam(f) &\leq diam(W_i) + diam(W_j) + diam(W_k) + diam(W_l) \\ &< 2^{-(m(x_i)+3)} + 2^{-(m(x_j)+3)} + 2^{-(m(x_k)+3)} + 2^{-(m(x_l)+3)} \\ &\leq 4(2^{-(m(x_i)+3)}) = 2^{-(m(x_i)+1)} \end{aligned}$$

So, $f$ is contained in $V_i$ and this implies that $f$ is nullhomotopic in $U_i$. By 1-connectivity of $X$, $f$ has a continuous extension over $E^2$. We denote the 2-cube $f(E^2)$ by $Y_{ijkl}$ if the family $\{h_{ij}, h_{jk}, h_{kl}, h_{li}\}$ of paths bound this cube. Note that any 2-cube $Y'_{ijkl}$ obtained by $g : E^2 \longrightarrow X$ which is contained in $W_i \bigcup W_j \bigcup W_k \bigcup W_l$ is homotopic to $Y_{ijkl}$. Indeed, suppose
$m(x_i) \leqslant min\{m(x_j), m(x_k), m(x_l)\}$. Then the 2-loop induced by $Y_{ijkl}$ and $Y'_{ijkl}$ is a 2-loop at $x_i$ with diameter less than $2^{-m(x_i+1)}$, and so it is homotopic to the constant loop at $x_i$ in $X$. An elementary manipulation of this homotopy gives a homotopy from $f$ to $g$. Now, connect each point $x_i \in W_i$ to $x_0$ by path $g_i$. Let $s_{ij}$ be a side of a cube with vertices $x_i$ and $x_j$. Then the paths $h_{ij}$ , $g_i$ and $g_j$ induce 1-loop $\lambda_{ij} : S^1 \longrightarrow X$. Again, by 1-connectivity of X, $\lambda_{ij}$ has a continuous extension over $E^2$. We denote the cube $\lambda_{ij}(E^2)$ by $Z_{ij}$. Each $Y$-cube together with associated $Z$-cubes induce an 2-loop. The 1-connectivity of $X$ implies that these 2-loops are uniquely determined up to homotopy. So we have a finite number of homotopy classes of these 2-loops and we denote them by $\alpha_1, ..., \alpha_N$. Moreover, corresponding to each $Y$-cube there is a homotopy class $\alpha_i$ which has the $Y$-cube as a side. Now, to get a contradiction, we show that $\pi_2(X)$ is generated by $\alpha_i$'s, $i = 1, ..., N$. For this, suppose that an 2-loop $\eta : (I^2, \dot{I}^2) \longrightarrow (X, x_0)$ is given. Let $\delta > 0$ be the Lebesgue's number of the covering $\{W_i : i = 1, ..., N_0\}$. We divide the cube $I^2$ to small subcubes $I_1, ..., I_l$ such that each $\eta(I_j)$ has diameter $< \delta$ , $j = 1, ..., l$ ; and so it is contained in some $W_{i_j}$. We denote $\eta(I_j)$ by $X_j$ and its corners by $v_{j_k}$ , $k = 1, ..., 4$. Now, connect each vertex $v_{j_k}$ to $x_0$ by a path $t_{j_k}$. By 1-connectivity of $X$, the triangles with vertices $v_{j_k}$, $v_{j_{k+1}}$ and $x_0$ and with sides $t_{j_k}$, $t_{j_{k+1}}$ and a side of $X_j$ induce a sequence of 1-loops which are homotopic to constant loop at $x_0$. We can fill inside them by homotopy. These filled triangles with the cube $X_j$ induce an 2-loop at $x_0$ which we denote its homotopy class by $\beta_j$. By 1-connectivity of $X$, $\beta_j$ is uniquely determined up to homotopy. So we can write $[\eta] = \beta_1 * ... * \beta_l$. We show that each $\beta_j$ is homotopic to one of $\alpha_i$'s, $i = 1, ..., N$. Indeed, corresponds to Corresponds to each $v_{j_k}$, there is an index $i_k \in 1, ..., N_0$ such that $v_{j_k} \in W_{i_j} \bigcap W_{i_k}$. We connect each $v_{i_k}$ to $x_{i_k}$ by a path $\theta_{i_k}$ and then we fill inside the 1-loops induced by paths $h_{i_k i_{k+1}}$ $\theta_{i_k}, \theta_{i_{k+1}}$ and a side of $X_j$ by homotopy. In this manner, we obtain an inessential 2-loop $\gamma_j$ which has the cube $Y_{i_1,i_2,i_3,i_4}$ as a side. (Note that $\gamma_j$ has diameter less than $2^{-(m(x_{i_j})+1)}$ for some $i_j$ and so it is nullhomotopic.) Let $\alpha_{i_j}$ be the homotopy class corresponding to the cube $Y_{i_1,i_2,i_3,i_4}$. Since $\gamma_j$ is nullhomotopic, then $\alpha_{i_j} = \beta_j$. So, we have

$$[\eta] = \beta_1 * ... * \beta_l = \alpha_{i_1} * ... * \alpha_{i_l}$$

Therefore, $\pi_2(X)$ is generated by $\alpha_i$'s and this is a contradiction. ∎

Suppose that the $n^{th}$homotopy group of $X$ is not finitely generated. Similar to [5], we define an equivalence relation $\{0,1\}^N$ via homotopic $n$-loops such as follows:
First, take a point $x \in X$ and a sequence $\{f_m\}_{m\in N}$ of $n$-loops as claimed in Lemma 2.1. For each $\alpha \in \{0,1\}^N$, let $f_m^\alpha$ be the constant $n$-loop at $x$, if $\alpha(m) = 0$, otherwise let $f_m^\alpha = f_m$. Define an $n$-loop $f_\alpha$ at $x$ as $f_0^\alpha * f_1^\alpha * \ldots$ . Write $\alpha \approx \beta$ if $f_\alpha \sim f_\beta$. Clearly $\approx$ is an equivalence relation and argument used in [5] , shows that it has continuum many equivalence classes; that is, there is a set of size of continuum of mutually non-homotopic $n$-loops. Therefore, we have the following theorem which is the extension of *Mycielski's conjecture* to higher homotopy groups.

**Theorem 2.2.** *Suppose $X$ is a compact metric space, which is $(n-1)$-connected, locally $(n-1)$-connected. Then $\pi_n(X)$ is either finitely generated or has the power of the continuum.*

In [3], we et al. clarified a relationship between the cardinality of $\pi_n(X,x)$ and discreteness of $\pi_n^{top}(X,x)$. We asserted that if $X$ is a connected separable metric space such that $\pi_n^{top}(X,x)$ is discrete, then $\pi_n(X,x)$ is countable, and as a result we showed that if $X$ is a connected locally $n$-connected separable metric space, then $\pi_n(X,x)$ is countable. First, we recall the following theorem from [3].

**Theorem 2.3.** *Suppose $X$ is a locally $(n-1)$-connected metrizable space and $x \in X$. Then the following are equivalent: (1) $\pi_n^{top}(X,x)$ is discrete. (2) $X$ is $n$-semilocally simply connected at $x$.*

Now, we show that the cardinality of $\pi_n(X,x)$ and discreteness of $\pi_n^{top}(X,x)$ are relevant.

**Theorem 2.4.** *Suppose $X$ is an $(n-1)$-connected locally $(n-1)$-connected compact metric space. Then $\pi_n^{top}(X)$ is discrete if and only if $\pi_n(X)$ is finitely generated. Moreover, $\pi_n^{top}(X)$ is not discrete if and only if $\pi_n(X)$ has the power of the continuum.*

*Proof.* The assertions follow immediately by Lemma 2.1 and Theorems 2.2 and 2.3. ∎

**Example 2.5.** *Let $X = \cup_{n\in\mathbb{N}} S_n$, where $S_n = \{(x,y,z) | (x-\frac{1}{n})^2 + y^2 + z^2 = \frac{1}{n^2}\}$, be a subspace of $\mathbb{R}^3$. It is easy to see that $X$ is 1-connected and locally 1-connected. However, the sequence $\{[S_n]\}$ is convergent to identity element of $\pi_2^{top}(X,0)$, implying that $\pi_2^{top}(X,0)$ is not discrete and has the power of continuum.*

## References

[1] D. Biss, The Topological Fundamental Group and Generalized Covering Spaces, Topology and its Applications, 124 (2002) 355-371.

[2] P. Fabel, Metric spaces with discrete topological fundamental group, Topology and its Applications 154 (2007) 635-638.

[3] F. H. Ghane, Z. Hamed, B. Mashayekhy, and H. Mirebrahimi, Topological homotopy groups, Bull. of the Belgian Math. Soc., To appear.

[4] J. Mycielski, Independent sets in topological algebras, Fund. Math. 55 (1964) 139-147.

[5] J. Pawlikowski, The fondamental group of a compact metric space, Proc. Amer. Math. Soc. 126 (10) (1998) 3083-3087.

[6] E. D. Spanier, Algebraic Topology, McGraw-Hill, 1966.

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