A note on the uniform limit of transitive dynamical systems

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Abstract

In this note we study the dynamical behaviour of the uniform limit of a sequence of continuous self-maps on a compact metric space satisfying (topological) transitivity or other related properties. Moreover, some conditions for the transitivity of a limit are given.

1 Introduction

Let X be a compact metric space and let $f_n : X \to X$ be a sequence of continuous maps uniformly convergent to a function f. An interesting problem, recently raised in [1], is the investigation of which dynamical properties possessed by the maps f_n are inherited by f. The main result in [1] (Theorem 3.1) states that the limit of a uniformly convergent sequence of continuous (topologically) transitive maps is transitive. As noted in [2] and [16], the result above is false when $X = S^1$. In this paper we go further in this topic investigating the behaviour of uniform limits in several ways : i) exhibiting some examples when X is the unit interval, ii) considering several properties naturally related to (topological) transitivity, iii) giving some conditions ensuring the transitivity of a limit.

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2 Preliminaries

Let (X,d) be a metric space and let (X,f) be the dynamical system given by the space (X,d) and a continuous map $f:(X,d)\to(X,d)$. According to Devaney [10] we say that the dynamical system (X,f) (or simply the map f) is chaotic if :

(i) f is (topologically) transitive, i.e., for every pair U and V of non-empty open subsets of (X,d) there is some $k\in N$ such that $f^k(U)\cap V\neq\emptyset$;

(ii) f is periodically dense, i.e., the set of periodic points of f is dense in X;

(iii) f has sensitive dependence on initial conditions, i.e., there is some $\delta>0$ such that, for any $x\in X$ and any neighbourhood V of x in (X,d), there exist $y\in V$ and some non-negative integer k for which $d(f^k(x),f^k(y))>\delta$.

It is worth noting that, in the definition above, condition (iii) is redundant if X is infinite ([5],[18]). Moreover if X is an interval of the real line, then f is chaotic if and only if it is transitive ([7], [17], [19], see also [3] for a more general result).

Let $f:I\to I$ be a continuous map. Let us recall that :

(i) f is topologically exact (or locally eventually onto) if for every non-degenerate subinterval J of I there exists a non-negative integer k such that $f^k(J)=I$;

(ii) f is strictly unimodal (see, e.g., [9]) if there exists a unique turning point $a\in(0,1)$ such that $f(0)=f(1)=0$, f is strictly increasing in $[0,a]$ and strictly decreasing in $[a,1]$.

Moreover a strictly unimodal map f with turning point a is called expanding if there is a constant $\lambda>1$ such that $|f(x)-f(y)|\geq\lambda|x-y|$ whenever x,y belong either to $[0,a]$ or to $[a,1]$ (see, e.g., [13]).

Observe that every topologically exact map $f:I\to I$ is chaotic in the sense of Devaney.

We refer the reader to [7], [8], [9],[10], [14] and [20] for more informations on topological dynamics.

3 The results

In [2] and [16] it is noted that a uniformly convergent sequence of transitive self-maps on the unit circle need not have a transitive limit. A more general statement is the content of the following

Remark 1. Let $f:X\to X$ be a continuous map, where X is a metric continuum. If such a map f is minimal and non-sensitive (like, e.g., an irrational rotation), then f is uniformly rigid, i.e., there exists a strictly increasing sequence $(n_k)_k$ of positive integers such that the sequence $(f^{n_k})_k$ converges uniformly to the identity on X (see [12]).

Nevertheless the maps $f^{n_k}:X\to X$ are topologically transitive [4].

As a consequence of the main result in [1], the authors state that if X is a compact interval of the real line and $f_n:X\to X$ is a uniformly convergent sequence of chaotic maps, then the limit f is chaotic ([1], Theorem 3.3). Unfortunately, even this particular and important case is false.

Before giving the first counterexample, we need the following

Lemma 2. *Every expanding strictly unimodal onto map is topologically exact.*

Proof. Although, as pointed out by the referee, this result can be deduced from, e.g., Propositions 45 and 53 in [7], we will give an easy proof for the sake of completeness.

Let $f : I \to I$ be an expanding strictly unimodal onto map with turning point a. Since f is onto, it follows that $f(a) = 1$.

Let λ be a real number greater than 1 such that

$$|f(x) - f(y)| \geq \lambda |x - y|$$

whenever either $x, y \in [0, a]$ or $x, y \in [a, 1]$ and let J be a non-degenerate subinterval of I. We claim that $a \in f^m(J)$ for some $m \in N_0$, where N_0 is the set of non-negative integers. If not, $f^n(J)$ is a subinterval either of $[0, a]$ or of $[a, 1]$, hence $\text{diam} f^{n+1}(J) \geq \lambda \text{ diam } f^n(J)$ for every $n \in N_0$.

Therefore $\text{diam} f^{n+1}(J) \geq \lambda^n \text{diam } J$ for every $n \in N_0$, hence there is some $n \in N_0$ such that $\text{diam} f^n(J) > 1$, a contradiction (recall that $\lambda > 1$).

So let us take some $k \in N_0$ such that $a \in f^k(J)$. Then $f(a) = 1 \in f^{k+1}(J)$ and $f(1) = 0 \in f^{k+2}(J)$.

Therefore by the argument above we obtain that $a \in f^{m+k+2}(J)$ for some $m \in N_0$.

Hence $0, 1 \in f^{m+k+3}(J)$ and $f^{m+k+3}(J) = I$.

Example 3. We will construct a sequence $f_n : I \to I$ of topologically exact maps uniformly convergent to a map $f : I \to I$ which does not have sensitive dependence on initial conditions (so, a fortiori, f is not topologically transitive).

For every $n \in N$ let $f_n : I \to I$ be the piecewise linear map defined by

$$f_n(0) = f_n(1) = 0, f_n(\tfrac{1}{3}) = \tfrac{1}{3} + \tfrac{1}{n+3}, f_n(\tfrac{2}{3}) = 1.$$

Every f_n is an expanding strictly unimodal onto map.

So by Lemma 2 every f_n is topologically exact.

On the other hand the uniform limit f is the identity on $[0, \frac{1}{3}]$, therefore f cannot exhibit sensitive dependence on initial conditions.

The following example will show that the situation, for interval maps, can be even worse than the one described in the previous example.

Example 4. We are going to exhibit a sequence of topologically exact self-maps on the unit interval which is uniformly convergent to the identity (an example of this type can be found in [6], Example 3).

Let $n \in N$ and let us set $x_i = \frac{i}{4n+1}$ for each $i \in 0, ..., 4n + 1$.

Now let $f_n : I \to I$ be the piecewise linear map defined by

$$f_n(x_{2i}) = \max\{0, x_{2i} - \tfrac{1}{n}\} \quad \text{and} \quad f_n(x_{2i+1}) = \min\{1, x_{2i+1} + \tfrac{1}{n}\}$$

for each $i \in 0, ..., 2n$.

In this way we obtain a sequence $(f_n)_n$ of continuous self-maps on the unit interval which is clearly uniformly convergent to the identity

We claim that each f_n is topologically exact. So let us fix some $n \in N$ and let us set $\epsilon = \frac{1}{2n}$ and $I_i = [x_i, x_{i+1}]$ for each $i \in \{0, ...4n\}$.

First let us show that for every non-degenerate subinterval J of I there are some $i, k \in N_0$ such that $I_i \subset f_n^k(J)$.

If $I_i \not\subset J$ for every i, then there is some j such that $J \subset I_j \cup I_{j+1}$. Let us set $J_1 = J \cap I_j$ $J_2 = J \cap I_{j+1}$.

Since $|f_n'(x)| > 4$ for every $x \in]x_i, x_{i+1}[$ and $i \in \{0, ...4n\}$, it follows that $\mathrm{diam} f_n(J) \geq \max\{\mathrm{diam} f_n(J_1), \mathrm{diam} f_n(J_2)\} > \max\{4\,\mathrm{diam} J_1, 4\,\mathrm{diam} J_2\} \geq 2\mathrm{diam}\, J$.

Therefore there are some $i, k \in N_0$ such that $I_i \subset f_n^k(J)$.

It remains to show that for every non-degenerate subinterval $J = [\alpha, \beta]$ of I which contains some I_i there exists a non-negative integer m such that $f_n^m(J) = I$. First let us check that $J \subset f_n(J)$ and $\mathrm{diam} f_n(J) \geq \mathrm{diam} J + \epsilon$ whenever $f_n(J) \neq I$.

Let us set $a = \min\{x_i \in J : i \text{ is even }\}$ and $b = \max\{x_i \in J : i \text{ is odd }\}$. Since $\alpha > a - \epsilon$ and $\beta < b + \epsilon$, it follows that

$$f_n(a) \leq \max\{0, \alpha - \epsilon\} \text{ and } f_n(b) \geq \min\{1, \beta + \epsilon\}.$$

Hence $J \subset f_n(J)$ and $\mathrm{diam} f_n(J) \geq \mathrm{diam} J + \epsilon$ whenever $f_n(J) \neq I$.

Therefore $\mathrm{diam} f_n^k(J) \geq \min\{\mathrm{diam} J + k\epsilon, 1\}$.

So $f_n^m(J) = I$ for some m.

In [16] the author gives some sufficient conditions for the transitivity of a limit function when X is a perfect (and complete) metric space. In the next remark we will exhibit a strong form of convergence which ensures the transitivity of a limit in the realm of all metric spaces.

Remark 5. Let $(f_n)_n$ be a sequence of continuous self-maps on a metric space (X, d). We say that the sequence $(f_n)_n$ is orbitally convergent to a map $f : (X, d) \to (X, d)$ if for every $\epsilon > 0$ there is some $n_0 \in N$ such that $d(f_n^k(x), f^k(x)) < \epsilon$ for every $x \in X$, $k \in N$ and every $n \geq n_0$.

Clearly orbital convergence is stronger than uniform convergence.

We claim that the limit f of an orbitally convergent sequence of topologically transitive maps $(f_n)_n$ is transitive.

In fact, let us consider two non-empty open subsets A and B of (X, d). Let $q \in X$ and $\epsilon > 0$ such that $B(q, \epsilon) = \{x \in X : d(q, x) < \epsilon\} \subset B$. Since $(f_n)_n$ is orbitally convergent to f, there is some $n_0 \in N$ such that $d(f_n^k(x), f^k(x)) < \frac{\epsilon}{2}$ for every $x \in X$, $k \in N$ and every $n \geq n_0$. Now let us take some $m \geq n_0$. Since f_m is transitive, there is some $k \in N$ and $a \in A$ such that $f_m^k(a) \in B(q, \frac{\epsilon}{2})$. Now $d(f^k(a), q) \leq d(f^k(a), f_m^k(a)) + d(f_m^k(a), q) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. So $f^k(a) \in B(q, \epsilon) \subset B$ and $f^k(A) \cap B \neq \emptyset$. Therefore f is transitive.

In general a transitive map need not have a dense orbit, however there are several relationships between these concepts (even for some kind of discontinuous maps, see [15] and [11]).

In particular let us consider a continuous self-map f on a compact metric space X and let $\mathrm{tr}(f)$ be the set of all points x of X whose orbit $O(f, x) = \{f^n(x) : n \geq 0\}$ is dense in X. It is well-known that the following conditions are equivalent (see, e.g., [13]) :

i) f is transitive;
ii) f is onto and there is a point with dense orbit;
iii) $\mathrm{tr}(f)$ is a dense G_δ-set in X.

In the next result we give, using the equivalent conditions listed above, a characterization of the transitive uniform limits of continuous self-maps on a compact metric space.

Proposition 6. *Let $\{f_n : n \in N\}$ be a sequence of transitive self-maps on a compact metric space (X, d) uniformly convergent to a map f. Then f is transitive if and only if there are $x_0 \in D = \bigcap\{tr(f_n) : n \in N\}$ and $x_1 \in X$ such that $\overline{\{f_n^{k_n}(x_0) : n \in N\}} \cap \overline{O(f, x_1)} \neq \emptyset$ for every sequence $(k_n)_{n\in N}$ of non-negative integers.*

Proof. Clearly f is onto. Moreover observe that D is dense in X, in fact every $\mathrm{tr}(f_n)$ is a dense G_δ-set and therefore D is the intersection of a countable family of open dense sets in the Baire space X.

If f is transitive, it is enough to take as x_0 any point of D and as x_1 a point whose orbit $O(f, x_1)$ is dense.

Now let us suppose that there are two points x_0 and x_1 satisfying the condition above. We claim that $O(f, x_1)$ is dense in X, and therefore f is transitive. So let us take an open disk $B(p, \epsilon)$ in X. Since every $O(f_n, x_0)$ is dense in X, there is some non-negative integer k_n such that $f_n^{k_n}(x_0) \in B(p, \frac{\epsilon}{2})$ for each $n \in N$. Since $\overline{\{f_n^{k_n}(x_0) : n \in N\}} \cap \overline{O(f, x_1)} \neq \emptyset$, it follows that there are $f_m^{k_m}(x_0)$and $f^s(x_1)$ such that $d(f_m^{k_m}(x_0), f^s(x_1) < \frac{\epsilon}{2}$. Therefore

$$d(f^s(x_1), p) \leq d(f^s(x_1), f_m^{k_m}(x_0)) + d(f_m^{k_m}(x_0), p) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $f^s(x_1) \in B(p, \epsilon)$ and $\overline{O(f, x_1)} = X$.

Although the previous results show that transitivity need not be inherited by uniform limits, nonetheless we can show the preservation of a related dynamical property : chain-transitivity.

Let f be a continuous self-map on a metric space (X, d) and let δ be a positive real number. A (finite or infinite) sequence $(x_n)_{n\geq 0}$ is a δ-chain if $d(f(x_n), x_{n+1}) < \delta$ for all n. The map f is called chain-transitive if for every $x, y \in X$ and every $\delta > 0$ there is a finite δ-chain x_0,..., x_n such that $x_0 = x$ and $x_n = y$ (see, e.g., [14]). It is well-known that every transitive self-map on a compact metric space is chain-transitive. Moreover it is worth noting that the identity map on a connected metric space is an example of a chain-transitive map which is not transitive.

Proposition 7. *Let $(f_n)_n$ be a sequence of chain-transitive maps on a metric space (X, d) uniformly convergent to a map f. Then f is chain-transitive.*

Proof. Let $a, b \in X$ and let $\epsilon > 0$. Since $(f_n)_n$ converges uniformly to f, let us take some k such that $d(f_k(x), f(x)) < \frac{\epsilon}{2}$ for every $x \in X$. Since f_k is chain-transitive, there are $x_0, ..., x_n$ such that $x_0 = a$, $x_n = b$ and $d(f_k(x_i), x_{i+1}) < \frac{\epsilon}{2}$ for every $i < n$. Therefore

$$d(f(x_i), x_{i+1}) \leq d(f(x_i), f_k(x_i)) + d(f_k(x_i), x_{i+1}) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ for every } i < n.$$

So $x_0, ..., x_n$ is an ϵ-chain (with respect to f) from a to b. Hence f is chain-transitive.

Corollary 8. *Let f be the uniform limit of a sequence of transitive self-maps on a compact metric space. Then f is chain-transitive.*

On the other hand, if we want to force a limit f to be transitive, we need to assume stronger dynamical properties on the maps f_n.

The final part of this paper is devoted to show that a variation of the well-known shadowing property, together with chain-transitivity, is good enough for our purposes.

Let f be a continuous self-map on a metric space (X, d) and let $\epsilon > 0$. A point $x \in X$ ϵ-shadows a finite sequence x_0,..., x_n if $d(f^i(x), x_i) < \epsilon$ for every $i \leq n$. The map f has the shadowing property if for every $\epsilon > 0$ there exists some $\delta > 0$ such that every finite δ-chain is ϵ-shadowed by some point (see, e.g., [14]). Note that if f is a continuous self-map on a compact metric space (X, d) which has the shadowing property, then for every $\epsilon > 0$ there exists some $\delta > 0$ such that for every infinite δ-chain $(x_n)_{n\geq 0}$ there exists $x \in X$ for which $d(f^i(x), x_i) < \epsilon$ for every $i \geq 0$.

Observe that every chain-transitive map which has the shadowing property is transitive (see, e.g., [14]). It is also worth noting that the shadowing property is not inherited by uniform limits. In fact let us consider, for every $n \in N$, the linear system $f_n : I \to I$ given by $f_n(x) = \frac{n}{n+1}x$ for every $x \in I$. Every f_n has the shadowing property, nonetheless the sequence $(f_n)_n$ converges uniformly to the identity map.

On the other hand the following strengthening of the shadowing property has a completely different behaviour with respect to uniform limits. A continuous self-map f on a metric space (X, d) is said to have the fine shadowing property if every finite ϵ-chain is ϵ-shadowed by some point, for every $\epsilon > 0$.

The expanding endomorphisms of the circle $E_m : S^1 \to S^1$ given by $E_m(x) = mx$ (mod 1), where $m \in Z$ and $|m| > 1$, are examples of (transitive) dynamical systems which have the fine shadowing property (see, e.g., [8]).

Proposition 9. *Let $(f_n)_n$ be a sequence of self-maps on a compact metric space (X, d) uniformly convergent to a map f. If every f_n has the fine shadowing property, then f has the shadowing property.*

Proof. Let $\epsilon > 0$ and set $\delta = \frac{\epsilon}{4}$. We claim that every δ-chain is ϵ-shadowed by some point (with respect to f). So let us take $x_0, ..., x_m$ such that $d(f(x_i), x_{i+1}) < \delta$ for every $i < m$. Since $(f^i_n)_n$ converges uniformly to f^i for every i, let us pick some k such that $d(f^i_k(x), f^i(x)) < \delta$ for every $x \in X$ and $i \in \{1, ..., m\}$. Now let us observe that $x_0, ..., x_m$ is an $\frac{\epsilon}{2}$-chain with respect to f_k. In fact, $d(f_k(x_i), x_{i+1}) \leq d(f_k(x_i), f(x_i)) + d(f(x_i), x_{i+1}) < \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}$ and f_k has the fine shadowing property, so there is some $w \in X$ which $\frac{\epsilon}{2}$-shadows $x_0, ..., x_m$ (with respect to f_k), i.e., $d(f^i_k(w), x_i) < \frac{\epsilon}{2}$ for every $i \leq m$.

So $d(f^i(w), x_i) \leq d(f^i(w), f^i_k(w)) + d(f^i_k(w), x_i) < \frac{\epsilon}{4} + \frac{\epsilon}{2} < \epsilon$.

Hence the δ-chain $x_0, ..., x_m$ is ϵ-shadowed by w with respect to f. Therefore f has the shadowing property.

By propositions 7 and 9 we obtain the following

Corollary 10. *Let* $(f_n)_n$ *be a sequence of chain-transitive self-maps on a compact metric space uniformly convergent to a map* f*. If every* f_n *has the fine shadowing property, then* f *is transitive.*

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