

Topological Homotopy Groups*

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Abstract

D. K. Biss (Topology and its Applications 124 (2002) 355-371) introduced the topological fundamental group and presented some interesting basic properties of the notion. In this article we intend to extend the above notion to homotopy groups and try to prove some similar basic properties of the topological homotopy groups. We also study more on the topology of the topological homotopy groups in order to find necessary and sufficient conditions for which the topology is discrete. Moreover, we show that studying topological homotopy groups may be more useful than topological fundamental groups.

1 Introduction and Motivation

Historically, J. Dugundji [3] in 1950, put, for the first time, a topology on fundamental groups of certain spaces and deduced a classification theorem for connected covers of a space.

Recently, Biss [1] generalized the results announced by J. Dugundji. He equipped the fundamental group of a pointed space (X, x) with the quotient topology inherited from $Hom((S^1, 1), (X, x))$ with compact-open topology and denoted by $\pi_1^{top}(X, x)$. He proved among other things that $\pi_1^{top}(X, x)$ is a topological group which is independent of the base point in path components and π_1^{top} is a functor from the homotopy category of based spaces to the category of topological groups which preserves the direct product. He showed that π_1^{top} is discrete if and only if the space X is semilocally simply connected. However, P. Fabel [5] mentioned that path connectedness and locally path connectedness of X is necessary.

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P. Fabel [4], using Biss' results, showed that the topological fundamental groups can distinguish the homotopy type of topological spaces when the algebraic structures fail to do. On the other hand, in some cases, this topology does not be able to separate spaces in terms of their homotopy types. For examples, if X is locally contractible, then $\pi_1^{top}(X)$ is discrete which is not interesting. In these situations the higher homotopy groups seems to be useful and this is a motivation to define a natural topology on homotopy groups of X . This topology, if it is compatible with the topology of fundamental groups, can also distinguish the spaces with distinct homotopy types even though the topology of their topological fundamental groups are the same.

In this article, we are going to extend some basic results of Biss by introducing a topology on n th homotopy group of a pointed space (X, x) as a quotient space of $Hom((I^n, \dot{I}^n), (X, x))$ equipped with compact-open topology and denote it by $\pi_n^{top}(X, x)$. We will show that $\pi_n^{top}(X, x)$ is a topological group which does not depend on based point x in a path component. We will also prove that π_n^{top} is a functor from the homotopy category of pointed spaces to the category of topological groups which preserves the direct product. Moreover, we present the notion of an n -semilocally simply connected space and prove that the discreteness of $\pi_n^{top}(X, x)$ implies that X is n -semilocally simply connected. By giving an example, it is shown that the converse is not true, in general. However, we show that the n th homotopy group of a locally n -connected metrizable space X is discrete.

Fabel in [5], clarified the relationship between the cardinality of $\pi_1(X, x)$ and discreteness of $\pi_1^{top}(X, x)$ and deduced that if X is a connected separable metric space and $\pi_1^{top}(X, x)$ is discrete, then $\pi_1(X, x)$ is countable. Here, we extend this result to higher homotopy groups and show that the n th homotopy group of a connected, locally n -connected separable metric space is countable. Finally, we give an example of a metric space X such that $\pi_1^{top}(X)$ is discrete whereas $\pi_2^{top}(X)$ is not. This shows that studying topological homotopy groups may be more useful than topological fundamental groups.

2 Topological Homotopy Groups

Let (X, x) be a pointed space. Then the space of continuous based maps $Hom((I^n, \dot{I}^n), (X, x))$ can be given the compact-open topology; this topology has as a subbase the sets $\langle K, U \rangle = \{f : (I^n, \dot{I}^n) \rightarrow (X, x) \mid f(K) \subseteq U\}$, where K ranges over all compact subsets of I^n and U ranges over all open subsets of X . By considering the natural projection

$$Hom((I^n, \dot{I}^n), (X, x)) \xrightarrow{p_n} [(I^n, \dot{I}^n), (X, x)] = \pi_n(X, x),$$

we are allowed to define a quotient topology on $\pi_n(X, x)$. By $\pi_n^{top}(X, x)$ we mean the topological space $\pi_n(X, x)$ equipped with the above topology. In the following, we are going to prove several basic properties of this topology.

Theorem 2.1. *Let (X, x) be a pointed space. Then $\pi_n^{top}(X, x)$ is a topological group for all $n \geq 1$.*

Proof. In order to show that the multiplication is continuous, we consider the following commutative diagram

$$\begin{array}{ccc} Hom((I^n, \dot{I}^n), (X, x)) \times Hom((I^n, \dot{I}^n), (X, x)) & \xrightarrow{\tilde{m}_n} & Hom((I^n, \dot{I}^n), (X, x)) \\ \downarrow p_n \times p_n & & \downarrow p_n \\ \pi_n^{top}(X, x) \times \pi_n^{top}(X, x) & \xrightarrow{m_n} & \pi_n^{top}(X, x), \end{array}$$

where \tilde{m}_n is concatenation of n -loops, and m_n is the multiplication in $\pi_n(X, x)$. Since $(p_n \times p_n)^{-1}m_n^{-1}(U) = \tilde{m}_n^{-1}p_n^{-1}(U)$ for every open subset U of $\pi_n^{top}(X, x)$, it is enough to show that \tilde{m}_n is continuous. Let $\langle K, U \rangle$ be a basis element in $Hom((I^n, \dot{I}^n), (X, x))$. Put

$$K_1 = \{(t_1, \dots, t_n) | (t_1, \dots, t_{n-1}, \frac{t_n}{2}) \in K\}$$

and

$$K_2 = \{(t_1, \dots, t_n) | (t_1, \dots, t_{n-1}, \frac{t_n + 1}{2}) \in K\}.$$

Then

$$\tilde{m}_n^{-1}(\langle K, U \rangle) = \{(f_1, f_2) | (f_1 * f_2)(K) \subseteq U\} = \langle K_1, U \rangle \times \langle K_2, U \rangle$$

is open in $Hom((I^n, \dot{I}^n), (X, x)) \times Hom((I^n, \dot{I}^n), (X, x))$ and so \tilde{m}_n is continuous. To prove that the operation of taking inverse is continuous, let K be any compact subset of I^n and put

$$K^{-1} = \{(t_1, \dots, t_{n-1}, 1 - t_n) | (t_1, \dots, t_n) \in K\}.$$

Clearly an n -loop α is in $\cap_{i=1}^m \langle K_i, U_i \rangle$ if and only if its inverse is in $\cap_{i=1}^m \langle K_i^{-1}, U_i \rangle$, where $\langle K_i, U_i \rangle$ are basis elements in $Hom((I^n, \dot{I}^n), (X, x))$. Hence the inverse map is continuous. ■

From now on, when we are dealing with π_n^{top} , by the notion \cong we mean the isomorphism in the sense of topological groups. The following result shows that the topological group π_n^{top} is independent of the base point x in the path component.

Theorem 2.2. *Let $\gamma : I \rightarrow X$ be a path with $\gamma(0) = x$ and $\gamma(1) = y$. Then $\pi_n^{top}(X, x) \cong \pi_n^{top}(X, y)$.*

Proof. Let α be a base n -loop at x in X . Define the following map

$$A_\alpha : I^n \times \{0\} \cup \dot{I}^n \times [0, 1] \rightarrow X$$

by $A_\alpha(s, 0) = \alpha(s)$ for all $s \in I^n$ and $A_\alpha(s, t) = \gamma^{-1}(t)$ for all $(s, t) \in \dot{I}^n \times [0, 1]$. By gluing lemma A_α is a continuous map. Now, we define

$$\gamma_\# : \pi_n^{top}(X, x) \longrightarrow \pi_n^{top}(X, y)$$

by $[\alpha] \mapsto [A_\alpha \circ r(-, 1)]$, where $r : I^n \times [0, 1] \rightarrow I^n \times [0, 1] \cup \dot{I}^n \times [0, 1]$ is the retraction introduced in [7]. It can be shown that $\gamma_\#$ is a group isomorphism (see [7]). Now, it is enough to show that $\gamma_\#$ is a homeomorphism. To prove $\gamma_\#$ is continuous, consider the following commutative diagram

$$\begin{CD} Hom((I^n, \dot{I}^n), (X, x)) @>\tilde{\gamma}>> Hom((I^n, \dot{I}^n), (X, y)) \\ @Vp_nVV @VVp_nV \\ \pi_n^{top}(X, x) @>\gamma_\#>> \pi_n^{top}(X, y), \end{CD}$$

where $\tilde{\gamma}(\alpha) = A_\alpha \circ r(-, 1)$ for all based n -loop α at x in X . For all basis elements $\langle K, U \rangle$ of $Hom((I^n, \dot{I}^n), (X, x))$ we have

$$\begin{aligned} \tilde{\gamma}(\langle K, U \rangle) &= \{ \tilde{\gamma} | \alpha : (I^n, \dot{I}^n) \rightarrow (X, x), \alpha(K) \subseteq U \} \\ &= \{ A_\alpha \circ r(-, 1) | \alpha : (I^n, \dot{I}^n) \rightarrow (X, x), \alpha(K) \subseteq U \} \\ &\subseteq \{ \beta | \beta : (I^n, \dot{I}^n) \rightarrow (X, y), \beta(K) \subseteq U \} \\ &= \langle K, U \rangle. \end{aligned}$$

Hence $\tilde{\gamma}$ is continuous and so is $\gamma_\#$. It is easy to see that $(\gamma^{-1})_\#$ is the inverse of $\gamma_\#$ and continuous. So the result holds. ■

It is known that π_n is a functor from the homotopy category of based spaces to the category of groups. So it is natural to ask whether π_n^{top} is a functor. Suppose $f : (X, x) \rightarrow (Y, y)$ is a pointed continuous map. It is enough to show that the induced homomorphism $f_* : \pi_n^{top}(X, x) \rightarrow \pi_n^{top}(Y, y)$ is continuous. Consider the following commutative diagram

$$\begin{CD} Hom((I^n, \dot{I}^n), (X, x)) @>f_\#>> Hom((I^n, \dot{I}^n), (X, y)) \\ @Vp_nVV @VVp_nV \\ \pi_n^{top}(X, x) @>f_*>> \pi_n^{top}(X, y), \end{CD}$$

where $f_\#(\alpha) = f \circ \alpha$. Clearly $f_\#^{-1}(\langle K, U \rangle) = \langle K, f^{-1}(U) \rangle$ for all basis elements $\langle K, U \rangle$ in $Hom((I^n, \dot{I}^n), (X, x))$. Since f is continuous, so are $f_\#$ and f_* . Hence we can consider π_n^{top} as a functor from $hTop^*$ to the category of topological groups. Now, we intend to show that the functor π_n^{top} preserves the direct product of spaces.

Theorem 2.3. *Let $\{(X_i, x_i) | i \in I\}$ be a family of pointed spaces. Then*

$$\pi_n^{top}(\prod_{i \in I} (X_i, x_i)) \cong \prod_{i \in I} \pi_n^{top}(X_i, x_i).$$

Proof. Consider the commutative diagram

$$\begin{CD} M = \prod_{i \in I} Hom((I^n, \dot{I}^n), (X_i, x_i)) @<\psi<< Hom((I^n, \dot{I}^n), (X_i, x_i)) \\ @VQ=\prod_{i \in I} p_n^iVV @VVp_nV \\ N = \prod_{i \in I} \pi_n^{top}(X_i, x_i) @<\varphi<< \pi_n^{top}(\prod_{i \in I} (X_i, x_i)), \end{CD}$$

where ψ and φ are natural homomorphism and isomorphism, respectively (see [8]). To show that φ is a homeomorphism, it is enough to show that Q is a quotient map. Let $Q^{-1}(U)$ be an open subset of M for some subset U of N . By product topology of M , $Q^{-1}(U)$ is the union of basic opens of the form $\prod_{i \in I} V_i$, where V_i 's are open in $Hom((I^n, \dot{I}^n), (X_i, x_i))$ and $V_i = Hom((I^n, \dot{I}^n), (X_i, x_i))$ for all $i \notin J$, for some finite subset J of I . Suppose $V = \prod_{i \in J} V_i'$ is a maximal open set having the property that $W = V \times \prod_{i \notin J} Hom((I^n, \dot{I}^n), (X_i, x_i))$ is a subset of $Q^{-1}(U)$. Since W is open and maximal, it is of the form $Q^{-1}(\prod_{i \in I} U_i)$, where $\prod_{i \in I} U_i$ is a basic open in N . Since $Q^{-1}(U)$ is constructed with open sets like the above, we have covered U with open sets and so U is open. ■

3 The Topology of $\pi_n^{top}(X)$

In this section, we are going to study more on the topology of $\pi_n^{top}(X)$, specially we intend to find necessary and sufficient conditions for which $\pi_n^{top}(X)$ is discrete.

Definition 3.1. A topological space X is called n -semilocally simply connected if for each $x \in X$ there exists an open neighborhood U of x for which any n -loop in U is nullhomotopic in X . In other words the induced homomorphism of the inclusion $i_* : \pi_n(U, x) \rightarrow \pi_n(X, x)$ is zero.

Theorem 3.2. *If $\pi_n^{top}(X)$ is discrete, then X is n -semilocally simply connected.*

Proof. For each $x \in X$, since $\pi_n^{top}(X, x)$ is discrete, there exists an open neighborhood W in $Hom((I^n, \dot{I}^n), (X, x))$ of the constant n -loop at x such that each element of W is homotopic to the constant loop at x . By compact-open topology, we can consider W as $\cap_{i=1}^m \langle K_i, U_i \rangle$, where K_i 's are compact subsets of I^n and U_i 's are open in X . Consider $U = \cap_{i=1}^m U_i$ as a nonempty open neighborhood, then $\langle I^n, U \rangle \subseteq W$. Therefore any n -loop in U at x belongs to W and so is nullhomotopic in X . Hence X is n -semilocally simply connected. ■

Note that the following examples show the inverse of Theorem 3.2 is not true, in general. In both of them, we use the fact that the compact-open topology on $Hom((I^2, \dot{I}^2), (X, 0))$ is equivalent to the uniformly convergence topology when X is a metric space [6].

Example 3.3. *Let $X = \cup_{n \in \mathbb{N}} S_n$, where*

$$S_1 = \{(x, y, z) \mid (x - \frac{1}{2})^2 + y^2 + z^2 = \frac{1}{4}\},$$

$$S_n = \{(x, y, z) \mid (x - \frac{n-1}{2n})^2 + y^2 + z^2 = (\frac{n-1}{2n})^2\},$$

for each $n \geq 2$. Then $\{S_n\}$ as a sequence of 2-loops in X at $p = (0, 0, 0)$ uniformly converges to S_1 . Now $[S_1]$ is a limit point in $\pi_2^{top}(X, x)$, nevertheless X is 2-semilocally simply connected.

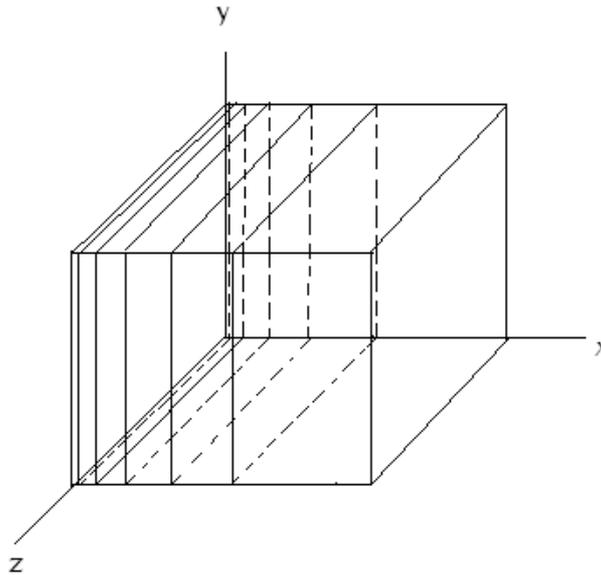


Figure-1

Example 3.4. Let X denotes the following subspace of \mathbb{R}^3 :

$$X = [0, 1] \times [0, 1] \times \{0, 1\} \cup [0, 1] \times \{0, 1\} \times [0, 1] \cup \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\} \times [0, 1] \times [0, 1]$$

Let $p = (0, 0, 0)$. Consider the following sequence of 2-loops at p

$$X_n = [0, \frac{1}{n}] \times [0, 1] \times \{0, 1\} \cup [0, \frac{1}{n}] \times \{0, 1\} \times [0, 1] \cup \{\frac{1}{n}\} \times [0, 1] \times [0, 1].$$

Obviously, this sequence is uniformly convergent to the nullhomotopic loop $X_0 = \{0\} \times [0, 1] \times [0, 1]$. Thus $\pi_2^{top}(X)$ is not discrete, however one can see that X is 2-semilocally simply connected. (see Figure-1)

We recall the following definitions in [9].

Definition 3.5. A space X is said to be n -connected for $n \geq 0$ if it is path connected and $\pi_k(X, x)$ is trivial for every base point $x \in X$ and $1 \leq k \leq n$. X is called locally n -connected if for each $x \in X$ and each neighborhood U of x , there is a neighborhood $V \subseteq U \subseteq X$ containing x so that $\pi_k(V) \rightarrow \pi_k(U)$ is zero map for all $0 \leq k \leq n$ and for all basepoint in V .

Theorem 3.6. Let X be a locally n -connected metric space. Then for any $x \in X$, $\pi_n^{top}(X, x)$ is discrete.

Proof. We know that $\pi_n^{top}(X, x)$ is the set of path components of loop space $\Omega^n(X, x)$ topologized with the quotient topology under the canonical surjection p_n satisfying $p_n(f) = p_n(g)$ if and only if the n -loops f and g belong to the same path component of $\Omega^n(X, x)$, see[7,Lemma 2.5.5]. To prove $\pi_n^{top}(X, x)$ is discrete, it is sufficient to show that the path components of $\Omega^n(X, x)$ are open. Suppose $f \in \Omega^n(X, x)$ and $f_k \rightarrow f$ uniformly. We must prove that f and f_k are homotopic, for sufficiently large k .

Since X is locally n -connected and $Im(f)$ is compact, so there exists $\epsilon > 0$ such that if $x \in Im(f)$ and α_x is an m -loop based at x ($m \leq n$), with $diam(\alpha_x) < \epsilon$, then α_x is null-homotopic.

Since $f \cup \{f_1, f_2, \dots\}$ is an equicontinuous collection of maps and $Im(f)$ and $Im(f_k)$'s are compact, then for the ϵ as above, there exists $\delta > 0$ such that each subcube $\tilde{I} \subseteq I^n$ with $diam(\tilde{I}) < \delta$, the images $f(\tilde{I})$ and $f_k(\tilde{I})$'s have diameters less than ϵ . Take a partition $\{I_1^n, \dots, I_l^n\}$ of I^n , with $diam(I_i^n) < \delta$, $i = 1, \dots, l$, then $diam(f(I_i^n)) < \epsilon$ and $diam(f_k(I_i^n)) < \epsilon$.

Let $v_j^i, j = 1, \dots, 2^n$ be the vertices of I_i^n . First, by local path connectivity of X , we can connect the vertices $f(v_j^i)$ and $f_k(v_j^i)$ by small path, for each $i = 1, \dots, l$ and $j = 1, \dots, 2^n$ and sufficiently large k . The boundary of the rectangles with corners $f(v_j^i)$ and $f_k(v_j^i)$ induce 1-loops which are homotopic to the constant loop. By local n -connectivity of X , we can fill in the homotopy across the sides of these rectangles. Similarly and by induction on k , we construct inessentials k -loops for each $k \leq n$ and then again fill in the homotopy across the sides of induced k -rectangles. In this way, in the n th step, we obtain a homotopy from $f | I_i^n$ to $f_k | I_i^n$. Now, the gluing lemma yields a homotopy from f to f_k . ■

With the added assumption that X is locally $(n - 1)$ -connected, the inverse of the Theorem 3.2 holds.

Theorem 3.7. *Suppose X is a locally $(n - 1)$ -connected metrizable space and $x \in X$. Then the following are equivalent:*

- (1) $\pi_n^{top}(X, x)$ is discrete.
- (2) X is n -semilocally simply connected at x .

The following result presents relationship between the cardinality of $\pi_n(X, x)$ and discreteness of $\pi_n^{top}(X, x)$.

Theorem 3.8. *Suppose X is a connected separable metric space such that $\pi_n^{top}(X, x)$ is discrete. Then $\pi_n(X, x)$ is countable.*

Proof. Since X is a separable metric space, it follows from the proof of the Urysohn metrization theorem [6, Theorem 4.1] that X can be embedded as a subspace of the Hilbert cube $Q = \prod_{i=1}^{\infty} [0, 1]$. The space $Hom(I^n, Q)$ is separable and metrizable, and hence the subspace $Hom((I^n, I^n), (X, x))$ is separable. Since $\pi_n(X, x)$ is the continuous image of $Hom((I^n, I^n), (X, x))$, the space $\pi_n^{top}(X, x)$ is separable. In particular, if $\pi_n^{top}(X, x)$ is discrete, then $\pi_n(X, x)$ is countable since $\pi_n^{top}(X, x)$ is the only dense subspace of $\pi_n^{top}(X, x)$. ■

Corollary 3.9. *If X is a connected, locally n -connected separable metric space, then $\pi_n(X, x)$ is countable.*

Proof. By Theorems 3.6 and 3.8, the result follows immediately. ■

Remark 3.10. It is known that if X is path connected, locally path connected, semilocally simply connected and $p : \tilde{X} \rightarrow X$ is a covering space of X , then $p_* : \pi_1^{top}(\tilde{X}) \rightarrow \pi_1^{top}(X)$ is an embedding, see [1,4,5]. However, by our results, if X is locally n -connected then $p_* : \pi_n^{top}(\tilde{X}) \rightarrow \pi_n^{top}(X)$ is also an embedding.

Recall a well-known theorem in Algebraic Topology [8] asserting the isomorphism

$$\pi_n(X, x) \cong \pi_1(\Omega^{n-1}(X, x), \tilde{x}),$$

where \tilde{x} is the constant loop $\tilde{x}(t) = x$ and $\Omega^{n-1}(X, x)$ is the $(n - 1)$ -loop space of X at x equipped with compact-open topology. Using the above isomorphism we can consider another topology on $\pi_n(X, x)$ induced by $\pi_1^{top}(\Omega^{n-1}(X, x), \tilde{x})$. We denote this topological group by $\pi_n^\Omega(X, x)$. Note that π_n^Ω is the composition of the two functors π_1^{top} and Ω^{n-1} [1,6]. Therefore the operation π_n^Ω is a functor from the homotopy category of topological based spaces to the category of topological groups. Now, it is natural and interesting to ask the relationship between the two topological groups $\pi_n^{top}(X, x)$ and $\pi_n^\Omega(X, x)$. Consider the following diagram:

$$\begin{array}{ccc} \text{Hom}((I^n, \dot{I}^n), (X, x)) & \xrightarrow{\psi} & \text{Hom}((I, \dot{I}), (\Omega^{n-1}(X, x), \tilde{x})) \\ \downarrow q_1 & & \downarrow q_2 \\ \pi_n^{top}(X, x) & \xrightarrow{\eta} & \pi_n^\Omega(X, x), \end{array}$$

where q_1 and q_2 are quotient maps which have been defined and the mapping ψ maps $f : ((I \times S^{n-1}) / \sim, *) \rightarrow (X, x)$ to $f^\# : (I, \dot{I}) \rightarrow (\Omega^{n-1}(X, x), \tilde{x})$, for which $f^\#(t) = f_t^\# \in \Omega^{n-1}(X, x)$ and $f_t^\#(z) = f([t, z])$ for each $z \in S^{n-1}$ (note that $(S^n, 1) \approx (I \times S^{n-1}) / \sim, *$). By [8, Theorem 11.12], there exists a bijection $\eta : \pi_n^{top}(X, x) \rightarrow \pi_n^\Omega(X, x)$ which commutes the above diagram. Thus, in order to show that η is a homeomorphism, it is enough to show that ψ is a homeomorphism. Now suppose X is a metric space, since I is locally compact and Hausdorff then the two compact-open topologies of the right hand side of the above diagram are equivalent to the uniform convergence topology [6]. It is easy to see that a sequence $\{f_n\}$ is convergent to f in $\text{Hom}((I^n, \dot{I}^n), (X, x))$ if and only if the sequence $\{f_n^\#\}$ is convergent to $f^\#$ in $\text{Hom}((I, \dot{I}), (\Omega^{n-1}(X, x), \tilde{x}))$. Hence the two topological groups $\pi_n^{top}(X, x)$ and $\pi_n^\Omega(X, x)$ are isomorphic and so we can consider two topologies on $\pi_n(X, x)$ which are equivalent when X is a metric space. By homeomorphism $\pi_n^{top}(X, x) \cong \pi_1^{top}(\Omega^{n-1}(X, x))$, we have the following assertions, (see [1,5]).

- (a) Let $\Omega^{n-1}(X, x)$ be path connected, locally path connected and semilocally simply connected. Then $\pi_n^{top}(X, x)$ is discrete.
- (b) Suppose $\Omega^{n-1}(X, x)$ is a connected separable space such that $\pi_n^{top}(X, x)$ is discrete. Then $\pi_n(X, x)$ is countable.
- (c) Let $\Omega^{n-1}(X, x)$ be connected, locally path connected and separable. Also, let $\pi_n(X, x)$ be free, then $\pi_n^{top}(X, x)$ is discrete.

The following example shows that studying topological homotopy groups may be more useful than topological fundamental groups.

Example 3.11. Let $X = \cup_{n \in \mathbb{N}} S_n$, where $S_n = \{(x, y, z) | (x - \frac{1}{n})^2 + y^2 + z^2 = \frac{1}{n^2}\}$, be a subspace of \mathbb{R}^3 . It is easy to see that each 1-loop in X is nullhomotopic (note that S_n 's are simply connected). Therefore $\pi_1^{top}(X)$ is trivial. the sequence $\{[S_n]\}$ is convergent to identity element of $\pi_2^{top}(X, 0)$, implying that $\pi_2^{top}(X, 0)$ is not discrete (see figure-2).

Remark 3.12. In general, if $\{X_i\}$ forms an inverse system of topological spaces for which each X_i contains an essential n -loop, then $\pi_n^{top}(\varprojlim X_i)$ is not discrete.

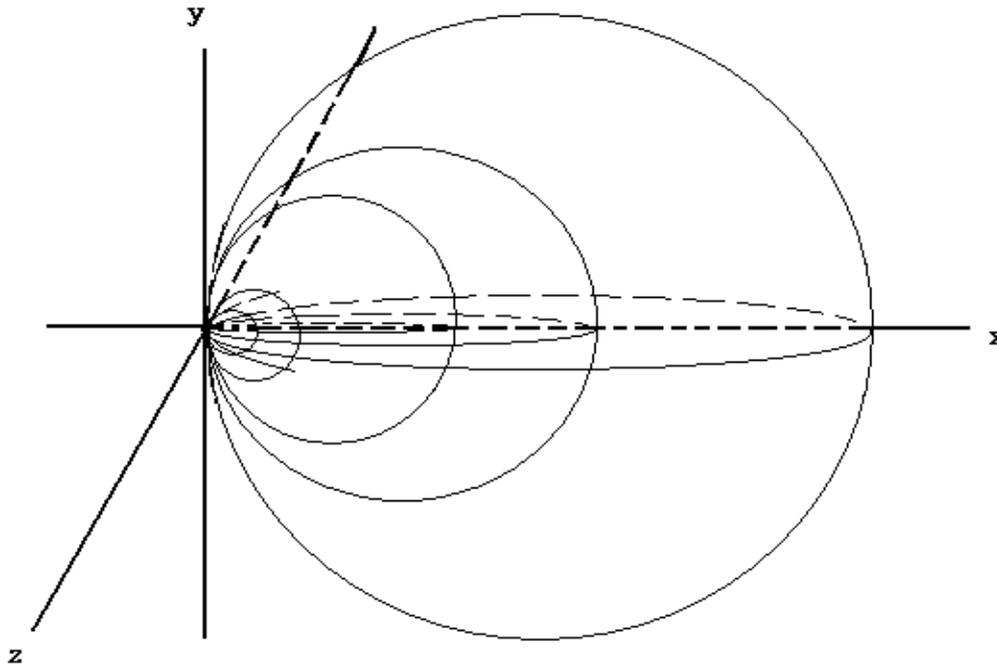


Figure-2

References

- [1] Biss, D. K., The Topological Fundamental Group and Generalized Covering Spaces, *Topology Appl.* 124, no.3 (2002) 355-371.
- [2] Conner, G. R. and Lamoreaux, J. W., On the Existence of the Universal Covering Spaces for Metric Spaces and Subsets of the Euclidean Plane, *Fundamenta Mathematicae* 187 (2005) 95-110.
- [3] Dugundji, J. A., Topological Fundamental Group, *Proc. Nat. Acad. Sci. USA*, 36 (1950) 141-143.
- [4] Fabel, P., Topological Fundamental Groups can Distinguish Spaces with Isomorphic Homotopy Groups, *Topology Proc.* 30, no.1 (2006) 185-195.
- [5] Fabel, P., Metric Spaces with Discrete Topological Fundamental Group, *Topology Appl.* 154 (2007) 635-638.
- [6] Munkers, J. R., *Topology: A First Course*, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1975.

- [7] Naber, G. L., *Topology, Geometry and Gauge Fields*, Springer, 1997.
- [8] Rottman, J.J., *An Introduction to Algebraic Topology*, Springer-Verlag, G.T.M. 119, 1988.
- [9] Spanier, E.D., *Algebraic Topology*, McGraw-Hill, 1966.

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