Integral characterizations for the dichotomy of evolution families

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Abstract

Some necessary and/or sufficient conditions for the uniform exponential dichotomy are pointed out. Thus are extended known results due to Barbashin[1], Datko [7], Lovelady [10], Pazy[14], Preda[15, 16, 17].

1 Introduction and Preliminaries

Let X be a real or complex Banach space and $B(X)$ the Banach algebra of all linear and bounded operators acting on X. We denote by $||\cdot||$ the norms of vectors and operators on X.

Consider the Cauchy Problem

$$\frac{du(t,x)}{dt} = A(t)u(t,x),\ u(0,x) = x \in X,\ t \geq 0$$

with $A(\cdot)$ locally integrable on R_+.

Roughly speaking, by dichotomy we understand the existence of a projector-valued function $P(\cdot)$, such that the solutions which start in $ImP(0)$ decay (in norm) to zero, and the solutions which start in $Im(I-P(0))$ are unbounded.

It is widely known that dichotomy and, in particular exponential dichotomy plays an important role in the investigation of the qualitative properties of nonlinear evolution equations such as linearized (in-)stability or the existence of the invariant and center manifolds (see for instance [5], [19]).

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In order to employ the modern techniques of functional analysis and operator theory in the study of the asymptotic behaviour of the solutions of the above system ($\dot{u}(t) = A(t)u(t)$) it can be associated a two-parameters family of bounded and linear operators $\Phi(t, t_0) = U(t)U^{-1}(t_0)$, where U is the unique solution of the Cauchy Problem denoted by $(A, 0, I)$:

$$\begin{cases} \dot{U}(t) = A(t)U(t) \\ U(0) = I, \text{ the identity on } X \end{cases}$$

We refer the reader to [3], [5], [12], [14], [20] for details.

Definition 1. 1. *An operator-valued two variables function* $\Phi : \{(t, s) \in \mathbb{R}\times\mathbb{R} : t \geq s \geq 0\} \mapsto B(X)$ *is called an evolution family if the following properties hold:*

e_1) $\Phi(t, t) = I$, *for all* $t \geq 0$*;*

e_2) $\Phi(t, s)\Phi(s, r) = \Phi(t, r)$, *for all* $t \geq s \geq r \geq 0$*;*

e_3) $\Phi(\cdot, s)x$ *is continuous on* $[s, \infty)$, *for all* $s \geq 0$, $x \in X$*;*
$\Phi(t, \cdot)x$ *is continuous on* $[0, t]$, *for all* $t \geq 0$, $x \in X$*;*

e_4) *there are* $M, \omega > 0$ *such that*

$$\|\Phi(t, s)\| \leq Me^{\omega(t-s)}, \quad \textit{for all} \quad t \geq s \geq 0.$$

Example 1.1. Consider the above operator Cauchy Problem $(A, 0, I)$. If $\sup\limits_{t\geq 0} \int\limits_t^{t+1} A(\tau)d\tau < \infty$ then $\Phi(t, t_0) = U(t)U^{-1}(t_0)$ is an evolution family which has the additional property that (e_2) holds for any $t, s, r \in \mathbb{R}_+$. See for instance [4], [5], [12].

Throughout in this paper we suppose that for every $t_0 \geq 0$ the vector subspace

$$X_1(t_0) = \{x_0 \in X : \ \Phi(\cdot, t_0) \in L^\infty_{[t_0,\infty)}(X)\}$$

is closed in X, where $L^\infty_{[t_0,\infty)}(X)$ is the Banach space of X-valued function f defined a.e. on $[t_0, \infty)$, such that f is strongly measurable and essentially bounded. Also we assume that $X_1(t_0)$ admits a complement $X_2(t_0)$ and we will denote by $P(t_0)$ a projector (that is $P(t_0) \in B(X)$, $P^2(t_0) = P(t_0)$) such that $Ker\ P(t_0) = X_2(t_0)$. Also we denote by $Q(t_0) = I - P(t_0)$.

Remark 1.1. For any evolution family Φ we have that
(i) $\Phi(t, t_0)\ X_1(t_0) \subset X_1(t)$ (or equivalent $\Phi(t, t_0)P(t_0) = P(t)\Phi(t, t_0)P(t_0)$), for all $t \geq t_0 \geq 0$;
(ii) $\Phi(t, s)P(s)\Phi(s, t_0)P(t_0) = \Phi(t, t_0)P(t_0)$, for all $t \geq s \geq t_0 \geq 0$;
(iii) $\Phi(t, t_0)Q(t_0)x \neq 0$, for all $t \geq t_0 \geq 0$ and $x \in X$ with $Q(t_0)x \neq 0$;

Remark 1.2. If Φ is the evolution family from the Example 1.1., then $X_1(t_0) = U(t_0)X_1(0), X_2(t_0) = U(t_0)X_2(0)$ and $P(t_0) = U(t_0)P(0)U^{-1}(t_0)$, for all $t_0 \geq 0$. Thus, in the case of the evolution families generated by differential systems the splitting at any moment $t_0 \geq 0$ can be obtained by the splitting at the moment zero.

We will assume in what follows that the projector-valued function $P(\cdot)$ is strongly continuous and bounded on R_+. Also, we will say that $P(\cdot)$ is a *dichotomy projector family* if in addition it satisfies

- $\Phi(t, t_0)P(t_0) = P(t)\Phi(t, t_0)$, for all $t \geq t_0 \geq 0$
- $\Phi(t, t_0) : KerP(t_0) \to KerP(t)$ is an isomorphism for all $t \geq t_0 \geq 0$;

Definition 1.2. *An evolution family Φ is said to be uniformly exponentially dichotomic (u.e.d) if there exist P a projector family and $N_1, N_2, \nu > 0$ such that*

d_1) $||\Phi(t, t_0)P(t_0)x|| \leq N_1 e^{-\nu(t-t_0)}||P(t_0)x||$, *for all $x \in X$ and all $t \geq t_0 \geq 0$.*

d_2) $||\Phi(t, t_0)Q(t_0)x|| \geq N_2 e^{\nu(t-t_0)}||Q(t_0)x||$, *for all $x \in X$ and all $t \geq t_0 \geq 0$.*

Recall now that the well-known theorem of A.M. Lyapunov states that if A is a $n \times n$ complex matrix then A has all its characteristic roots with real parts negative if only if for any positive definite Hermitian matrix H there exists a unique positive definite Hermitian matrix W satisfying the equation $A^*W + WA = -H$ (where * denotes the conjugate transpose of a matrix) (see [2]).

The passing to the infinite-dimensional Hilbert spaces is due to Krein and Daleckij in [5] by considering a time-invariant linear system which generates one-parameter semigroups $T(t) = e^{tA}$ where A is a bounded linear operator. Their result is elegantly extended by R. Datko in [6], for the general case of linear time-invariant systems $\dot{u}(t) = Au(t)$ where A is an unbounded linear operator which generates a C_0- semigroup.

Interesting to note here is that in the proof of the main theorem from [6], R. Datko establishes a result which have come into widespread usage in the theory of stability for strongly continuous semigroups of linear operators. More exactly, Datko states that the semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is exponentially stable if and only if, for each vector x from a general Hilbert space X, the function $t \to \|T(t)x\|$ lies in $L^2(\mathbb{R}_+, \mathbb{R}_+)$ (where $\mathbb{R}_+ = [0, \infty)$). Later, A.Pazy (see for instance [14]) shows that the result remains valid even if $L^2(\mathbb{R}_+, \mathbb{R}_+)$ is replaced by any $L^p(\mathbb{R}_+, \mathbb{R}_+)$, where $p \in [1, \infty)$ and X is a general Banach space. In 1973, R.Datko [7] generalize the results above, stating that an evolutionary process $\{\Phi(t, s)\}_{t\geq s\geq 0}$, on a Banach space X, is uniformly exponentially stable if and only if there is $p \in [1, \infty)$ such that $\sup_{s\geq 0} \int_s^\infty \|\Phi(t, s)x\|^p dt < \infty$, for each $x \in X$. Also, a nonlinear version of Datko's theorem is obtained in [8] by Ichikawa in 1984. It is worth to mention here that a version of Datko's result could be already found in the monograph of Daleckij and Krein (see Theorem 6.2., page 133 from [5]) for evolution families generated by differential systems (see Example 1.1.).

Datko-Pazy theorem was improved by Rolewicz in 1986(see [18]) when he proved that if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, nondecreasing function with $\varphi(0) = 0$ and $\varphi(u) > 0$ for each strictly positive u, and $\{\Phi(t, s)\}_{t\geq s\geq 0}$ is an evolution family on X such that $\sup\limits_{s\geq 0}\int\limits_s^\infty \varphi(\|\Phi(t,s)x\|dt < \infty$ for each $x \in X$ then Φ is uniformly exponentially stable. A shorter proof of the Rolewicz theorem was given in [22] by Q. Zheng in 1988 removing also the continuity assumption on φ. Also, we note that an analogous result to Rolewicz theorem was obtained independently by Littman [9] in 1989, in the case of C_0-semigroup, but also without continuity hypothesis for φ. Also, a discrete-time version, for the case of C_0-semigroups was provided by Zabczyk [21] in 1974, with the additional requirement that the above function φ is also convex. Jan van Neerven generalize the results above in the case of C_0-semigroups and he present an unified treatment in terms of Banach function spaces as in [13], Theorem 3.1.5. In fact he states that the semigroup $\mathcal{T} = \{T(t)\}_{t\geq 0}$ is uniformly exponentially stable if there exists a Banach function space E over R_+ with the property that $\lim\limits_{t\to\infty} ||\chi_{[0,t]}|| = \infty$ such that $||T(\cdot)x|| \in E$ for all $x \in X$. The Datko-Pazy theorem follows from this by taking $E = L^p(\mathbb{R}_+)$ and Rolewicz's result can be derived as well by taking for E a suitable Orlicz space over $\mathbb{R}_+$.

A first extension of Datko theorem to the general case of exponential dichotomy is due to Popescu and Preda in [15], where it is analyzed the case of differential systems. Later Preda and Megan generalize the Datko-Pazy theorem for dichotomy, first for C_0-semigroups [16] and second for evolution families [17]. Since we use essentially the result from [17] in our proofs, we will list it bellow.

Theorem 1.1. (Preda and Megan, 1985, [17]) *The evolution family Φ is uniformly exponentially dichotomic if there exist $m, N > 0$ and $p \geq 1$ such that*

- $(\int\limits_{t_0}^{\infty} \|\Phi(t,t_0)P(t_0)x_0\|^p dt)^{\frac{1}{p}} \leq N\|P(t_0)x_0\|$,
- $(\int\limits_{t_0}^{t} \|\Phi(s,t_0)Q(t_0)x_0\|^p dt)^{\frac{1}{p}} \leq N\|\Phi(t,t_0)Q(t_0)x_0\|$,
- $\|\Phi(t_0+1,t_0)Q(t_0)x_0\| \geq m\|Q(t_0)x_0\|$,

for all $t \geq t_0 \geq 0$ and $x_0 \in X$.

Note that, in all above integral conditions, the integrand is the first parameter of the evolution family. Integral characterizations with the second parameter as integrand are obtained firstly by Barbashin in 60's. Thus in [1], E.A. Barbashin proved that an evolution family Φ is uniformly exponentially stable if and only if there is $K > 0$ such that $\int_{t_0}^t ||\Phi(t,\tau)||\ d\tau \leq K$, for all $t \geq t_0 \geq 0$. This result is extended to the general case of dichotomy for differential systems in [10], [17], [4] and it states that an evolution family Φ is uniformly exponentially dichotomic if and only if there is $K > 0$ and $p > 0$ such that $(\int_{t_0}^t ||\Phi(t)P_1\Phi^{-1}(\tau)||^p\ d\tau)^{\frac{1}{p}} + (\int_t^\infty ||\Phi(t)P_2\Phi^{-1}(\tau)||^p\ d\tau)^{\frac{1}{p}} \leq K$, for all $t \geq t_0 \geq 0$.

Analyzing the technique of proof of the above results we can distinguish that Datko's type related results are connected with the Lyapunov method for the study of the asymptotic behaviour if differential systems and Barbashin's type results are related to Perron's method (test functions). The present approach extends both Datko-Pazy and Barbashin line of results to the general case of uniform exponential dichotomy of abstract evolution families.

Thus, Theorem 2.1. is a version of Theorem 1.1. [17] when it is assumed the existence of a dichotomic projector family (i.e. we assume the invertibility of the operator $\Phi(t, t_0)$ on $KerP(t_0)$). Theorem 2.2. is a generalization of Datko Theorem from [7] to the general case of uniform exponential dichotomy. Theorems 2.3, 2.4., 2.5. are extension of the above Barbashin's result from [1], and respectively of some theorems pointed out by Coppel [5], Lovelady [10], Preda-Megan [16, 17].

2 Main Results

Lemma 2.1. *Let* $f,\ g : \mathbb{R}_+ \to \mathbb{R}_+$ *with* g *continuous on* R_+. *If*

i) $f(t) \le g(t - t_0) f(t_0)$, *for each* $t \ge t_0 \ge 0$;

ii) there exists $\delta > 0$ *with* $g(\delta) < 1$.

Then there exist $N,\ \nu > 0$, *independently of* f, *such that* $f(t) \le N e^{-\nu(t-t_0)} f(t_0)$, *for each* $t \ge t_0 \ge 0$.

Proof. See for instance [11].

Theorem 2.1 *Assume that* $\{P(t)\}_{t\ge 0}$ *is a dichotomic projector family associated with the evolution family* Φ. *Then* Φ *is uniformly exponential dichotomic if and only if there exist the constants* $p, k, m > 0$ *such that:*

i) $\left(\int_t^\infty ||\Phi(\tau, t)P(t)x||^p d\tau\right)^{\frac{1}{p}} + \left(\int_0^t ||\Phi^{-1}(t, \tau)Q(t)x||^p d\tau\right)^{\frac{1}{p}} \le k||x||$, *for each* $t \ge 0$ *and* $x \in X$;

ii) $m||Q(t)x|| \le ||\Phi(t + 1, t)Q(t)x||$, *for each* $t \ge 0$ *and* $x \in X$.

Proof. Sufficiency: By (i) we have that

$$\left(\int_t^\infty ||\Phi(\tau, t)P(t)x||^p d\tau\right)^{\frac{1}{p}} \le k||x||, \text{ for all } t \ge 0, \text{ and } x \in X.$$

Applying (i) for $t \ge t_0 \ge 0$ and $x = \Phi(t, t_0)Q(t_0)y,\ y \in X$ we obtain that

$$\left(\int_{t_0}^t ||\Phi(\tau, t_0)Q(t_0)y||^p d\tau\right)^{\frac{1}{p}} \le k||\Phi(t, t_0)Q(t_0)y||,$$

for all $t \ge t_0 \ge 0$ and $y \in X$. Using Theorem 1.1. we get that Φ is uniformly exponentially dichotomic.

Necessity. It follows easily by taking into account that

$$||\Phi^{-1}(t, t_0)Q(t)y|| \le \frac{1}{N_2} e^{-\nu(t-t_0)}||y||.$$

(see Definition 1.2.) and by setting $k = \frac{1}{(\nu p)^{\frac{1}{p}}}(N_1 \sup\limits_{t\ge 0} \|P(t)\| + \frac{1}{N_2} \sup\limits_{t\ge 0} \|Q(t)\|)$.

Theorem 2.2 *Assume that* $\{P(t)\}_{t\ge 0}$ *is a dichotomic projector family associated with the evolution family* Φ. *If there exist* $k, p > 0$ *with*

$$\left(\int_t^\infty ||\Phi(\tau, t)P(t)x||^p d\tau\right)^{\frac{1}{p}} + \left(\int_0^t ||\Phi^{-1}(t, \tau)Q(t)x||^p d\tau\right)^{\frac{1}{p}} \le k||x||,$$

for all $t \ge 0$ *and* $x \in X$, *then there exist* $N_1, \nu > 0$ *such that for each* $\alpha > 0$ *we can find* $N_\alpha > 0$ *with:*

i) $||\Phi(t,t_0)P(t_0)x|| \le N_1 e^{-\nu(t-s)}||\Phi(s,t_0)P(t_0)x||$, *for all* $t \ge s \ge t_0 \ge 0$ *and* $x \in X$*;*

ii) $||\Phi(t,t_0)Q(t_0)x|| \ge N_\alpha e^{\nu(t-s)}||\Phi(s,t_0)Q(t_0)x||$, *for all* $t \ge s \ge t_0 + \alpha$ *and* $x \in X$*.*

Proof. By hypothesis we get that

$$\left(\int_t^\infty ||\Phi(\tau,t)P(t)x||^p d\tau\right)^{\frac{1}{p}} \le k||x||, \text{ for all } t \ge 0, \text{ and } x \in X.$$

Applying Theorem 1.1. we get that there exist $N_1,\ \nu_1 > 0$ such that

$$||\Phi(t,t_0)P(t_0)x|| \le N_1 e^{-\nu_1(t-t_0)}||x||, \text{ for all } t \ge t_0 \ge 0, \text{ and } x \in X.$$

Writing the hypothesis for $t \ge t_0 \ge 0$ and $x = \Phi(t,t_0)Q(t_0)y,\ y \in X$, it follows that

$$\left(\int_{t_0}^t ||\Phi^{-1}(t,\tau)Q(t)\Phi(t,t_0)Q(t_0)y||^p d\tau\right)^{\frac{1}{p}} \le k||\Phi(t,t_0)Q(t_0)y||$$

and hence

$$\int_{t_0}^t ||\Phi(\tau,t_0)Q(t_0)y||^p d\tau \le k^p||\Phi(t,t_0)Q(t_0)y||^p, \text{ for all } t \ge t_0 \ge 0, \text{ and } y \in X.$$

Therefore we deduce that

$$(\diamond) \qquad \varphi(s)e^{\frac{1}{k^p}(t-s)} \le \varphi(t) \le k^p||\Phi(t,t_0)y||^p, \text{ for all } t \ge s \ge t_0 \ge 0, \text{ and } y \in X,$$

where

$$\varphi(t) = \int_{t_0}^t ||\Phi(\tau,t_0)Q(t_0)y||^p d\tau.$$

For $s \ge t_0 \ge 0$ and $\tau \in [t_0, s]$, we have that

$$||\Phi(s,t_0)Q(t_0)y||^p \le ||\Phi(s,\tau)||^p\ ||\Phi(\tau,t_0)Q(t_0)y||^p \le$$
$$\le M^p e^{\omega p(s-\tau)}||\Phi(\tau,t_0)Q(t_0)y||^p.$$

Then

$$\frac{1}{M^p\omega p}(1 - e^{-\omega p(s-t_0)})||\Phi(s,t_0)Q(t_0)y||^p \le \varphi(s).$$

For $\alpha > 0$ and $s - t_0 \ge \alpha$ we obtain that

$$(1 - e^{-\omega p\alpha})\frac{1}{M^p\omega p}||\Phi(s,t_0)Q(t_0)y||^p \le \varphi(s), \quad \text{for all } s \ge t_0 + \alpha,$$

and by $(\diamond)$ we get that for each $\alpha > 0$, there exists

$$N_\alpha = (1 - e^{-\omega p\alpha})^{\frac{1}{p}}\frac{1}{kM(\omega p)^{\frac{1}{p}}},$$

with

$$N_\alpha e^{\frac{1}{pk^p}(t-s)}||\Phi(s,t_0)Q(t_0)y|| \le ||\Phi(t,t_0)Q(t_0)y||, \text{ for all } t \ge s \ge t_0 + \alpha.$$

Thus there exists $\nu_2 = \frac{1}{pk^p} > 0$ such that for each $\alpha > 0$, there exists $N_\alpha > 0$ with

$$||\Phi(t,t_0)Q(t_0)x|| \geq N_\alpha e^{\nu_2(t-s)}||\Phi(s,t_0)Q(t_0)x||, \text{ for all } t \geq s \geq t_0+\alpha, \text{ for all } x \in X.$$

Thus there exist $\nu = \min\{\nu_1, \nu_2\} > 0$ and $N_1 > 0$ such that for any $\alpha > 0$, we can find $N_\alpha > 0$ with

$$||\Phi(t,t_0)P(t_0)x|| \leq N_1 e^{-\nu(t-s)}||\Phi(s,t_0)P(t_0)x||, \text{ for all } t \geq s \geq t_0$$

and

$$||\Phi(t,t_0)Q(t_0)x|| \geq N_\alpha e^{\nu(t-s)}||\Phi(s,t_0)Q(t_0)x||, \text{ for all } t \geq s \geq t_0+\alpha, \text{ and } x \in X.$$

Theorem 2.3. *Let Φ be an evolution family. If there exist $p, k > 0$ such that:*

i) $\left(\int_0^t ||\Phi(t,\tau)P(\tau)||^p d\tau\right)^{\frac{1}{p}} \leq k$, *for each $t \geq 0$;*

ii) $\left(\int_t^\infty \frac{d\tau}{||\Phi(\tau,t_0)Q(t_0)x||^p}\right)^{\frac{1}{p}} \leq \frac{k}{||\Phi(t,t_0)Q(t_0)x||}$, *for each $t \geq t_0 \geq 0$ and $x \in X$ with $Q(t_0)x \neq 0$;*

then Φ is uniformly exponentially dichotomic.

Proof. Let $t \geq t_0 + 1$ and $r(t) = M \sup_{\tau \geq 0} ||P(\tau)||e^{\omega t}$. Then

$$||\Phi(t,t_0)P(t_0)||^p \int_{t_0}^t r^{-p}(\tau - t_0)d\tau = \int_{t_0}^t ||\Phi(t,\tau)P(\tau)\Phi(\tau,t_0)P(t_0)||^p r^{-p}(\tau-t_0)d\tau$$

$$\leq \int_{t_0}^t ||\Phi(t,\tau)P(\tau)||^p d\tau \leq \int_0^t ||\Phi(t,\tau)P(\tau)||^p d\tau \leq k^p.$$

But

$$\int_{t_0}^t r^{-p}(\tau-t_0)d\tau = \int_0^{t-t_0} r^{-p}(s)ds \geq \int_0^1 r^{-p}(s)ds = \alpha > 0$$

and hence

$$\alpha^{\frac{1}{p}}||\Phi(t,t_0)P(t_0)|| \leq k, \quad \text{for all } t \geq t_0+1,$$

which implies that

$$||\Phi(t,t_0)P(t_0)|| \leq \frac{k}{\alpha^{\frac{1}{p}}}, \quad \text{for all } t \geq t_0+1.$$

Taking now $t_0 \leq t < t_0+1$ we have that

$$||\Phi(t,t_0)P(t_0)|| \leq Me^\omega \sup_{t\geq 0} ||P(t)||.$$

Denoting

$$L = \max\{\frac{k}{\alpha^{\frac{1}{p}}}, Me^\omega \sup_{t \geq 0}||P(t)||\},$$

we obtain that

$$(\diamond\diamond) \qquad ||\Phi(t,t_0)P(t_0)|| \le L, \text{ for all } t \ge t_0 \ge 0.$$

Taking by this time $t \ge t_0 \ge 0$ and $\tau \in [t_0, t]$ we get that

$$||\Phi(t,t_0)P(t_0)|| \le L||\Phi(t,\tau)P(\tau)||,$$

which implies that

$$(\diamond\diamond\diamond) \qquad (t-t_0)^{\frac{1}{p}}||\Phi(t,t_0)P(t_0)|| \le Lk, \text{ for all } t \ge t_0 \ge 0.$$

Adding up $(\diamond\diamond)$ with $(\diamond\diamond\diamond)$ we deduce that

$$||\Phi(t,t_0)P(t_0)|| \le \frac{L(1+k)}{1+(t-t_0)^{\frac{1}{p}}}, \text{ for all } t \ge t_0 \ge 0,$$

and hence

$$||\Phi(t,t_0)P(t_0)|| \le ||\Phi(t,\tau)P(\tau)||\ ||\Phi(\tau,t_0)P(t_0)|| \le$$
$$\le \frac{L(1+k)}{1+(t-\tau)^{\frac{1}{p}}}||\Phi(\tau,t_0)P(t_0)||, \text{ for all } t \ge \tau \ge t_0 \ge 0.$$

Applying Lemma 2.1. we have that there exist $N_1, \nu_1 > 0$ such that

$$||\Phi(t,t_0)P(t_0)|| \le N_1 e^{-\nu_1(t-t_0)}, \text{ for all } t \ge t_0 \ge 0.$$

Take now $x \in X$ with $Q(t_0)x \ne 0$ and $t \ge t_0 \ge 0$. Denoting

$$\varphi(t) = \int_t^\infty \frac{d\tau}{||\Phi(\tau,t_0)Q(t_0)x||^p},$$

by (ii) we have that

$$\varphi(t) \le -k^p\dot{\varphi}(t),$$

and hence

$$\frac{1}{k^p} \le -\frac{\dot{\varphi}(t)}{\varphi(t)},$$

which implies

$$e^{\frac{1}{k^p}(t-t_0)} \le \frac{\varphi(t_0)}{\varphi(t)}, \ \forall\ t \ge t_0 \ge 0.$$

Thus

$$\int_t^\infty \frac{d\tau}{||\Phi(\tau,t_0)Q(t_0)x||^p} e^{\frac{1}{k^p}(t-t_0)} \le \frac{k^p}{||Q(t_0)x||^p}, \text{ for all } t \ge t_0 \ge 0.$$

From here it follows that

$$\int_t^{t+1} \frac{d\tau}{||\Phi(\tau,t_0)Q(t_0)x||^p} e^{\frac{1}{k^p}(t-t_0)} \le \frac{k^p}{||Q(t_0)x||^p}.$$

Taking into account that

$$||\Phi(\tau,t_0)Q(t_0)x|| = ||\Phi(\tau,t)\Phi(t,t_0)Q(t_0)x|| \le$$

$$\leq Me^{\omega}||\Phi(t,t_0)Q(t_0)x||, \quad \text{for all } \tau \in [t, t+1].$$

Thus, we obtain that

$$\frac{1}{M^p e^{\omega p}||\Phi(t,t_0)Q(t_0)x||^p} \leq \int_t^{t+1} \frac{d\tau}{||\Phi(\tau,t_0)Q(t_0)x||^p},$$

which implies

$$\frac{1}{M^p e^{\omega p}||\Phi(t,t_0)Q(t_0)x||^p} e^{\frac{1}{k^p}(t-t_0)} \leq \frac{k^p}{||Q(t_0)x||^p},$$

and hence

$$||\Phi(t,t_0)Q(t_0)x|| \geq \frac{1}{kMe^{\omega}} e^{\frac{1}{pk^p}(t-t_0)}||Q(t_0)x||, \quad \text{for all } t \geq t_0 \geq 0.$$

Denoting

$$N_2 = \frac{1}{kMe^{\omega}} \text{ and } \nu_2 = \frac{1}{pk^p},$$

we have that

$$||\Phi(t,t_0)Q(t_0)x|| \geq N_2 e^{\nu_2(t-t_0)}||Q(t_0)x||, \quad \text{for all } t \geq t_0 \geq 0.$$

Thus Φ is uniformly exponentially dichotomic.

Theorem 2.4*Assume that $\{P(t)\}_{t\geq 0}$ be a projector-family associated to Φ with the property that*

$$P(t)\Phi(t,t_0) = \Phi(t,t_0)P(t_0), \quad \textit{for all}\, t \geq t_0 \geq 0.$$

Then Φ is uniformly exponentially dichotomic if and only if there exist $p > 0,\ k > 0$ such that:

i) $\left(\int_0^t ||\Phi(t,\tau)P(\tau)||^p d\tau\right)^{\frac{1}{p}} \leq k$, *for each $t \geq 0$;*

ii) $\left(\int_t^{\infty} \frac{d\tau}{||\Phi(\tau,t_0)Q(t_0)x||^p}\right)^{\frac{1}{p}} \leq \frac{k}{||\Phi(t,t_0)Q(t_0)x||}$, *for each $t \geq t_0 \geq 0$ and $x \in X$ with $Q(t_0)x \neq 0$.*

Proof. Necessity. Since Φ is uniformly exponentially dichotomic, then for each $p > 0$ we have that

$$\left(\int_0^t ||\Phi(t,\tau)P(\tau)||^p d\tau\right)^{\frac{1}{p}} \leq N_1\left(\int_0^t e^{-\nu p(t-\tau)} d\tau\right)^{\frac{1}{p}} \sup_{t\geq 0}||P(t)|| \leq$$

$$\leq \frac{N_1 \sup_{t\geq 0}||P(t)||}{(\nu p)^{\frac{1}{p}}}, \quad \text{for all } t \geq 0.$$

Let $x \in X$ with $Q(t_0)x \neq 0$ and $\tau \geq t \geq t_0 \geq 0$. Then

$$||\Phi(\tau,t_0)Q(t_0)x|| = ||\Phi(\tau,t)Q(t)\Phi(t,t_0)x|| \geq N_2 e^{\nu(\tau-t)}||\Phi(t,t_0)Q(t_0)x||,$$

for all $\tau \geq t \geq t_0 \geq 0$. Thus

$$\Big(\int_t^\infty \frac{1}{||\Phi(\tau,t_0)Q(t_0)x||^p} d\tau\Big)^{\frac{1}{p}} \leq \frac{1}{N_2}\Big(\int_t^\infty e^{-\nu p(\tau-t)} d\tau\Big)^{\frac{1}{p}} \frac{1}{||\Phi(t,t_0)Q(t_0)x||}.$$

From here we have that

$$\Big(\int_t^\infty \frac{d\tau}{||\Phi(\tau,t_0)Q(t_0)x||^p}\Big)^{\frac{1}{p}} \leq \frac{1}{N_2(\nu p)^{\frac{1}{p}}} \frac{1}{||\Phi(t,t_0)Q(t_0)x||},$$

for all $t \geq t_0 \geq 0$, and $x \in X$ with $Q(t_0)x \neq 0$. Setting now

$$k = \max\{N_1 \sup_{t\geq 0} ||P(t)||, \frac{1}{N_2}\} \frac{1}{(\nu p)^{\frac{1}{p}}}$$

we complete the proof.

Sufficiency: It follows by Theorem 2.3.

Theorem 2.5. *Assume that $\{P(t)\}_{t\geq 0}$ is a dichotomic projector family associated with the evolution family Φ. Then Φ is uniformly exponentially dichotomic if and only if there exist $p, k > 0$ such that*

$$\Big(\int_0^t ||\Phi(t,\tau)P(\tau)||^p d\tau\Big)^{\frac{1}{p}} + \Big(\int_t^\infty ||\Phi^{-1}(\tau,t)Q(\tau)||^p d\tau\Big)^{\frac{1}{p}} \leq k, \ \forall\, t \geq 0.$$

Proof. Sufficiency. It follows identically as in Theorem 2.3. that there exist $N_1, \nu_1 > 0$ such that

$$||\Phi(t,t_0)P(t_0)|| \leq N_1 e^{-\nu_1(t-t_0)}, \quad \text{for all } t \geq t_0 \geq 0.$$

Take now $\tau \geq t \geq t_0 \geq 0$ and $x \in X \setminus \{0\}$. Then

$$\Phi(\tau,t_0)Q(t_0) = \Phi(\tau,t)\Phi(t,t_0)Q(t_0),$$

which implies that

$$\Phi^{-1}(\tau,t)\Phi(\tau,t_0)Q(t_0) = \Phi(t,t_0)Q(t_0),$$

and hence

$$||\Phi(t,t_0)Q(t_0)x||^p \int_t^\infty \frac{d\tau}{||\Phi(\tau,t_0)Q(t_0)x||^p} = \int_t^\infty \frac{||\Phi^{-1}(\tau,t)Q(\tau)\Phi(\tau,t_0)Q(t_0)x||^p}{||\Phi(\tau,t_0)Q(t_0)x||^p} d\tau \leq$$

$$\leq \int_t^\infty ||\Phi^{-1}(\tau,t)Q(\tau)||^p d\tau \leq k^p, \quad \text{for all } t \geq t_0 \geq 0 \text{ and } x \in X \setminus \{0\}.$$

Thus we get that

$$\Big(\int_t^\infty \frac{d\tau}{||\Phi(\tau,t_0)Q(t_0)x||^p}\Big)^{\frac{1}{p}} \leq \frac{k}{||\Phi(t,t_0)Q(t_0)x||}, \quad \text{for all } t \geq t_0 \geq 0, \ \text{ and } x \in X\setminus\{0\}.$$

Applying Theorem 2.3. we obtain that Φ is uniformly exponentially dichotomic.

Necessity. Since Φ is exponentially dichotomic we have that

$$\Big(\int_0^t ||\Phi(t,\tau)P(\tau)||^p d\tau\Big)^{\frac{1}{p}} \leq N_1\Big(\int_0^t e^{-\nu p(t-\tau)}d\tau\Big)^{\frac{1}{p}} \sup_{\tau\geq 0}||P(\tau)|| \leq$$

$$\leq \frac{N_1 \sup_{t\geq 0}||P(t)||}{(\nu p)^{\frac{1}{p}}}, \quad \text{for all } t \geq 0.$$

Moreover, if

$$\Phi(t,t_0)Q(t_0)x = y,$$

then

$$||y|| \geq N_2 e^{\nu(t-t_0)}||\Phi^{-1}(t,t_0)Q(t)y||$$

hence

$$||\Phi^{-1}(t,t_0)Q(t)|| \leq \frac{1}{N_2}e^{-\nu(t-t_0)}||Q(t)||, \quad \text{for all } t \geq t_0 \geq 0.$$

Thus, for each $p > 0$ we have that

$$\Big(\int_t^\infty ||\Phi^{-1}(\tau,t)Q(\tau)||^p d\tau\Big)^{\frac{1}{p}} \leq \frac{1}{N_2}\Big(\int_t^\infty e^{-\nu p(\tau-t)}d\tau\Big)^{\frac{1}{p}} \sup_{t\geq 0}||Q(t)|| =$$

$$= \frac{1}{N_2(\nu p)^{\frac{1}{p}}}\sup_{t\geq 0}||Q(t)||.$$

Denoting by

$$k = (N_1 \sup_{t\geq 0}||P(t)|| + \frac{1}{N_2}\sup_{t\geq 0}||Q(t)||)\frac{1}{(\nu p)^{\frac{1}{p}}},$$

we obtain that

$$\Big(\int_0^t ||\Phi(t,\tau)P(\tau)||^p d\tau\Big)^{\frac{1}{p}} + \Big(\int_t^\infty ||\Phi^{-1}(\tau,t)Q(\tau)||^p d\tau\Big)^{\frac{1}{p}} \leq k,$$

for all $t \geq 0$ and $p > 0$.

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