Differentially transcendental functions

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Abstract

The aim of this article is to exhibit a method for proving that certain analytic functions are not solutions of algebraic differential equations. The method is based on model-theoretic properties of differential fields and properties of certain known transcendental differential functions, as of $\Gamma(x)$. Furthermore, it also determines differential transcendence of solution of some functional equations.

1   Notation and preliminaries

The theory DF_0 of differential fields of characteristic 0 is the theory of fields with additional two axioms that relate to the derivative D:

$$D(x + y) = Dx + Dy, \quad D(xy) = xDy + yDx.$$

Thus, a model of DF_0 is a differential field $\mathbf{K} = (K, +, \cdot, D, 0, 1)$ where $(K, +, \cdot, 0, 1)$ is a field and D is a differential operator satisfying the above axioms. A. Robinson proved that DF_0 has a model completion, and then defined DCF_0 to be the model completion of DF_0. Subsequently, L. Blum found simple axioms of DFC_0 without refereeing to differential polynomials in more than one variable, see [22]. In the following, if not otherwise stated, $\mathbf{F}, \mathbf{K}, \mathbf{L}, \ldots$ will denote differential fields, $F, L, K, \ldots$ their domains while $\mathbf{F}^*, \mathbf{K}^*, \mathbf{L}^*, \ldots$ will denote their field parts, i.e. $\mathbf{F}^* = (F, +, \cdot, 0, 1)$. Thus, $\mathbf{F}^*[x_1, x_2, \ldots, x_n]$ denotes the set of (ordinary) algebraic polynomials over $\mathbf{F}^*$ in variables $x_1, x_2, \ldots, x_n$. The symbol $\mathbf{L}\{X\}$ denotes the ring of differential polynomials over $\mathbf{L}$ in the variable X. Hence, if $f \in L\{X\}$ then for some $n \in N$, $N = \{0, 1, 2, \ldots\}$, $f = f(X, DX, \ldots, D^n X)$ where $f \in \mathbf{F}^*(x, y_1, y_2, \ldots, y_n)$.

Suppose $\mathbf{L} \subseteq \mathbf{K}$. The symbol $\mathrm{td}(\mathbf{K}|\mathbf{L})$ denotes the transcendental degree of $\mathbf{K}^*$ over $\mathbf{L}^*$. The basic properties of td are described in the following proposition.

Proposition 1.1. *Let* $\mathbf{A} \subseteq \mathbf{B} \subset \mathbf{C}$ *be ordinary algebraic fields. Then*
a. $\mathrm{td}(\mathbf{B}|\mathbf{A}) \leq \mathrm{td}(\mathbf{C}|\mathbf{A})$.   **b.** $\mathrm{td}(\mathbf{C}|\mathbf{A}) = \mathrm{td}(\mathbf{C}|\mathbf{B}) + \mathrm{td}(\mathbf{B}|\mathbf{A})$.

The statement follows from the fact that every transcendental base of $\mathbf{B}$ over $\mathbf{A}$ can be extended to a transcendental base of $\mathbf{C}$ over $\mathbf{A}$, see [15].

If $b \in K$, then $\mathbf{L}(b)$ denotes the simple differential extension of $\mathbf{L}$ in $\mathbf{K}$, i.e. $\mathbf{L}(b)$ is the smallest differential subfield of $\mathbf{K}$ containing both L and b. If $\mathbf{L} \subseteq \mathbf{K}$ then $b \in K$ is differential algebraic over $\mathbf{L}$ if and only if there is a non-zero differential polynomial $f \in L\{X\}$ such that $f(b, Db, D^2b, \ldots, D^nb) = 0$; if b is not differential algebraic over $\mathbf{L}$ then b is differential transcendental over $\mathbf{L}$. $\mathbf{K}$ is a differential algebraic extension of $\mathbf{L}$ if every $b \in K$ is differential algebraic over $\mathbf{L}$. Next, $f = 0$ is algebraic differential equation if $f \in L\{X\}$. If $f \in L\{X\}\backslash L$, the order of f, denoted by $\mathrm{ord}f$, is the largest n such that D^nX occurs in f. If $f \in L$ we put $\mathrm{ord}\, f = -1$ and then we write $f(a) = f$ for each a. For $f \in L\{X\}$ we shall write occasionally f' instead of Df, and $f(a)$ instead of $f(a, Da, D^2a, \ldots, D^na)$ for each a and $n = \mathrm{ord}\, f$.

Models of DCF_0 are differentially closed fields. A differential field $\mathbf{K}$ is *differentially closed* if whenever $f, g \in \mathbf{K}\{X\}$, g is non-zero and $\mathrm{ord}\, f > \mathrm{ord}\, g$, there is an $a \in K$ such that $f(a) = 0$ and $g(a) \neq 0$. Let us observe that any differentially closed field is algebraically closed. The theory DCF_0 admits elimination of quantifiers and it is submodel complete (A. Robinson): if $\mathbf{F} \subseteq \mathbf{L}, \mathbf{K}$ then $\mathbf{K}_F \equiv \mathbf{L}_F$, i.e. $\mathbf{K}$ and $\mathbf{L}$ are elementary equivalent over $\mathbf{F}$.

Other notations, notions and results concerning differential fields that will be used in this article will be as in [22] or [18].

2   Differential algebraic extensions

In this section we shall state certain properties of extensions of differential fields, which we need to prove the main theorem 2.8. These properties, described by propositions 2.1 – 2.5, parallel in most cases properties of algebraic fields (without differential operators), by replacing the notion of the algebraic degree with the notion of the transcendental degree of fields and $\mathrm{deg}f$ by $\mathrm{ord}f$. Proofs of these propositions are standard and can be found in the basic literature on differential fields and therefore they are omitted.

Proposition 2.1. *Suppose* $\mathbf{L} \subseteq \mathbf{K}$ *and let* $b \in K$. *Then* b *is differentially algebraic over* $\mathbf{F}$ *if and only if* $\mathrm{td}(\mathbf{L}(b)|\mathbf{L}) < \infty$.

Suppose $\mathbf{L} \subseteq \mathbf{K}$ and let $a_1, a_2, \ldots, a_n \in K$ be differentially algebraic over $\mathbf{L}$. Then

$$\mathbf{L}(a_1, a_2, \ldots, a_i) = \mathbf{L}(a_1, a_2, \ldots, a_{i-1})(a_i)$$

and a_i is differentially algebraic over $\mathbf{L}(a_1, a_2, \ldots, a_{i-1})$ for each $i = 1, 2, \ldots, n$. Thus, by propositions 1.1 and 2.1 we have

$$\begin{aligned} \mathrm{td}(\mathbf{L}(a_1, a_2, \ldots, a_n)|\mathbf{L}) = \ &\mathrm{td}(\mathbf{L}(a_1, a_2, \ldots, a_n)|\mathbf{L}(a_1, a_2, \ldots, a_{n-1})) + \\ &\ldots + \mathrm{td}(\mathbf{L}(a_1)|\mathbf{L}) < \infty, \end{aligned}$$

so every $b \in L(a_1, \ldots, a_n)$ is differentially algebraic over $\mathbf{L}$. Hence $\mathbf{L}(a_1, \ldots, a_n)$ is differentially algebraic over $\mathbf{L}$.

Proposition 2.2. *Let* $\mathbf{L} \subseteq \mathbf{K}$ *and suppose* $b, a_1, \ldots, a_n \in K$. *If* $a_1, \ldots, a_n$ *are differentially algebraic over* $\mathbf{L}$ *and* b *is differentially algebraic over* $\mathbf{L}(a_1, \ldots, a_n)$, *then* b *is differentially algebraic over* $\mathbf{L}$.

Proposition 2.3. *If* $\mathbf{F} \subseteq \mathbf{L} \subseteq \mathbf{K}$ *and* $\mathbf{L}$ *is differentially algebraic over* $\mathbf{F}$ *and* $\mathbf{K}$ *is differentially algebraic over* $\mathbf{L}$ *then* $\mathbf{K}$ *is differentially algebraic over* $\mathbf{L}$.

Proposition 2.4. *Suppose* $\mathbf{L} \subseteq \mathbf{K}$ *and* $b \in K$. *Then* b *is a differentially algebraic over* $\mathbf{L}$ *if and only if* Db *is differentially algebraic over* $\mathbf{L}$.

Proposition 2.5. *Suppose* $\mathbf{F} \subseteq \mathbf{K}$ *and let*

$$L = \{b \in K \colon b \text{ is differentially algebraic over } \mathbf{F}\}.$$

Then

a. *L is a differential subfield of $\mathbf{K}$ extending differentially algebraic $\mathbf{F}$.*

b. *If $\mathbf{K}$ is differentially algebraic closed then $\mathbf{L}$ is differentially algebraic closed.*

Remark 2.6. *There are differential fields with elements not having functional representation. One example of this kind is the Hardy field of germs of functions* [5].

Let us denote by $\mathcal{M}_D$ the class of complex functions meromorphic on a complex domain D. If $D = C$ then we shall write $\mathcal{M}$ instead of $\mathcal{M}_D$. Let $\mathcal{C} = \mathbf{C}(z)$ be the differential field of complex rational functions. Then $\mathcal{M}$ is differential field and $\mathcal{C} \subseteq \mathcal{M}$. Further, let $\mathcal{L} = \{f \in \mathcal{M} : f \text{ differentially algebraic over } \mathcal{C}\}$. By Proposition 2.5 $\mathcal{L}$ is a differential subfield of $\mathcal{M}$ extending differentially algebraic $\mathcal{C}$.

Assume that e, g are complex functions. If e is an entire function (i.e. holomorphic in entire finite complex plane) and g is meromorphic, then their composition $e \circ g$ is not necessary meromorphic, as the example $e(z) = e^z$, $g(z) = 1/z$ shows. However, $g \circ e$ is meromorphic since g is a quotient of entire functions, and obviously entire functions are closed under compositions. The next proposition states that a similar property holds for algebraic differential functions.

Proposition 2.7. *Let* e, g *be complex algebraic differential functions over* $\mathcal{C}$. *If* e *is entire and* g *is meromorphic, then* $g \circ e$ *is meromorphic and differentially algebraic over* $\mathcal{C}$.

Proof. Suppose that g is a solution of an algebraic differential equation over $\mathcal{C}$, i.e.

$$f(z, g, Dg, \ldots, D^n g) \equiv 0 \text{ for some } f \in \mathcal{C}[z, u_0, u_1, \ldots, u_n].$$

If $e' = 0$ then e is a constant, so we may assume the nontrivial case, that $e' \neq 0$. There are rational expressions λ_{kj} in $e', e'', \ldots$ such that

$$(D^k u) \circ e = \sum_{j=1}^{k} \lambda_{kj} D^j(u \circ e), \quad u \text{ is any meromorphic function.}$$

In fact, the sequence λ_{kj} satisfies the recurrent identity

$$\lambda_{k+1,j} = (\lambda'_{kj} + \lambda_{k,j-1})/e', \ \lambda_{11} = 1/e', \ \lambda_{k0} = 0, \ \lambda_{kj} = 0 \text{ if } j > k.$$

It is easy to see that $\lambda_{kk} = (1/e')^k$. Let

$$f_1 = f(e, y, \lambda_{11}Dy, \lambda_{21}Dy + \lambda_{22}D^2y, \ldots, \lambda_{n1}Dy + \ldots + \lambda_{nn}D^ny).$$

Then $g \circ e$ is a solution of $f_1 = 0$: since $(D^kg) \circ e = \sum_{j=1}^k \lambda_{kj}D^j(g \circ e)$, it follows, taking $h = g \circ e$,

$$\begin{aligned} f_1(z, h, Dh, \ldots, D^nh) &= f(e, g \circ e, (Dg) \circ e, \ldots, (D^ng) \circ e) \\ &= f(z, g, Dg, \ldots, D^ng)|_{z=e} \equiv 0. \end{aligned}$$

Therefore, $g \circ e$ is meromorphic and differentially algebraic over $\mathcal{C}$.   ∎

Let us mention that N. Steinmetz [24, Satz 2.] has considered a proposition which is the converse of the one we proved above.

The next theorem is a direct corollary of the propositions 2.5 and 2.7, since $\mathcal{L}$ is a field and therefore it is closed under values of rational expressions.

Theorem 2.8. *Let $a(z)$ be a complex differential transcendental function over $\mathcal{C}$, $f(z, u_0, u_1, \ldots, u_m, y_1, \ldots, y_n)$ a rational expression over $\mathcal{L}$, and assume that $e_1, \ldots, e_m$ are entire functions which are differentially algebraic over $\mathcal{C}$, $e_i' \neq 0$, $1 \leq i \leq m$. If b is meromorphic and $f(z, b, b \circ e_1, \ldots, b \circ e_m, Db, \ldots, D^nb) \equiv a(z)$ then b is differential transcendental over $\mathcal{C}$.*

We shall use it in proofs of differential transcendentality of certain complex functions. In most cases functions $e_i(z)$ will be linear functions $\alpha z + \beta$, $\alpha \neq 0$, and f will be a polynomial over $\mathcal{C}$ (observe that $\mathcal{C} \subset \mathcal{L}$). Also, it is possible to consider differential transcendental functions when f is a rational expression over $\mathcal{L}$, for example as a solution of the equation $y''(z) + y'(z)/y(z) + y(\sin z) = \Gamma(z+1)$.

3   Differentially transcendental functions

Suppose $\mathbf{L} \subseteq \mathbf{K}$. Let $\mathbf{R}(x)$ be the differential field of real rational functions. The following Hölder's famous theorem asserts the differential transcendentality of Gamma function, see [9].

Theorem 3.1. a. *$\Gamma(x)$ is not differentially algebraic over $\mathbf{R}(x)$.* **b.** *$\Gamma(z)$ is not differentially algebraic over $\mathbf{C}(z)$.* ▽

Now we shall use the differential transcendence of $\Gamma(z)$ and properties of differential fields developed in the previous section to prove differential transcendentality over $\mathcal{C}$ of some analytic functions. $\Gamma(z)$ is meromorphic and by Hölder's theorem and Proposition 2.4, $\Gamma(z) \notin \mathcal{L}$.

Example 3.2. This example is an archetype of proofs, based on properties of differential fields exhibited in the previous section, of differential transcendentality of some well known complex functions. The Riemann zeta function is differentially transcendental over $\mathcal{C}$ (Hilbert). First we observe that $\zeta(s)$ is meromorphic and that $\zeta(s)$ satisfies the well-known functional equation:

$$\zeta(s) = \chi(s)\zeta(1 - s), \quad \text{where} \quad \chi(s) = \frac{(2\pi)^s}{2\Gamma(s)\cos(\frac{\pi s}{2})}.$$

Now, suppose that $\zeta(s)$ is differentially algebraic over $\mathcal{C}$, i.e. that $\zeta(s) \in \mathcal{L}$. Then $\zeta(1-s)$ and $\zeta(s)/\zeta(1-s)$ belong to $\mathcal{L}$, too, so $\chi(s)$ belongs to $\mathcal{L}$. The elementary functions $(2\pi)^s$, and $\cos(\frac{\pi s}{2})$ obviously are differentially algebraic over $\mathcal{C}$ i.e. they belong to $\mathcal{L}$. As $\mathcal{L}$ is a field, it follows that $\Gamma(z)$ is differentially algebraic over $\mathcal{C}$. Hence, $\Gamma(z) \in \mathcal{L}$ but this yields a contradiction. Therefore $\zeta(s)$ is differentially transcendental function over $\mathcal{C}$. Generally, Dirichlet L-series

$$L_k(s) = \sum_{n=1}^{\infty} \chi_k(n)\frac{1}{n^s} \quad (k \in \mathbf{Z}),$$

where $\chi_k(n)$ is Dirichlet character see [12], is differentially transcendental function over $\mathcal{C}$. This follows from a well-known functional equations

$$L_{-k}(s) = 2^s \pi^{s-1} k^{-s+\frac{1}{2}} \Gamma(1-s) \cos(\frac{\pi s}{2}) L_{-k}(1-s)$$

and

$$L_{+k}(s) = 2^s \pi^{s-1} k^{-s+\frac{1}{2}} \Gamma(1-s) \sin(\frac{\pi s}{2}) L_{+k}(1-s),$$

where $k \in \mathbf{N}$. D. Gokhman also proved in [5], but in entirely different way, that Dirichlet series are differentially transcendental. Besides Riemann zeta function $\zeta(s) = L_{+1}(s)$, Dirichlet eta function

$$\eta(s) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n^s} = (1 - 2^{1-s})L_{+1}(s)$$

and Dirichlet beta function

$$\beta(s) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)^s} = L_{-4}(s)$$

are transcendental differential functions as examples of Dirichlet series see [2, page 289]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\triangledown$

Example 3.3. Kurepa's Function related to the Kurepa Left Factorial Hypothesis, see [7, problem B44], is defined by [14]:

$$(1) \qquad\qquad K(z) = \int_0^{\infty} e^{-t}\frac{t^z - 1}{t - 1} \, dt.$$

Kurepa established in [14] that $K(z)$ can be continued meromorphically to the whole complex plane. Also, it satisfies the recurrence relation

$$(2) \qquad\qquad K(z) - K(z-1) = \Gamma(z),$$

hence, by Theorem 2.8, $K(z)$ is differential transcendental over $\mathcal{C}$. The functional equation (2), besides Kurepa's function $K(z)$, has another solution which is given by the series

$$(3) \qquad\qquad K_1(z) = \sum_{n=0}^{\infty} \Gamma(z - n).$$

This function has simple poles at integer points. We can conclude similarly that $K_1(z)$ is transcendental differential function. Between functions $K(z)$ and $K_1(z)$ the following relation is true see [23] and [16]:

$$K(z) = \frac{\text{Ei}(1)}{e} - \frac{\pi}{e} \operatorname{ctg} \pi z + K_1(z),$$

where $\text{Ei}(x)$ is the exponential integral see [6]. Each meromorphic solution of the functional equation (2) is differential transcendental over $\mathcal{C}$.

Alternating Kurepa's function related to the alternating sums of factorials see [7, problem B43], is defined by [20]:

$$(4) \qquad\qquad A(z) = \int\limits_0^\infty e^{-t} \frac{t^{z+1} - (-1)^z t}{t + 1}\, dt.$$

$A(z)$ is meromorphic and it satisfies the functional equation

$$(5) \qquad\qquad A(z) + A(z - 1) = \Gamma(z + 1).$$

The functional equation (5), besides alternating Kurepa's function $A(z)$, has another solution which is given by the series

$$(6) \qquad\qquad A_1(z) = \sum_{n=0}^\infty (-1)^n \Gamma(z + 1 - n).$$

As in the case of $K(z)$, we see that $A(z)$ and $A_1(z)$ are differential transcendental over $\mathcal{C}$. The following identity holds, see [17]:

$$A(z) = -\Big(1 + e\,\text{Ei}(-1)\Big)(-1)^z + \frac{\pi e}{\sin \pi z} + A_1(z).$$

Generally, each meromorphic solution of a functional equation (5) is transcendental differential function over the field $\mathcal{C}$.

G. Milovanović introduced in [19] a sequence of meromorphic functions

$$K_m(z) - K_m(z - 1) = K_{m-1}(z), \quad K_{-1}(z) = \Gamma(z), \quad K_0(z) = K(z).$$

By use of induction and Theorem 2.8 we can conclude that $K_m(z)$ are differential transcendental over $\mathcal{C}$. Analogously, a sequence of meromorphic functions defined by

$$A_m(z) + A_m(z - 1) = A_{m-1}(z), \quad A_{-1}(z) = \Gamma(z + 1), \quad A_0(z) = A(z)$$

is a sequences of transcendental differential functions over $\mathcal{C}$. ∇

Example 3.4. The meromorphic function

$$H_1(z) = \sum_{n=0}^\infty \frac{1}{(n + z)^2}$$

is differentially transcendental over $\mathcal{C}$. Really, $D^2 \ln(\Gamma(z)) = H_1(z)$, i.e. $\Gamma(z)$ satisfies the algebraic differential equation $(D^2\Gamma)\Gamma - (D\Gamma)^2 - H_1\Gamma^2 = 0$ over $\mathcal{C}(H_1)$. Thus, if H_1 would be differentially algebraic over $\mathcal{C}$, then by Proposition 2.3, Γ would be too, what yields a contradiction. Hence, $H_1(z)$ is differentially transcendental function over $\mathcal{C}$. ∇

Remark 3.5. We see that in Example 3.4 functions $\Gamma(z)$ and $H_1(z)$ are differentially algebraically dependent, i.e. $g(\Gamma, H_1) = 0$, where $g(x, y) = x''x - (x')^2 - yx^2$. We do not know if similar dependencies exist for pairs (K, Γ) and (ζ, Γ). It is very likely that these pairs are in fact differentially transcendental.

Example 3.6. W. Gautschi and J. Waldvogel considered in [4] a family of functions which satisfies

$$I_\alpha(x) = xI_{\alpha-1}(x) + \Gamma(\alpha), \quad \alpha > -1, \ x > 0.$$

For each $x_0 > 0$ function $f(\alpha) = I_\alpha(x_0)$ can be continued meromorphically (with respect to the complex α, see [4]). By Theorem 2.8 the function $f(\alpha)$ is differential transcendental over $\mathcal{C}$. $\qquad\qquad \nabla$

Example 3.7. The Barnes G-function is defined by [1]:

$$G(z + 1) = (2\pi)^{z/2} e^{-(z(z+1)+\gamma z^2)/2} \prod_{n=1}^{\infty} \left(\left(1 + \frac{z}{n}\right)^n e^{-z+z^2/(2n)} \right),$$

where γ is the Euler constant. Barnes G-function is an entire function and it satisfies following functional equation

(7) $$G(z + 1) = \Gamma(z)G(z),$$

By Theorem 2.8 each meromorphic solution of the functional equation (7) is differential transcendental over $\mathcal{C}$. $\qquad\qquad \nabla$

Example 3.8. The Hadamard's factorial function is defined by:

$$H(z) = \frac{1}{\Gamma(1-z)} \frac{d}{dz} \ln \left(\Gamma\left(\frac{1-z}{2}\right) \Big/ \Gamma\left(1 - \frac{z}{2}\right) \right),$$

see [3]. Hadamard's function is an entire function and it satisfies following functional equation

(8) $$H(z + 1) = zH(z) + \frac{1}{\Gamma(1-z)}.$$

By Theorem 2.8 each meromorphic solution of the functional equation (8) is differential transcendental over $\mathcal{C}$. $\qquad\qquad \nabla$

Example 3.9. Let $\tau(n)$ be defined by $\sum_{n=1}^{\infty} \tau(n)q^n = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$, $q = e^{2\pi iz}$. This function was first studied by [21]. Ramanujan's Dirichlet L-series defined by

$$f(s) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s},$$

is differentially transcendental function over $\mathcal{C}$. This follows from functional equation see [8, page 173]:

$$f(s) = \frac{(2\pi)^s}{(2\pi)^{12-s}} \frac{\Gamma(12-s)}{\Gamma(s)} f(12 - s)$$

i.e.
$$f(s) = -\frac{(2\pi)^s}{(2\pi)^{12-s}} \frac{\pi}{\sin(\pi s)} \left(\prod_{i=1}^{11} (s - i) \right) \frac{1}{\Gamma(s)^2} f(12 - s). \qquad \nabla$$

Example 3.10. Let us note that function [6] (integral 3.411/1):

$$\int\limits_0^\infty \frac{x^{p-1}}{e^x-1}\,dx = \Gamma(p)\zeta(p)$$

is differentially transcendental over $\mathcal{C}$. This follows from the functional equation $\zeta(1-s) = 2(2\pi)^{-s}\cos(\frac{s\pi}{2})\big(\Gamma(s)\zeta(s)\big)$. Analogously function $\Gamma(p)\beta(p)$ is differentially transcendental over $\mathcal{C}$. On the basis of differentially transcendental functions $\Gamma(p), \Gamma(p)\zeta(p), \Gamma(p)\beta(p)$ it is possible to find large number of examples differentially transcendental functions from table of integrals [6] (e.g. integrals: 3.411/3, 3.523/1, 3.523/3, 3.551/2, 3.769/1, 3.769/2, 3.944/5, 3.944/6). ∇

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