Fredholm Theory in Hilbert Space — A Concise Introductory Exposition

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Abstract

This is a brief introduction to Fredholm theory for Hilbert space operators organized into ten sections. The classical partition of the spectrum into point, residual, and continuous spectra is reviewed in Section 1. Fredholm operators are introduced in Section 2, and Fredholm index in Section 3. The essential spectrum is considered in Section 4, the spectral picture is presented in Section 5, and Riesz points are discussed in Section 6. Weyl spectrum is the subject of Section 7 and, after bringing some basic results on ascent and descent in Section 8, Browder spectrum is investigated in Section 9. Finally, Weyl and Browder theorems close this expository paper in Section 10.

1 The Spectrum

Throughout this paper $\mathcal{H}$ stands for a nonzero complex Hilbert space. A subspace of $\mathcal{H}$ is a *closed* linear manifold of $\mathcal{H}$. The closure of a linear manifold $\mathcal{M}$ is denoted by $\mathcal{M}^-$ and the orthogonal complement of $\mathcal{M}$ is denoted by $\mathcal{M}^\perp$, both are subspaces of $\mathcal{H}$. By an operator on $\mathcal{H}$ we mean a bounded linear (equivalently, a continuous linear) transformation of $\mathcal{H}$ into itself. Let $\mathcal{B}[\mathcal{H}]$ be the unital Banach algebra of all operators on $\mathcal{H}$, and let $T^* \in \mathcal{B}[\mathcal{H}]$ stand for the adjoint of $T \in \mathcal{B}[\mathcal{H}]$. The kernel or null space (which is a subspace of $\mathcal{H}$) and the range (which is a linear manifold of $\mathcal{H}$) of $T \in \mathcal{B}[\mathcal{H}]$ are denoted by $\mathcal{N}(T)$ and $\mathcal{R}(T)$, respectively:

$$\mathcal{N}(T) = T^{-1}\{0\} = \Big\{x \in \mathcal{H} \colon\; Tx = 0\Big\},$$

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$$\mathcal{R}(T) = T(\mathcal{H}) = \Big\{y \in \mathcal{H}\colon\; y = Tx \text{ for some } x \in \mathcal{H}\Big\}.$$

To begin with we consider a classical partition of the spectrum of a Hilbert space operator. Take an arbitrary operator T in the unital Banach algebra $\mathcal{B}[\mathcal{H}]$. Let

$$\rho(T) = \Big\{\lambda \in \mathbb{C}\colon\; \mathcal{N}(\lambda I - T) = \{0\} \;\text{ and }\; \mathcal{R}(\lambda I - T) = \mathcal{H}\Big\}$$

be the *resolvent set* of T, which according to the Open Mapping Theorem is precisely the set of all $\lambda \in \mathbb{C}$ such that $\lambda I - T$ is invertible (i.e., such that $\lambda I - T$ has an inverse in the algebra $\mathcal{B}[\mathcal{H}]$). Its complement,

$$\sigma(T) = \mathbb{C}\backslash\rho(T) = \Big\{\lambda \in \mathbb{C}\colon\; \lambda I - T \text{ is not invertible}\Big\},$$

is the *spectrum* of T. Recall that the spectrum of $T \in \mathcal{B}[\mathcal{H}]$ is a nonempty and compact subset of the complex plane $\mathbb{C}$. Let $\big\{\sigma_P(T), \sigma_R(T), \sigma_C(T)\big\}$ be the classical partition of $\sigma(T)$ consisting of the point spectrum, residual spectrum and continuous spectrum, which are defined as follows.

$$\sigma_P(T) = \Big\{\lambda \in \mathbb{C}\colon\; \mathcal{N}(\lambda I - T) \neq \{0\}\Big\}$$

is the *point spectrum* of T (i.e., the set of all *eigenvalues* of T). If λ is an eigenvalue of T (i.e., if $\lambda \in \sigma_P(T)$), then $\mathcal{N}(\lambda I - T)$ is an *eigenspace* of T. The set

$$\sigma_R(T) = \Big\{\lambda \in \mathbb{C}\colon\; \mathcal{N}(\lambda I - T) = \{0\} \;\text{ and }\; \mathcal{R}(\lambda I - T)^- \neq \mathcal{H}\Big\}$$

is the *residual spectrum* of T, and the *continuous spectrum* of T is given by

$$\sigma_C(T) = \Big\{\lambda \in \mathbb{C}\colon\; \mathcal{N}(\lambda I - T) = \{0\},\; \mathcal{R}(\lambda I - T)^- = \mathcal{H} \;\text{ and }\; \mathcal{R}(\lambda I - T) \neq \mathcal{H}\Big\}.$$

The diagram below summarizes such a partition of the spectrum [17, p.5]. Here the residual spectrum is split into two disjoint parts, $\sigma_R(T) = \sigma_{R_1}(T) \cup \sigma_{R_2}(T)$, the point spectrum is split into four disjoint parts, $\sigma_P(T) = \bigcup_{i=1}^4 \sigma_{P_i}(T)$, and we adopt the following abbreviated notation: $T_\lambda = (\lambda I - T)$, $\mathcal{N}_\lambda = \mathcal{N}(T_\lambda)$, and $\mathcal{R}_\lambda = \mathcal{R}(T_\lambda)$. Also $\mathcal{B}[\mathcal{R}_\lambda, \mathcal{H}]$ denotes the Banach space of all continuous (i.e., bounded) linear transformation of $\mathcal{R}_\lambda$ into $\mathcal{H}$ — so that if T_λ is injective (i.e., if $\mathcal{N}(T_\lambda) = \{0\}$), then its linear inverse T_λ^{-1} on $\mathcal{R}_\lambda$ is continuous if and only if $\mathcal{R}_\lambda$ is closed (by the Banach Continuous Inverse Theorem, see e.g., [18, p.228]).

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<tr><td rowspan="2">$\mathcal{N}_\lambda = \{0\}$</td><td>$T_\lambda^{-1} \in \mathcal{B}[\mathcal{R}_\lambda, \mathcal{H}]$</td><td>$\rho(T)$</td><td>$\varnothing$</td><td>$\varnothing$</td><td>$\sigma_{R_1}(T)$</td><td></td></tr>
<tr><td>$T_\lambda^{-1} \notin \mathcal{B}[\mathcal{R}_\lambda, \mathcal{H}]$</td><td>$\varnothing$</td><td>$\sigma_C(T)$</td><td>$\sigma_{R_2}(T)$</td><td>$\varnothing$</td><td rowspan="2">$\Big\}\; \sigma_{AP}(T)$</td></tr>
<tr><td colspan="2">$\mathcal{N}_\lambda \neq \{0\}$</td><td>$\sigma_{P_1}(T)$</td><td>$\sigma_{P_2}(T)$</td><td>$\sigma_{P_3}(T)$</td><td>$\sigma_{P_4}(T)$</td></tr>
<tr><td colspan="4"></td><td colspan="2">$\underbrace{\qquad\qquad}_{\sigma_{CP}(T)}$</td><td></td></tr>
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There are however overlapping parts of the spectrum that are commonly used too (see e.g., [14, Chapter 9]). For instance, the *compression spectrum* $\sigma_{CP}(T)$ and the *approximate point spectrum* (or *approximation spectrum*) $\sigma_{AP}(T)$, defined by

$$\begin{aligned}\sigma_{CP}(T) &= \Big\{\lambda \in \mathbb{C}\colon\; \mathcal{R}(\lambda I - T) \text{ is not dense in } \mathcal{H}\Big\}\\ &= \sigma_{P_3}(T) \cup \sigma_{P_4}(T) \cup \sigma_R(T),\end{aligned}$$

$$\sigma_{AP}(T) = \Big\{\lambda \in \mathbb{C} \colon\; (\lambda I - T) \text{ is not bounded below}\Big\}$$
$$= \sigma_P(T) \cup \sigma_C(T) \cup \sigma_{R_2}(T) = \sigma(T)\backslash\sigma_{R_1}(T).$$

Recall that the approximate point spectrum is nonempty, closed in $\mathbb{C}$, and includes the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$. Moreover, $\sigma_{R_1}(T)$ is open in $\mathbb{C}$, and so is $\sigma_{P_1}(T)$. Furthermore, the residual spectrum is given by the formula

$$\sigma_R(T) = \sigma_P(T^*)^*\backslash\sigma_P(T),$$

and so

$$\sigma_C(T) = \sigma(T)\Big\backslash\Big(\sigma_P(T) \cup \sigma_P(T^*)^*\Big),$$

where we are using the standard notation Λ^* for the set of all complex conjugates of a given subset Λ of $\mathbb{C}$ (i.e., $\Lambda^* = \{\lambda \in \mathbb{C} \colon\; \overline{\lambda} \in \Lambda\}$) — see e.g., [18, Propositions 6.16 and 6.17]. Indeed,

$$\rho(T) = \rho(T^*)^*, \qquad \sigma(T) = \sigma(T^*)^*, \qquad \sigma_C(T) = \sigma_C(T^*)^*,$$

the subparts of the point and residual spectrum are related by the expressions

$$\sigma_{P_1}(T) = \sigma_{R_1}(T^*)^*, \qquad \sigma_{P_2}(T) = \sigma_{R_2}(T^*)^*,$$

$$\sigma_{P_3}(T) = \sigma_{P_3}(T^*)^*, \qquad \sigma_{P_4}(T) = \sigma_{P_4}(T^*)^*,$$

and the compression and approximate point spectrum are such that

$$\sigma_{CP}(T) = \sigma_P(T^*)^*,$$

$$\partial\sigma(T) \subseteq \sigma_{AP}(T) \cap \sigma_{AP}(T^*)^* = \sigma(T)\Big\backslash\Big(\sigma_{P_1}(T) \cup \sigma_{R_1}(T)\Big).$$

Remark 1.1. Take an arbitrary $T \in \mathcal{B}[\mathcal{H}]$. Since in a finite-dimensional space every linear manifold is closed and every injective operator is invertible, it follows by the preceding diagram that

$$\dim \mathcal{H} < \infty \quad \Longrightarrow \quad \sigma(T) = \sigma_P(T) = \sigma_{P_4}(T),$$

which is a finite set — this extends to finite-rank operators but not to compact operators on a infinite-dimensional space. However,

$$T \text{ is compact} \quad \Longrightarrow \quad \sigma(T)\backslash\{0\} = \sigma_P(T)\backslash\{0\} \subseteq \sigma_{P_4}(T),$$

and, in this case, $\sigma(T)$ is a countable set for which 0 is the only possible accumulation point (see e.g., [7, Section VII.7] or [18, Section 6.6] — this in fact is a consequence of the Fredholm Alternative as stated in the forthcoming Remark 3.2).

2 Fredholm Operators

Let $\mathcal{B}_\infty[\mathcal{H}]$ be the (two-sided) ideal of all compact operators from $\mathcal{B}[\mathcal{H}]$. An operator T in $\mathcal{B}[\mathcal{H}]$ is *left semi-Fredholm* if there exists $S \in \mathcal{B}[\mathcal{H}]$ and $K \in \mathcal{B}_\infty[\mathcal{H}]$ such that $ST = I + K$, and *right semi-Fredholm* if there exists $S \in \mathcal{B}[\mathcal{H}]$ and $K \in \mathcal{B}_\infty[\mathcal{H}]$ such that $TS = I + K$. We say that T in $\mathcal{B}[\mathcal{H}]$ is *semi-Fredholm* if it is either left or right semi-Fredholm, and *Fredholm* if it is both left and right semi-Fredholm. Let $\mathcal{F}_\ell$ be the class of all left semi-Fredholm operators and let $\mathcal{F}_r$ be the class of all right semi-Fredholm operators:

$$\mathcal{F}_\ell = \Big\{T \in \mathcal{B}[\mathcal{H}]\colon\; ST = I + K \;\text{ for some }\; S \in \mathcal{B}[\mathcal{H}] \;\text{ and some }\; K \in \mathcal{B}_\infty[\mathcal{H}]\Big\},$$

$$\mathcal{F}_r = \Big\{T \in \mathcal{B}[\mathcal{H}]\colon\; TS = I + K \;\text{ for some }\; S \in \mathcal{B}[\mathcal{H}] \;\text{ and some }\; K \in \mathcal{B}_\infty[\mathcal{H}]\Big\}.$$

The classes of all semi-Fredholm and Fredholm operators from $\mathcal{B}[\mathcal{H}]$ will be denoted by $\mathcal{SF}$ and $\mathcal{F}$, respectively;

$$\mathcal{SF} = \mathcal{F}_\ell \cup \mathcal{F}_r \quad \text{and} \quad \mathcal{F} = \mathcal{F}_\ell \cap \mathcal{F}_r.$$

It is clear that

$$T \in \mathcal{F}_\ell \quad \text{if and only if} \quad T^* \in \mathcal{F}_r.$$

Thus $T \in \mathcal{SF}$ if and only if $T^* \in \mathcal{SF}$, and $T \in \mathcal{F}$ if and only if $T^* \in \mathcal{F}$. Recall the following elementary properties involving kernel and range of a Hilbert space operator and its adjoint (see e.g., [18, pp.393,394]):

$$\mathcal{N}(T^*) = \mathcal{H} \ominus \mathcal{R}(T)^- = \mathcal{R}(T)^\perp,$$

$$\mathcal{R}(T^*) \;\text{ is closed if and only if }\; \mathcal{R}(T) \;\text{ is closed.}$$

Proposition 2.1. [7, p.351]. *An operator $T \in \mathcal{B}[\mathcal{H}]$ is left semi-Fredholm if and only if $\mathcal{R}(T)$ is closed and $\mathcal{N}(T)$ is finite-dimensional.*

Therefore, since that $T \in \mathcal{F}_\ell$ if and only if $T^* \in \mathcal{F}_r$,

$$\mathcal{F}_\ell = \Big\{T \in \mathcal{B}[\mathcal{H}]\colon\; \mathcal{R}(T) \;\text{ is closed and }\; \dim \mathcal{N}(T) < \infty\Big\},$$

$$\mathcal{F}_r = \Big\{T \in \mathcal{B}[\mathcal{H}]\colon\; \mathcal{R}(T) \;\text{ is closed and }\; \dim \mathcal{N}(T^*) < \infty\Big\}.$$

Corollary 2.2. *Take any operator T in $\mathcal{B}[\mathcal{H}]$. It is semi-Fredholm if and only if $\mathcal{R}(T)$ is closed and $\mathcal{N}(T)$ or $\mathcal{N}(T^*)$ is finite-dimensional. It is Fredholm if and only if $\mathcal{R}(T)$ is closed and both $\mathcal{N}(T)$ and $\mathcal{N}(T^*)$ are finite-dimensional.*

Thus, according to Corollary 2.2, the class of all Fredholm operators is given by

$$\mathcal{F} = \Big\{T \in \mathcal{B}[\mathcal{H}]\colon\; \mathcal{R}(T) \;\text{ is closed, }\; \dim \mathcal{N}(T) < \infty \;\text{ and }\; \dim \mathcal{N}(T^*) < \infty\Big\}$$

and its complement,

$$\mathcal{B}[\mathcal{H}]\backslash\mathcal{F} = \mathcal{B}[\mathcal{H}]\backslash(\mathcal{F}_\ell \cap \mathcal{F}_r) = (\mathcal{B}[\mathcal{H}]\backslash\mathcal{F}_\ell) \cup (\mathcal{B}[\mathcal{H}]\backslash\mathcal{F}_r),$$

is the union of

$$\mathcal{B}[\mathcal{H}]\backslash\mathcal{F}_\ell = \Big\{T \in \mathcal{B}[\mathcal{H}]\colon\; \mathcal{R}(T) \;\text{ is not closed or }\; \dim \mathcal{N}(T) = \infty\Big\},$$

$$\mathcal{B}[\mathcal{H}]\backslash\mathcal{F}_r = \Big\{T \in \mathcal{B}[\mathcal{H}]\colon\; \mathcal{R}(T) \;\text{ is not closed or }\; \dim \mathcal{N}(T^*) = \infty\Big\}.$$

3 Fredholm Index

Let $\mathbb{Z}$ be the set of all integers and put $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty\} \cup \{+\infty\}$, the set of all extended integers. Take any operator T in $\mathcal{SF}$ so that $\mathcal{N}(T)$ or $\mathcal{N}(T^*)$ is finite-dimensional. The *Fredholm index* $\operatorname{ind}(T)$ of any T in $\mathcal{SF}$ is defined in $\overline{\mathbb{Z}}$ by

$$\operatorname{ind}(T) = \dim \mathcal{N}(T) - \dim \mathcal{N}(T^*).$$

It is usual to write

$$\alpha(T) = \dim \mathcal{N}(T) \quad \text{and} \quad \beta(T) = \dim \mathcal{N}(T^*)$$

so that

$$\operatorname{ind}(T) = \alpha(T) - \beta(T).$$

Since T^* and T lie in $\mathcal{SF}$ together, we get $\operatorname{ind}(T^*) = -\operatorname{ind}(T)$. Note that

$$\beta(T) = \alpha(T^*) = \dim \mathcal{R}(T)^\perp = \dim(\mathcal{H} \ominus \mathcal{R}(T)^-).$$

Remark 3.1. (Finite-dimensional). Take an arbitrary $T \in \mathcal{B}[\mathcal{H}]$ and recall that

$$\dim \mathcal{H} = \dim \mathcal{N}(T) + \dim \mathcal{R}(T) \quad \text{and} \quad \dim \mathcal{H} = \dim \mathcal{R}(T) + \dim \mathcal{R}(T)^\perp.$$

Hence, if $\mathcal{H}$ is finite-dimensional, then $\operatorname{ind}(T) = 0$. Indeed,

$$\dim \mathcal{N}(T) - \dim \mathcal{R}(T)^\perp = \dim \mathcal{N}(T) + \dim \mathcal{R}(T) - \dim \mathcal{H} = \dim \mathcal{H} - \dim \mathcal{H} = 0.$$

But linear manifolds of finite-dimensional spaces are closed, and therefore on a finite-dimensional space every operator is Fredholm with a null index:

$$\dim \mathcal{H} < \infty \quad \Longrightarrow \quad \Big\{T \in \mathcal{F} \colon\ \operatorname{ind}(T) = 0\Big\} = \mathcal{B}[\mathcal{H}].$$

Remark 3.2. (Fredholm Alternative). If $K \in \mathcal{B}_\infty[\mathcal{H}]$ and $\lambda \neq 0$, then

$$\mathcal{R}(\lambda I - K) \text{ is closed} \quad \text{and} \quad \dim \mathcal{N}(\lambda I - K) = \dim \mathcal{N}(\overline{\lambda} I - K^*) < \infty.$$

This is the so-called *Fredholm Alternative* for compact operators (see e.g., [1, p.87], [7, p.217] or [18, p.480]), which can be restated in terms of Fredholm indices:

If $K \in \mathcal{B}_\infty[\mathcal{H}]$ and $\lambda \neq 0$, then $\lambda I - K$ is Fredholm with $\operatorname{ind}(\lambda I - K) = 0$.

Remark 3.3. Observe that Corollary 2.2 and some of its straightforward consequences can also be naturally rephrased in terms of Fredholm indices.

(a) *An operator $T \in \mathcal{B}[\mathcal{H}]$ is semi-Fredholm if and only if $\mathcal{R}(T)$ is closed and $\alpha(T)$ or $\beta(T)$ is finite; an operator $T \in \mathcal{B}[\mathcal{H}]$ is Fredholm if and only if $\mathcal{R}(T)$ is closed and both $\alpha(T)$ and $\beta(T)$ are finite.*

Note that $\alpha(T)$ and $\beta(T)$ are both finite if and only if $\operatorname{ind}(T)$ is finite (reason: α and β were defined for semi-Fredholm operators so that if one is $\pm\infty$, then the other must be finite). Therefore,

(b) *T is Fredholm if and only if it is semi-Fredholm with finite Fredholm index.*

A nonzero scalar operator (i.e., a nonzero multiple of the identity) is Fredholm with a null Fredholm index (by Corollary 2.2). This is readily generalized as follows.

(c) *If* $T \in \mathcal{F}$, *then* $\gamma T \in \mathcal{F}$ *and* $\operatorname{ind}(\gamma T) = \operatorname{ind}(T)$ *for every* $\gamma \in \mathbb{C}\backslash\{0\}$.

Proposition 3.4. [7, p.354].

$$S, T \in \mathcal{F}_\ell \quad \Longrightarrow \quad ST \in \mathcal{F}_\ell \quad \text{and} \quad \operatorname{ind}(ST) = \operatorname{ind}(S) + \operatorname{ind}(T).$$

Corollary 3.5. *Take an arbitrary nonnegative integer* n.

$$T \in \mathcal{F} \quad \Longrightarrow \quad T^n \in \mathcal{F} \quad \text{and} \quad \operatorname{ind}(T^n) = n \operatorname{ind}(T).$$

Proof. The result holds trivially for $n = 0$ (cf. Corollary 2.2) and tautologically for $n = 1$. Thus suppose $n \geq 2$. Proposition 3.4 says that the claimed result holds for $n = 2$ if $\mathcal{F}$ is replaced with $\mathcal{F}_\ell$. Then a trivial induction ensures that it holds for each $n \geq 2$, if $\mathcal{F}$ is still replaced with $\mathcal{F}_\ell$. That is, for every integer $n \geq 2$,

$$T \in \mathcal{F}_\ell \quad \Longrightarrow \quad T^n \in \mathcal{F}_\ell \quad \text{and} \quad \operatorname{ind}(T^n) = n \operatorname{ind}(T).$$

Suppose $T \in \mathcal{F} = \mathcal{F}_\ell \cap \mathcal{F}_r$. The above implication holds and, since $T^* \in \mathcal{F}_\ell$, it also holds for T^* so that $T^{*n} \in \mathcal{F}_\ell$. Thus $T^n \in \mathcal{F}_r$, and hence $T^n \in \mathcal{F} = \mathcal{F}_\ell \cap \mathcal{F}_r$. ∎

The null operator on an infinite-dimensional space is not Fredholm, and so a compact operator may not be Fredholm. However, the sum of a Fredholm and a compact is a Fredholm operator with the same index.

Proposition 3.6. [1, p.98]

$$T \in \mathcal{F} \quad \text{and} \quad K \in \mathcal{B}_\infty[\mathcal{H}] \quad \Longrightarrow \quad T + K \in \mathcal{F} \quad \text{and} \quad \operatorname{ind}(T + K) = \operatorname{ind}(T).$$

A *Weyl operator* is a semi-Fredholm operator with null Fredholm index. Equivalently, a Weyl operator is a Fredholm operator with null Fredholm index (cf. Remark 3.3(b)). Let $\mathcal{W}$ denote the class of all Weyl operators from $\mathcal{B}[\mathcal{H}]$:

$$\mathcal{W} = \Big\{T \in \mathcal{F} \colon \operatorname{ind}(T) = 0\Big\}.$$

Since $T \in \mathcal{F}$ if and only if $T^* \in \mathcal{F}$ and $\operatorname{ind}(T^*) = -\operatorname{ind}(T)$, it follows that $T \in \mathcal{W}$ if and only if $T^* \in \mathcal{W}$. Observe from Remark 3.1 that every operator on a finite-dimensional space is a Weyl operator;

$$\dim \mathcal{H} < \infty \quad \Longrightarrow \quad \mathcal{W} = \mathcal{B}[\mathcal{H}].$$

Moreover, the Fredholm Alternative in Remark 3.2 can be rephrased as follows. If K is a compact operator, then $\lambda I - K$ is a Weyl operator for every nonzero λ:

$$K \in \mathcal{B}_\infty[\mathcal{H}] \quad \text{and} \quad \lambda \neq 0 \quad \Longrightarrow \quad (\lambda I - K) \in \mathcal{W}.$$

Also, by Remark 3.3(c), every nonzero multiple of a Weyl operator is again a Weyl operator; in particular, every nonzero scalar operator is a Weyl operator. In fact, the product of a couple of Weyl operators is again a Weyl operator (by Proposition 3.4). Note that every compact Fredholm operator is a Weyl operator by Proposition 4.4 and 3.6. Also note that every self-adjoint operator with a closed range and a finite-dimensional kernel is a Weyl operator. Actually, every normal operator with a closed range and a finite-dimensional kernel is a Weyl operator (reason: since $\mathcal{N}(T) = \mathcal{N}(T^*T)$, so that $\mathcal{N}(T^*) = \mathcal{N}(TT^*)$, for every $T \in \mathcal{B}[\mathcal{H}]$ — see e.g., [18, p.393] — it follows that $\mathcal{N}(T^*) = \mathcal{N}(T)$ whenever T is normal.)

4 Essential Spectrum

The *left spectrum* $\sigma_\ell(T)$ and *right spectrum* $\sigma_r(T)$ of $T \in \mathcal{B}[\mathcal{H}]$ are defined by

$$\sigma_\ell(T) = \Big\{\lambda \in \mathbb{C}\colon\; \lambda I - T \text{ is not left invertible}\Big\} = \Big\{\lambda \in \mathbb{C}\colon\; \mathcal{N}(\lambda I - T) \neq \{0\}\Big\},$$

$$\sigma_r(T) = \Big\{\lambda \in \mathbb{C}\colon\; \lambda I - T \text{ is not right invertible}\Big\} = \Big\{\lambda \in \mathbb{C}\colon\; \mathcal{R}(\lambda I - T) \neq \mathcal{H}\Big\},$$

so that

$$\sigma(T) = \Big\{\lambda \in \mathbb{C}\colon\; \lambda I - T \text{ is not invertible}\Big\} = \sigma_\ell(T) \cup \sigma_r(T).$$

Note that the left spectrum coincides with the point spectrum $\sigma_P(T)$ (i.e., with the set of all eigenvalues of T). In fact,

$$\sigma_\ell(T) = \sigma_P(T) \quad \text{ and } \quad \sigma_r(T) = \sigma(T)\backslash\sigma_{P_1}(T),$$

where $\sigma_{P_1}(T) = \Big\{\lambda \in \sigma_P(T)\colon\; \mathcal{R}(\lambda I - T)^- = \mathcal{R}(\lambda I - T) = \mathcal{H}\Big\}$ or, equivalently, $\sigma_{P_1}(T) = \Big\{\lambda \in \sigma_P(T)\colon\; \mathcal{R}(\lambda I - T) = \mathcal{H}\Big\}$ — see diagram of Section 1 — and hence

$$\sigma_\ell(T) \cap \sigma_r(T) = \sigma_P(T)\backslash\sigma_{P_1}(T).$$

Consider the Calkin algebra $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$ (i.e., the quotient algebra of $\mathcal{B}[\mathcal{H}]$ modulo the ideal $\mathcal{B}_\infty[\mathcal{H}]$ of all compact operators), which is a unital Banach algebra whenever $\mathcal{H}$ is infinite-dimensional. In this case, let $\pi\colon \mathcal{B}[\mathcal{H}] \to \mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$ be the natural map from $\mathcal{B}[\mathcal{H}]$ to $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$ (i.e., $\pi(T) = [T] = T + \mathcal{B}_\infty[\mathcal{H}]$ for every T in $\mathcal{B}[\mathcal{H}]$). The *essential spectrum* (or the *Calkin spectrum*) $\sigma_e(T)$ of $T \in \mathcal{B}[\mathcal{H}]$ is the spectrum of $\pi(T)$ in the Calkin algebra $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$; that is,

$$\sigma_e(T) = \sigma(\pi(T)).$$

Similarly, the *left essential spectrum* $\sigma_{\ell e}(T)$ and *right essential spectrum* $\sigma_{re}(T)$ of $T \in \mathcal{B}[\mathcal{H}]$ are defined as the left and right spectrum of $\pi(T)$ in the Calkin algebra $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$, respectively; that is,

$$\sigma_{\ell e}(T) = \sigma_\ell(\pi(T)) \quad \text{ and } \quad \sigma_{re}(T) = \sigma_r(\pi(T)),$$

and so

$$\sigma_e(T) = \sigma_{\ell e}(T) \cup \sigma_{re}(T).$$

Proposition 4.1. [7, p.359]. *If* $T \in \mathcal{B}[\mathcal{H}]$*, then*

$$\sigma_{\ell e}(T) = \Big\{\lambda \in \mathbb{C}\colon\; \mathcal{R}(\lambda I - T) \text{ is not closed } \text{ or } \dim \mathcal{N}(\lambda I - T) = \infty\Big\},$$

$$\sigma_{re}(T) = \Big\{\lambda \in \mathbb{C}\colon\; \mathcal{R}(\lambda I - T) \text{ is not closed } \text{ or } \dim \mathcal{N}(\overline{\lambda} I - T^*) = \infty\Big\}.$$

Let the *essential point spectrum* $\sigma_{Pe}(T)$ of $T \in \mathcal{B}[\mathcal{H}]$ be the point spectrum of $\pi(T)$ in the Calkin algebra $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$,

$$\sigma_{Pe}(T) = \sigma_P(\pi(T)),$$

and put $\sigma_{P_1 e}(T) = \sigma_{P_1}(\pi(T))$ so that, by Proposition 4.1,

$$\sigma_C(T) \subseteq \sigma_{\ell e}(T) = \sigma_{Pe}(T) \subseteq \sigma(T)\backslash\sigma_{R_1}(T) = \sigma_{AP}(T),$$

$$\sigma_C(T) \subseteq \sigma_{re}(T) = \sigma_e(T)\backslash\sigma_{P_1 e}(T) \subseteq \sigma(T)\backslash\sigma_{P_1}(T) = \sigma_{AP}(T^*)^*,$$

with $\sigma_{R_1}(T) = \big\{\lambda \in \sigma(T)\colon\ \mathcal{R}(\lambda I - T)^- = \mathcal{R}(\lambda I - T) \neq \mathcal{H}$ and $\mathcal{N}(\lambda I - T) = \{0\}\big\}$ and $\sigma_{P_1}(T) = \big\{\lambda \in \sigma(T)\colon\ \mathcal{R}(\lambda I - T) = \mathcal{H}$ and $\mathcal{N}(\lambda I - T) \neq \{0\}\big\}$ — see diagram of Section 1 — so that

$$\sigma_C(T) \subseteq \sigma_{\ell e}(T) \cap \sigma_{re}(T) = \sigma_{Pe}(T)\backslash\sigma_{P_1 e}(T) \subseteq \sigma(T)\Big\backslash\big(\sigma_{P_1}(T) \cup \sigma_{R_1}(T)\big).$$

Corollary 4.2. (Atkinson Theorem).

$$\sigma_e(T) = \big\{\lambda \in \mathbb{C}\colon\ (\lambda I - T) \notin \mathcal{F}\big\}.$$

Proof. According to Proposition 4.1,

$$\sigma_{\ell e}(T) = \big\{\lambda \in \mathbb{C}\colon\ (\lambda I - T) \in \mathcal{B}[\mathcal{H}]\backslash\mathcal{F}_\ell\big\} = \big\{\lambda \in \mathbb{C}\colon\ (\lambda I - T) \notin \mathcal{F}_\ell\big\},$$

$$\sigma_{re}(T) = \big\{\lambda \in \mathbb{C}\colon\ (\lambda I - T) \in \mathcal{B}[\mathcal{H}]\backslash\mathcal{F}_r\big\} = \big\{\lambda \in \mathbb{C}\colon\ (\lambda I - T) \notin \mathcal{F}_r\big\}.$$

Since $\sigma_e(T) = \sigma_{\ell e}(T) \cup \sigma_{re}(T)$ and $\mathcal{F} = \mathcal{F}_\ell \cap \mathcal{F}_r$, we get the desired identity. ∎

Observe that what the Atkinson Theorem says is that *an operator T is Fredholm if and only if its image $\pi(T)$ in the Calkin algebra $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$ is invertible* (see e.g., [14, Problem 181]). This is usually referred to by saying that T is *essentially invertible.* Thus the essential spectrum is the set of all scalars λ for which $(\lambda I - T)$ is not a Fredholm operator (i.e., not essentially invertible), and so the essential spectrum is also called *Fredholm spectrum.* Since an operator lies in $\mathcal{F}$ together with its adjoint, it follows by Corollary 4.2 that $\lambda \in \sigma_e(T)$ if and only if $\overline{\lambda} \in \sigma_e(T^*)$:

$$\sigma_e(T) = \sigma_e(T^*)^*.$$

It is clear from Proposition 4.1 that

$$\sigma_e(T) \subseteq \sigma(T).$$

Remark 4.3. Take an arbitrary $T \in \mathcal{B}[\mathcal{H}]$. Since $\sigma_e(T) = \sigma(\pi(T))$ if $\mathcal{H}$ is infinite-dimensional, it follows that

$$\dim \mathcal{H} = \infty \quad \Longrightarrow \quad \sigma_e(T) \neq \varnothing,$$

and the converse holds by Remark 3.1 and Corollary 4.2,

$$\dim \mathcal{H} < \infty \quad \Longrightarrow \quad \sigma_e(T) = \varnothing.$$

In both cases $\sigma_e(T)$ is a compact set and, by Proposition 3.6 and Corollary 4.2, $\sigma_e(T+K) = \sigma_e(T)$ — indeed, $\pi(T+K) = \pi(T)$ — for all $K \in \mathcal{B}_\infty[\mathcal{H}]$.

5 Spectral Picture

Take any operator T in $\mathcal{B}[\mathcal{H}]$. For each $k \in \overline{\mathbb{Z}}\backslash\{0\}$ put

$$\sigma_k(T) = \Big\{\lambda \in \mathbb{C}\colon\; (\lambda I - T) \in \mathcal{SF} \;\text{ and }\; \operatorname{ind}(\lambda I - T) = k\Big\}.$$

Recall that $(\lambda I - T) \in \mathcal{SF}$ if and only if $(\overline{\lambda} I - T^*) \in \mathcal{SF}$ with $\operatorname{ind}(\lambda I - T) = -\operatorname{ind}(\overline{\lambda} I - T^*)$, and so $\lambda \in \sigma_k(T)$ if and only if $\overline{\lambda} \in \sigma_{-k}(T^*)$. Thus, for $k \in \overline{\mathbb{Z}}\backslash\{0\}$,

$$\sigma_k(T) = \sigma_{-k}(T^*)^*.$$

These $\sigma_k(T)$ are open subsets of $\mathbb{C}$ for each nonzero k in $\overline{\mathbb{Z}}$ [7, p.366]. If $\lambda \in \sigma_{+\infty}(T)$, then $0 \leq \dim \mathcal{N}(\overline{\lambda} I - T^*) < \dim \mathcal{N}(\lambda I - T) = \infty$, which implies that λ is an eigenvalue of T of infinite multiplicity. Dually, If $\lambda \in \sigma_{-\infty}(T) = \sigma_{+\infty}(T^*)^*$, then $\overline{\lambda}$ is an eigenvalue of T^* of infinite multiplicity. Hence,

$$\sigma_{+\infty}(T) \subseteq \sigma_P(T) \quad\text{ and }\quad \sigma_{-\infty}(T) \subseteq \sigma_P(T^*)^*.$$

Now take an arbitrary nonzero integer $k \in \mathbb{Z}\backslash\{0\}$. If $k > 0$ and $\lambda \in \sigma_k(T)$, then $0 < \operatorname{ind}(\lambda I - T) < \infty$, and so $0 \leq \dim \mathcal{N}(\overline{\lambda} I - T^*) < \dim \mathcal{N}(\lambda I - T) < \infty$, which implies $0 < \dim \mathcal{N}(\lambda I - T) < \infty$ so that λ is an eigenvalue of T of finite multiplicity. Dually, if $k < 0$ (i.e., $-k > 0$) and $\lambda \in \sigma_k(T) = \sigma_{-k}(T^*)^*$, then $\overline{\lambda}$ is an eigenvalue of T^* of finite multiplicity. Outcome: If $k \in \mathbb{Z}\backslash\{0\}$, then

$$\sigma_k(T) \;\subseteq \begin{cases} \sigma_{PF}(T), & k > 0, \\ \sigma_{PF}(T^*)^*, & k < 0, \end{cases}$$

where $\sigma_{PF}(T)$ denotes the set of all eigenvalues of finite multiplicity,

$$\sigma_{PF}(T) = \Big\{\lambda \in \sigma_P(T)\colon\; \dim \mathcal{N}(\lambda I - T) < \infty\Big\},$$

that is, $\sigma_{PF}(T) = \{\lambda \in \mathbb{C}\colon\; 0 < \dim \mathcal{N}(\lambda I - T) < \infty\}$, and so

$$\sigma_k(T) \subseteq \sigma_{PF}(T) \cup \sigma_{PF}(T^*)^*$$

for all $k \in \mathbb{Z}\backslash\{0\}$. Therefore, since $\sigma_R(T) = \sigma_P(T^*)^* \backslash \sigma_P(T)$, we get

$$\bigcup_{k \in \overline{\mathbb{Z}}\backslash\{0\}} \sigma_k(T) \subseteq\; \sigma_P(T) \cup \sigma_P(T^*)^* =\; \sigma_P(T) \cup \sigma_R(T) \;\subseteq\; \sigma(T).$$

Observe that, if $k \in \mathbb{Z}\backslash\{0\}$ (i.e., if k is a nonzero integer), then (cf. Remark 3.3(b))

$$\sigma_k(T) = \Big\{\lambda \in \mathbb{C}\colon\; (\lambda I - T) \in \mathcal{F} \;\text{ and }\; \operatorname{ind}(\lambda I - T) = k\Big\}.$$

Now, for $k = 0$, we define $\sigma_0(T)$ as the following subset of $\sigma(T)$:

$$\begin{aligned} \sigma_0(T) &= \Big\{\lambda \in \sigma(T)\colon\; (\lambda I - T) \in \mathcal{SF} \;\text{ and }\; \operatorname{ind}(\lambda I - T) = 0\Big\} \\ &= \Big\{\lambda \in \sigma(T)\colon\; (\lambda I - T) \in \mathcal{F} \;\text{ and }\; \operatorname{ind}(\lambda I - T) = 0\Big\} \\ &= \Big\{\lambda \in \sigma(T)\colon\; (\lambda I - T) \in \mathcal{W}\Big\}, \end{aligned}$$

which can be rewritten as

$$\begin{aligned}\sigma_0(T) \;&=\; \Big\{\lambda\in\sigma(T)\colon\; \mathcal{R}(\lambda I-T) \text{ is closed } \text{ and }\\ &\qquad\qquad\qquad \dim\mathcal{N}(\lambda I-T)=\dim\mathcal{N}(\overline{\lambda}I-T^*)<\infty\Big\}\\ &=\Big\{\lambda\in\sigma_P(T)\colon\; \mathcal{R}(\lambda I-T)^-=\mathcal{R}(\lambda I-T)\neq\mathcal{H} \text{ and }\\ &\qquad\qquad\qquad \dim\mathcal{N}(\lambda I-T)=\dim\mathcal{N}(\overline{\lambda}I-T^*)<\infty\Big\}\\ &=\Big\{\lambda\in\sigma_{P_4}(T)\colon\; \dim\mathcal{N}(\lambda I-T)=\dim\mathcal{N}(\overline{\lambda}I-T^*)<\infty\Big\},\end{aligned}$$

with $\sigma_{P_4}(T) = \Big\{\lambda\in\sigma_P(T)\colon\; \mathcal{R}(\lambda I-T)^- = \mathcal{R}(\lambda I-T)\neq\mathcal{H}\Big\}$ — see diagram of Section 1. Therefore,

$$\sigma_0(T)\subseteq\sigma_{P_4}(T)\cap\sigma_{PF}(T)\cap\sigma_{PF}(T^*)^*,$$

and so

$$\sigma_0(T)=\sigma_0(T^*)^*.$$

The partition of the spectrum $\sigma(T)$ obtained in the next proposition is called the *spectral picture* of T [23].

Proposition 5.1. *If $T\in\mathcal{B}[\mathcal{H}]$, then*

$$\sigma(T)=\sigma_e(T)\cup\bigcup_{k\in\mathbb{Z}}\sigma_k(T)$$

and

$$\sigma_e(T)=\Big(\sigma_{\ell e}(T)\cap\sigma_{re}(T)\Big)\cup\sigma_{+\infty}(T)\cup\sigma_{-\infty}(T),$$

where

$$\sigma_e(T)\cap\bigcup_{k\in\mathbb{Z}}\sigma_k(T)=\varnothing,\quad \sigma_k(T)\cap\sigma_j(T)=\varnothing,\quad \big(\sigma_{\ell e}(T)\cap\sigma_{re}(T)\Big)\cap\sigma_{\pm\infty}(T)=\varnothing.$$

Proof. The collection $\{\sigma_k(T)\}_{k\in\mathbb{Z}}$ of subsets of $\sigma(T)$ is pairwise disjoint and

$$\bigcup_{k\in\mathbb{Z}}\sigma_k(T)=\Big\{\lambda\in\sigma(T)\colon\; (\lambda I-T)\in\mathcal{F}\Big\}.$$

Since

$$\sigma_e(T)=\Big\{\lambda\in\sigma(T)\colon\; (\lambda I-T)\notin\mathcal{F}\Big\},$$

we get the claimed partition of the spectrum $\sigma(T)$. The partition of the essential spectrum $\sigma_e(T)$ into $\Big(\sigma_{\ell e}(T)\cap\sigma_{re}(T)\Big)\cup\sigma_{+\infty}(T)\cup\sigma_{-\infty}(T)$ is readily verified by the proof of Corollary 4.2. ∎

Recall that $\sigma_k(T)$ is an open subset of $\mathbb{C}$ for each $k\in\overline{\mathbb{Z}}\backslash\{0\}$. The pairwise disjoint sets $\{\sigma_k(T)\}_{k\in\mathbb{Z}\backslash\{0\}}$, which are subsets of $\sigma(T)\backslash\sigma_e(T)$, are called the *holes* in the essential spectrum $\sigma_e(T)$, while $\sigma_{\pm\infty}(T)$, which are subsets of $\sigma_e(T)$, are called the *pseudoholes* in $\sigma_e(T)$. Thus the spectral picture of T consists of the essential spectrum $\sigma_e(T)$, the holes $\sigma_k(T)$ and pseudoholes $\sigma_{\pm\infty}(T)$ (to each is associated a nonzero index k in $\overline{\mathbb{Z}}\backslash\{0\}$), and the set $\sigma_0(T)=\{\lambda\in\sigma(T)\colon\; (\lambda I-T)\in\mathcal{W}\}$. It is worth noticing that any spectral picture can be attained [6] by an operator in $\mathcal{B}[\mathcal{H}]$.

6 Riesz Points

The set $\sigma_0(T)$ will play a rather important role in the sequel. It consists of an open set $\tau_0(T)$ and the set $\pi_0(T)$ of isolated points of $\sigma(T)$ for which the Riesz idempotents have finite rank [7, p.366]. The next proposition says that $\pi_0(T)$ is precisely the set of all isolated points of $\sigma(T)$ that lie in $\sigma_0(T)$.

Proposition 6.1. [7, p.366]. *If λ is an isolated point of $\sigma(T)$, then the following assertions are pairwise equivalent.*

(a) $\lambda \in \sigma_0(T)$.

(b) $\lambda \notin \sigma_{\ell e}(T) \cap \sigma_{re}(T)$.

(c) *The Riesz idempotent E_λ has finite rank.*

Recall the definition of *Riesz idempotents* associated with isolated points of the spectrum: Let λ be an isolated point of $\sigma(T)$ and consider the spectral projection

$$E_\lambda = \tfrac{1}{2\pi i} \int_{\Gamma_\lambda} (\gamma I - T)^{-1} d\gamma,$$

where Γ_λ is a positively (i.e., counterclockwise) oriented circle enclosing λ but no other point of $\sigma(T)$. (This is a straightforward generalization of the Riemann integral of a scalar-valued function on Γ_λ, which extends naturally for any vector-valued function and, in particular, for any function from Γ_λ to the Banach space $\mathcal{B}[\mathcal{H}]$.) The operator $E_\lambda \colon \mathcal{H} \to \mathcal{H}$ is indeed a projection (i.e., a linear idempotent, bounded but not necessarily orthogonal) that commutes with every operator that commutes with T, so that the $\mathcal{R}(E_\lambda)$ is T-invariant. Moreover, $\sigma(T|_{\mathcal{R}(E_\lambda)}) = \{\lambda\}$. The spectral projection E_λ is the *Riesz idempotent* corresponding to λ [7, p.210].

Thus $\pi_0(T)$ is a subset of $\sigma_{P_4}(T)$ consisting of those isolated points λ of the spectrum for which $\mathcal{R}(E_\lambda)$ is finite-dimensional or, equivalently, consisting of those isolated points of the spectrum that lie in $\sigma_0(T)$. Summing up:

$$\sigma_0(T) = \tau_0(T) \cup \pi_0(T),$$

where $\tau_0(T)$ is an open subset of the complex plane included in $\sigma_{P_4}(T)$ and

$$\begin{aligned}
\pi_0(T) &= \Big\{\lambda \in \sigma_P(T)\colon\ \lambda \text{ is an isolated point of } \sigma(T) \text{ and } \dim \mathcal{R}(E_\lambda) < \infty\Big\} \\
&= \Big\{\lambda \in \sigma_P(T)\colon\ \lambda \text{ is an isolated point of } \sigma(T) \text{ and } \lambda \notin \Big(\sigma_{\ell e}(T) \cap \sigma_{re}(T)\Big)\Big\} \\
&= \Big\{\lambda \in \sigma_P(T)\colon\ \lambda \text{ is an isolated point of } \sigma(T) \text{ and } \lambda \in \sigma_0(T)\Big\} \\
&= \Big\{\lambda \in \sigma_{P_4}(T)\colon\ \lambda \text{ is an isolated point of } \sigma(T) \text{ and} \\
&\qquad\qquad \dim \mathcal{N}(\lambda I - T) = \dim \mathcal{N}(\overline{\lambda} I - T^*) < \infty\Big\}.
\end{aligned}$$

The set $\pi_0(T)$ is called the set of *Riesz points* of T, which is sometimes also referred to as the set of isolated eigenvalues of T of *finite algebraic multiplicity.*

For any operator T let $\sigma_{\rm iso}(T)$ denote the set of all isolated points of $\sigma(T)$, and so $\sigma_{\rm acc}(T) = \sigma(T)\backslash\sigma_{\rm iso}(T)$, its complement in $\sigma(T)$, is precisely the set of all accumulation points of $\sigma(T)$. Thus the Riesz points of T can be simply written as

$$\pi_0(T) = \sigma_{\rm iso}(T) \cap \sigma_0(T).$$

Now let $\pi_{00}(T)$ denote the set of all isolated eigenvalues of T of finite multiplicity,

$$\pi_{00}(T) = \sigma_{\text{iso}}(T) \cap \sigma_{PF}(T),$$

which is sometimes also referred to as the set of isolated eigenvalues of T of *finite geometric multiplicity.* Since $\sigma_0(T) \subseteq \sigma_{PF}(T)$, it is clear that

$$\pi_0(T) \subseteq \pi_{00}(T).$$

Proposition 6.2. $\quad \pi_0(T) = \Big\{\lambda \in \pi_{00}(T)\colon\; \mathcal{R}(\lambda I - T) \text{ is closed}\Big\}.$

Proof. If $\lambda \in \pi_0(T)$, then $\lambda \in \pi_{00}(T)$ and $\mathcal{R}(\lambda I - T)$ is closed (because $\lambda \in \sigma_0(T)$). Conversely, suppose $\mathcal{R}(\lambda I - T)$ is closed and $\lambda \in \pi_{00}(T)$. Thus $(\lambda I - T) \in \mathcal{F}_\ell$ (since $\mathcal{R}(\lambda I - T)$ is closed and $\dim \mathcal{N}(\lambda I - T) < \infty$) so that $\lambda \notin \sigma_{\ell e}(T)$, and hence $\lambda \notin \sigma_{\ell e}(T) \cap \sigma_{re}(T)$, which means that $\lambda \in \pi_0(T)$ by Proposition 6.1. (Recall that λ is an isolated point of $\sigma(T)$ since it lies in $\pi_{00}(T)$.) ∎

7 Weyl Spectrum

The *Weyl spectrum* of an operator $T \in \mathcal{B}[\mathcal{H}]$ is the set

$$\sigma_w(T) \;=\; \bigcap_{K \in \mathcal{B}_\infty[\mathcal{H}]} \sigma(T + K),$$

which is the largest part of $\sigma(T)$ that remains unchanged under compact perturbations. Clearly, $\sigma_w(T) = \sigma_w(T + K)$ for all $K \in \mathcal{B}_\infty[\mathcal{H}]$. Another characterization of it is given by the Schechter Theorem (cf. [24], [25]) as in Proposition 7.1 below.

Proposition 7.1. [7, p.367]. *If* $T \in \mathcal{B}[\mathcal{H}]$, *then*

$$\sigma_w(T) = \sigma_e(T) \cup \bigcup_{k \in \mathbb{Z}\setminus\{0\}} \sigma_k(T).$$

Since $\sigma_k(T) \subseteq \sigma(T)\backslash\sigma_e(T)$ for all $k \in \mathbb{Z}$, it follows by Proposition 7.1 that,

$$\sigma_w(T)\backslash\sigma_e(T) \;=\; \bigcup_{k \in \mathbb{Z}\setminus\{0\}} \sigma_k(T) \quad \text{and} \quad \sigma_e(T) = \sigma_w(T) \iff \bigcup_{k \in \mathbb{Z}\setminus\{0\}} \sigma_k(T) = \varnothing.$$

Corollary 7.2. $\quad \sigma_w(T) = \Big\{\lambda \in \mathbb{C}\colon\; (\lambda I - T) \notin \mathcal{W}\Big\}.$

Proof. $\lambda \in \sigma_w(T)$ if and only if either $\lambda \in \sigma_e(T)$ or $\lambda \in \sigma_k(T)$ for some $k \neq 0$ in $\mathbb{Z}$ by Proposition 7.1. Thus $\lambda \in \sigma_w(T)$ if and only if either $(\lambda I - T) \notin \mathcal{F}$ (Corollary 4.2) or $(\lambda I - T) \in \mathcal{F}$ with $\operatorname{ind}(\lambda I - T) \neq 0$, which means that $(\lambda I - T) \notin \mathcal{W}$. ∎

Then the Weyl spectrum $\sigma_w(T)$ is the set of all scalars λ for which $(\lambda I - T)$ is not a Weyl operator (i.e., for which $(\lambda I - T)$ is not a Fredholm operator of index zero). Since an operator lies in $\mathcal{W}$ together with its adjoint, it follows by Corollary 7.2 that $\lambda \in \sigma_w(T)$ if and only if $\overline{\lambda} \in \sigma_w(T^*)$:

$$\sigma_w(T) = \sigma_w(T^*)^*.$$

Recall that the essential (or Calkin, or Fredholm) spectrum $\sigma_e(T)$ is the set of all scalars λ for which $(\lambda I - T)$ is not a Fredholm operator. Therefore,

$$\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma(T)$$

since it is clear that $\sigma_w(T) \subseteq \sigma(T)$ by the very definition of $\sigma_w(T)$, which can also be verified by Corollaries 4.2 and 7.2: if $(\lambda I - T) \notin \mathcal{W}$, then either $(\lambda I - T) \notin \mathcal{F}$, and hence $\lambda \in \sigma_e(T) \subseteq \sigma(T)$, or $(\lambda I - T) \in \mathcal{F}$ and $\operatorname{ind}(\lambda I - T) \neq 0$, which implies that $\lambda \in \sigma_k(T) \subseteq \sigma(T)$ for some $k \in \mathbb{Z}\backslash\{0\}$ — cf. Proposition 7.1.

Remark 7.3. Take an arbitrary $T \in \mathcal{B}[\mathcal{H}]$. By Remark 3.1 and Corollary 7.2,

$$\dim \mathcal{H} < \infty \quad \Longrightarrow \quad \sigma_w(T) = \varnothing.$$

The converse holds by Remark 4.3 since $\sigma_e(T) \subseteq \sigma_w(T)$:

$$\dim \mathcal{H} = \infty \quad \Longrightarrow \quad \sigma_w(T) \neq \varnothing,$$

and $\sigma_w(T)$ is compact because it is the intersection $\bigcap_{K \in \mathcal{B}_\infty[\mathcal{H}]} \sigma(T+K)$ of compact sets. The preceding implication, Remark 3.2 (Fredholm Alternative), Remark 4.3, and Corollary 7.2 ensure that, if $\dim \mathcal{H} = \infty$, then

$$T \text{ is compact} \quad \Longrightarrow \quad \sigma_e(T) = \sigma_w(T) = \{0\}.$$

Moreover, if $\mathcal{H}$ is separable and infinite-dimensional, then [3]

$$0 \in \sigma_w(T) \quad \Longrightarrow \quad T \text{ is a commutator}$$

(i.e., the exist operators A and B such that $T = AB - BA$).

Another characterization of the Weyl spectrum says that $\sigma_w(T)$ is precisely the complement of $\sigma_0(T)$ in $\sigma(T)$.

Corollary 7.4. $\sigma_w(T) = \sigma(T)\backslash\sigma_0(T)$.

Proof. Immediate from Corollary 7.2 and the very definition of $\sigma_0(T)$. That is,

$$\sigma_w(T) = \big\{\lambda \in \sigma(T) \colon (\lambda I - T) \notin \mathcal{W}\big\} \text{ and } \sigma_0(T) = \big\{\lambda \in \sigma(T) \colon (\lambda I - T) \in \mathcal{W}\big\}.$$

∎

Since $\sigma_w(T)$ and $\sigma_0(T)$ are both subsets of $\sigma(T)$, it then follows that $\sigma_0(T)$ is the complement of $\sigma_w(T)$ in $\sigma(T)$,

$$\sigma_0(T) = \sigma(T)\backslash\sigma_w(T),$$

and therefore $\{\sigma_w(T), \sigma_0(T)\}$ forms a partition of the spectrum $\sigma(T)$:

$$\sigma(T) = \sigma_w(T) \cup \sigma_0(T) \quad \text{and} \quad \sigma_w(T) \cap \sigma_0(T) = \varnothing.$$

Thus $\sigma_w(T) = \sigma(T)$ if and only if $\sigma_0(T) = \varnothing$ and so, by Proposition 5.1,

$$\sigma_e(T) = \sigma_w(T) = \sigma(T) \quad \Longleftrightarrow \quad \bigcup_{k \in \mathbb{Z}} \sigma_k(T) = \varnothing.$$

Corollary 7.5.

$$\pi_0(T) = \sigma_{\rm iso}(T)\backslash\sigma_w(T) = \sigma_{\rm iso}(T)\backslash\sigma_e(T)$$
$$= \pi_{00}(T)\backslash\sigma_w(T) = \pi_{00}(T)\backslash\sigma_e(T).$$

Proof. Since $\sigma_0(T) = \sigma(T)\backslash\sigma_w(T)$ by Corollary 7.4, we get the first identity:

$$\sigma_{\rm iso}(T)\backslash\sigma_w(T) = \sigma_{\rm iso}(T) \cap \Big(\sigma(T)\backslash\sigma_w(T)\Big) = \sigma_{\rm iso}(T) \cap \sigma_0(T) = \pi_0(T).$$

Proposition 7.1 says that $\sigma_w(T) = \sigma_e(T) \cup G(T)$, where $G(T)$ is a subset of $\sigma(T)$ that is open in $\mathbb{C}$ (union of open sets). Thus

$$\sigma_{\rm iso}(T)\backslash\sigma_w(T) = \sigma_{\rm iso}(T)\Big\backslash\Big(\sigma_e(T) \cup G(T)\Big) = \Big(\sigma_{\rm iso}(T)\backslash\sigma_e(T)\Big) \cap \Big(\sigma_{\rm iso}(T)\backslash G(T)\Big).$$

Since $G(T)$ is a subset of $\sigma(T)$ that is open in $\mathbb{C}$, it follows that $\sigma_{\rm iso}(T) \cap G(T) = \varnothing$. Hence $\sigma_{\rm iso}(T)\backslash G(T) = \sigma_{\rm iso}(T)$, and so

$$\sigma_{\rm iso}(T)\backslash\sigma_w(T) = \sigma_{\rm iso}(T)\backslash\sigma_e(T),$$

which proves the second identify. Finally, to verify the remaining identities recall that $\pi_0(T) \subseteq \pi_{00}(T) \subseteq \sigma_{\rm iso}(T)$, $\pi_0(T) \subseteq \sigma_0(T) = \sigma(T)\backslash\sigma_w(T)$ and $\sigma_e(T) \subseteq \sigma_w(T)$. Thus the above identities ensure that

$$\begin{aligned}\pi_{00}(T)\backslash\sigma_w(T) &\subseteq \pi_{00}(T)\backslash\sigma_e(T) \subseteq \sigma_{\rm iso}(T)\backslash\sigma_e(T)\\ &= \sigma_{\rm iso}(T)\backslash\sigma_w(T) = \pi_0(T) = \pi_0(T)\backslash\sigma_w(T) \subseteq \pi_{00}(T)\backslash\sigma_w(T),\end{aligned}$$

and hence

$$\pi_0(T) = \pi_{00}(T)\backslash\sigma_w(T) = \pi_{00}(T)\backslash\sigma_e(T). \qquad \blacksquare$$

The equivalent assertions in the next proposition define an important class of operators. An operator for which any of those equivalent assertions holds is said to satisfy Weyl's theorem. This will be discussed later in Section 10.

Proposition 7.6. *For any $T \in \mathcal{B}[\mathcal{H}]$ the assertions below are pairwise equivalent.*

(a) $\sigma(T)\backslash\sigma_w(T) = \pi_{00}(T)$.

(b) $\sigma_0(T) = \pi_{00}(T)$.

(c) $\sigma(T)\backslash\pi_{00}(T) = \sigma_w(T)$.

Proof. According to Corollary 7.4 we have $\sigma(T)\backslash\sigma_w(T) = \sigma_0(T)$. Thus (a) and (b) are equivalent. Corollary 7.4 says that $\sigma(T)\backslash\sigma_0(T) = \sigma_w(T)$, and so (b) implies (c) and, since $\pi_{00}(T) \subseteq \sigma(T)$, it follows that (c) implies (b) as well. Indeed,

$$\sigma_0(T) = \sigma(T)\backslash\sigma_w(T) = \sigma(T)\Big\backslash\Big(\sigma(T)\backslash\pi_{00}(T)\Big) = \pi_{00}(T). \qquad \blacksquare$$

Remark 7.7. If any of the equivalent assertions in Proposition 7.6 holds, then

$$\pi_0(T) = \pi_{00}(T).$$

Indeed, since $\pi_0(T) = \sigma_{\rm iso}(T) \cap \sigma_0(T)$ and $\pi_{00}(T) = \sigma_{\rm iso}(T) \cap \sigma_{PF}(T)$, it follows that, if $\sigma_0(T) = \pi_{00}(T)$, then $\pi_0(T) = \sigma_{\rm iso}(T) \cap \pi_{00}(T) = \pi_{00}(T)$.

8 Ascent and Descent

Take an arbitrary operator T in $\mathcal{B}[\mathcal{H}]$, and let $\mathbb{N}_0$ be the set of all nonnegative integers. For every $n \in \mathbb{N}_0$,

$$\mathcal{N}(T^n) \subseteq \mathcal{N}(T^{n+1}) \quad \text{and} \quad \mathcal{R}(T^{n+1}) \subseteq \mathcal{R}(T^n)$$

trivially. Moreover, it is readily verified by induction (see e.g., [28, p.271]) that

(a) if $\mathcal{N}(T^{n_0+1}) = \mathcal{N}(T^{n_0})$ for some n_0, then $\mathcal{N}(T^n) = \mathcal{N}(T^{n_0})$ for all $n \geq n_0$;

(b) if $\mathcal{R}(T^{n_0+1}) = \mathcal{R}(T^{n_0})$ for some n_0, then $\mathcal{R}(T^n) = \mathcal{R}(T^{n_0})$ for all $n \geq n_0$.

Now put $\overline{\mathbb{N}}_0 = \mathbb{N}_0 \cup \infty$, the set of all extended nonnegative integers with its natural (extended) ordering. The *ascent* of T is the least nonnegative integer such that $\mathcal{N}(T^{n+1}) = \mathcal{N}(T^n)$,

$$\operatorname{asc}(T) = \min\Big\{n \in \overline{\mathbb{N}}_0 \colon\; \mathcal{N}(T^{n+1}) = \mathcal{N}(T^n)\Big\}.$$

The *descent* of T is the least nonnegative integer such that $\mathcal{R}(T^{n+1}) = \mathcal{R}(T^n)$,

$$\operatorname{dsc}(T) = \min\Big\{n \in \overline{\mathbb{N}}_0 \colon\; \mathcal{R}(T^{n+1}) = \mathcal{R}(T^n)\Big\}.$$

Note that $\operatorname{asc}(T) = 0$ if and only if T is injective (i.e., $\mathcal{N}(T) = \{0\}$) and $\operatorname{dsc}(T) = 0$ if and only if T is surjective (i.e., $\mathcal{R}(T) = \mathcal{H}$). The next proposition exhibits a duality between ascent and descent for Fredholm operators [13].

Proposition 8.1. *Take any operator T in $\mathcal{B}[\mathcal{H}]$.*

(a) $\operatorname{asc}(T) < \infty \iff \operatorname{dsc}(T^*) < \infty$, *and* $\operatorname{dsc}(T) < \infty \iff \operatorname{asc}(T^*) < \infty$.

(b) $\operatorname{dsc}(T) < \infty \implies \operatorname{asc}(T^*) \leq \operatorname{dsc}(T)$, *and* $\operatorname{dsc}(T^*) < \infty \implies \operatorname{asc}(T) \leq \operatorname{dsc}(T^*)$.

(c) *If* $T \in \mathcal{F}$, *then* $\operatorname{asc}(T) = \operatorname{dsc}(T^*)$ *and* $\operatorname{dsc}(T) = \operatorname{asc}(T^*)$.

Proof. We shall use freely the relations between range and kernel, involving adjoints and orthogonal projections (see e.g., [18, p.393]). Take any operator T in $\mathcal{B}[\mathcal{H}]$ and an arbitrary integer n in $\mathbb{N}_0$. If $\operatorname{asc}(T) = \infty$, then $\mathcal{N}(T^n) \subset \mathcal{N}(T^{n+1})$ so that $\mathcal{N}(T^{n+1})^\perp \subset \mathcal{N}(T^n)^\perp$ or, equivalently, $\mathcal{R}(T^{*(n+1)})^- \subset \mathcal{R}(T^{*n})^-$, which implies that $\mathcal{R}(T^{*(n+1)}) \subset \mathcal{R}(T^{*n})$, and hence $\operatorname{dsc}(T^*) = \infty$. Dually, if $\operatorname{asc}(T^*) = \infty$, then $\operatorname{dsc}(T) = \infty$. That is,

$$\operatorname{asc}(T) = \infty \implies \operatorname{dsc}(T^*) = \infty, \qquad \text{and} \qquad \operatorname{asc}(T^*) = \infty \implies \operatorname{dsc}(T) = \infty.$$

If $\operatorname{dsc}(T) = \infty$, then $\mathcal{R}(T^{n+1}) \subset \mathcal{R}(T^n)$ so that $\mathcal{R}(T^n)^\perp \subset \mathcal{R}(T^{n+1})^\perp$ or, equivalently, $\mathcal{N}(T^{*n}) \subset \mathcal{N}(T^{*(n+1)})$, and hence $\operatorname{asc}(T^*) = \infty$. Dually, if $\operatorname{dsc}(T^*) = \infty$, then $\operatorname{asc}(T) = \infty$. That is,

$$\operatorname{dsc}(T) = \infty \implies \operatorname{asc}(T^*) = \infty, \qquad \text{and} \qquad \operatorname{dsc}(T^*) = \infty \implies \operatorname{asc}(T) = \infty.$$

Summing up:

(a) $\operatorname{asc}(T) = \infty \iff \operatorname{dsc}(T^*) = \infty$, $\quad$ and $\quad$ $\operatorname{dsc}(T) = \infty \iff \operatorname{asc}(T^*) = \infty$.

If $\mathrm{dsc}\,(T) < \infty$, then put $n_0 = \mathrm{dsc}\,(T)$ in $\mathbb{N}_0$, and take an arbitrary integer $n \ge n_0$. Thus $\mathcal{R}(T^n) = \mathcal{R}(T^{n_0})$, and hence $\mathcal{R}(T^n)^- = \mathcal{R}(T^{n_0})^-$ so that $\mathcal{N}(T^{*n})^\perp = \mathcal{N}(T^{*n_0})^\perp$, which implies $\mathcal{N}(T^{*n}) = \mathcal{N}(T^{*n_0})$. Therefore, $\mathrm{asc}\,(T^*) \le n_0$, and so $\mathrm{asc}\,(T^*) \le \mathrm{dsc}\,(T)$. Dually, if $\mathrm{dsc}\,(T^*) < \infty$, then $\mathrm{asc}\,(T) \le \mathrm{dsc}\,(T^*)$. That is,

(b) $\mathrm{dsc}\,(T) < \infty \Rightarrow \mathrm{asc}\,(T^*) \le \mathrm{dsc}\,(T)$, and $\mathrm{dsc}\,(T^*) < \infty \Rightarrow \mathrm{asc}\,(T) \le \mathrm{dsc}\,(T^*)$.

On the other hand, if $\mathrm{asc}\,(T) < \infty$, then put $n_0 = \mathrm{asc}\,(T)$ in $\mathbb{N}_0$ and take any integer $n \ge n_0$. Thus $\mathcal{N}(T^n) = \mathcal{N}(T^{n_0})$ or, equivalently, $\mathcal{R}(T^{*n})^\perp = \mathcal{R}(T^{*n_0})^\perp$ so that $\mathcal{R}(T^{*n})^- = \mathcal{R}(T^{*n_0})^-$. Now we use the assumption that T is Fredholm. Indeed, if $T \in \mathcal{F}$, then $T^n \in \mathcal{F}$ by Corollary 3.5, which implies that $\mathcal{R}(T^n)$ is closed, for every nonnegative integer n (by the way, this is what is actually necessary) and so is $\mathcal{R}(T^{*n})$ (see e.g., [18, p.394]). Thus $\mathcal{R}(T^{*n}) = \mathcal{R}(T^{*n_0})$. Hence $\mathrm{dsc}\,(T^*) \le n_0$ so that $\mathrm{dsc}\,(T^*) \le \mathrm{asc}\,(T)$. Dually, if $\mathrm{asc}\,(T^*) < \infty$, then $\mathrm{dsc}\,(T) \le \mathrm{asc}\,(T^*)$. That is,

(b′) $\mathrm{asc}\,(T) < \infty \Rightarrow \mathrm{dsc}\,(T^*) \le \mathrm{asc}\,(T)$, and $\mathrm{asc}\,(T^*) < \infty \Rightarrow \mathrm{dsc}\,(T) \le \mathrm{asc}\,(T^*)$.

Therefore, since $\mathrm{asc}\,(T) < \infty$ if and only if $\mathrm{dsc}\,(T^*) < \infty$, and $\mathrm{dsc}\,(T) < \infty$ if and only if $\mathrm{asc}\,(T^*) < \infty$ by (a), it follows by (b) and (b') that

(c) $$\mathrm{asc}\,(T) = \mathrm{dsc}\,(T^*) \quad \text{and} \quad \mathrm{dsc}\,(T) = \mathrm{asc}\,(T^*)$$

whenever T is Fredholm. ∎

Proposition 8.2. [28, pp.272,273]. *Let T be an arbitrary operator in $\mathcal{B}[\mathcal{H}]$. If $\mathrm{asc}\,(T) < \infty$ and $\mathrm{dsc}\,(T) < \infty$, then $\mathrm{asc}\,(T) = \mathrm{dsc}\,(T)$.*

Proposition 8.3. [8]. *Take any operator T in $\mathcal{B}[\mathcal{H}]$.*

(a) *If $\dim \mathcal{N}(T) < \infty$ or $\dim \mathcal{N}(T^*) < \infty$, then* ($a_1$) $\mathrm{asc}\,(T) < \infty$ *implies* $\dim \mathcal{N}(T) \le \dim \mathcal{N}(T^*)$, ($a_2$) $\mathrm{dsc}\,(T) < \infty$ *implies* $\dim \mathcal{N}(T^*) \le \dim \mathcal{N}(T)$.

(b) *If $\dim \mathcal{N}(T) = \dim \mathcal{N}(T^*) < \infty$, then $\mathrm{asc}\,(T) < \infty$ if and only if $\mathrm{dsc}\,(T) < \infty$.*

Corollary 8.4. *Suppose $T \in \mathcal{B}[\mathcal{H}]$ is a Fredholm operator.*

(a) *If $\mathrm{asc}\,(T) < \infty$ and $\mathrm{dsc}\,(T) < \infty$, then $\mathrm{ind}\,(T) = 0$.*

(b) *If $\mathrm{ind}\,(T) = 0$, then $\mathrm{asc}\,(T) < \infty$ if and only if $\mathrm{dsc}\,(T) < \infty$.*

Proof. Immediate from Proposition 8.3. ∎

A *Browder operator* is a Fredholm operator with finite ascent and descent. Let $\mathcal{B}$ denote the class of all Browder operators from $\mathcal{B}[\mathcal{H}]$:

$$\mathcal{B} = \Big\{T \in \mathcal{F} \colon\; \mathrm{asc}\,(T) < \infty \;\text{ and }\; \mathrm{dsc}\,(T) < \infty\Big\};$$

equivalently, according to Proposition 8.2,

$$\mathcal{B} = \Big\{T \in \mathcal{F} \colon\; \mathrm{asc}\,(T) = \mathrm{dsc}\,(T) < \infty\Big\}.$$

Thus

$$\mathcal{F}\backslash\mathcal{B} = \Big\{T \in \mathcal{F} \colon\; \mathrm{asc}\,(T) = \infty \;\text{ or }\; \mathrm{dsc}\,(T) = \infty\Big\}$$

and, by Proposition 8.3, $T \in \mathcal{B}$ if and only if $T^* \in \mathcal{B}$. Observe that, according to Corollary 8.4, every Browder operator is a Weyl operator,

$$\mathcal{B} \subset \mathcal{W} \subset \mathcal{F},$$

and

$$T \in \mathcal{W} \implies \Big\{ \operatorname{asc}(T) < \infty \iff \operatorname{dsc}(T) < \infty \Big\}$$

so that

$$\mathcal{W} \backslash \mathcal{B} = \Big\{ T \in \mathcal{W} \colon\; \operatorname{asc}(T) = \operatorname{dsc}(T) = \infty \Big\}.$$

Since $\dim \mathcal{N}(T^n) \leq \dim \mathcal{N}(T^{n+1}) \leq \dim \mathcal{H}$, it follows that if $\mathcal{H}$ is finite-dimensional, then $\operatorname{asc}(T) < \infty$. Thus, according to Remark 3.1, every operator on a finite-dimensional space is a Browder operator.

Recall that $\rho(T) = \mathbb{C} \backslash \sigma(T)$ is the resolvent set of T, and $\sigma_{\text{iso}}(T) = \sigma(T) \backslash \sigma_{\text{acc}}(T)$ is the set of all isolated points of the spectrum $\sigma(T)$ — the complement in $\sigma(T)$ of the set $\sigma_{\text{acc}}(T)$ of all accumulation points of the spectrum. The class $\mathcal{B}$ of all Browder operators can be equivalently described as follows.

Proposition 8.5. $\quad \mathcal{B} = \Big\{ T \in \mathcal{F} \colon\; 0 \in \rho(T) \cup \sigma_{\text{iso}}(T) \Big\}.$

Proof. An operator T in $\mathcal{B}[\mathcal{H}]$ is Browder if and only if it is Fredholm and, $\lambda I - T$ is invertible for sufficiently small $\lambda \neq 0$ [15]. Equivalently, T is Fredholm and, for some $\varepsilon > 0$, $\lambda \in \rho(T)$ for every $0 \neq |\lambda| < \varepsilon$; that is, $B_\varepsilon(0) \backslash \{0\} \subset \rho(T)$, where $B_\varepsilon(0)$ is the open ball centered at 0 with radius ε. Since $\rho(T)$ is an open subset of $\mathbb{C}$, this simply means that either $0 \in \rho(T)$ or 0 is an isolated point of $\sigma(T) = \mathbb{C} \backslash \rho(T)$. ∎

Thus, since $\mathcal{W} \backslash \mathcal{B} = \mathcal{W} \cap (\mathcal{F} \backslash \mathcal{B})$, we get from Proposition 8.5 that

$$\mathcal{F} \backslash \mathcal{B} = \Big\{ T \in \mathcal{F} \colon\; 0 \in \sigma_{\text{acc}}(T) \Big\} \quad \text{and} \quad \mathcal{W} \backslash \mathcal{B} = \Big\{ T \in \mathcal{W} \colon\; 0 \in \sigma_{\text{acc}}(T) \Big\}.$$

9 Browder Spectrum

The *Browder spectrum* of an operator $T \in \mathcal{B}[\mathcal{H}]$ is the set

$$\sigma_b(T) \;=\; \bigcap_{K \in \mathcal{B}_\infty[\mathcal{H}] \cap \{T\}'} \sigma(T + K),$$

where $\{T\}'$ denote the commutant of T (i.e., the collection of all operators in $\mathcal{B}[\mathcal{H}]$ that commute with T). Thus $\sigma_b(T)$ is the largest part of $\sigma(T)$ that remains unchanged under compact perturbations in the commutant of T, and this coincides with the set of all scalars λ for which $(\lambda I - T)$ is not a Browder operator (i.e., for which $(\lambda I - T)$ is not a Fredholm operator with a finite ascent and a finite descent).

Proposition 9.1. [8]. $\sigma_b(T) = \big\{\lambda \in \mathbb{C}\colon\; (\lambda I - T) \notin \mathcal{B}\big\}.$

Since an operator lies in $\mathcal{B}$ together with its adjoint, it follows by Proposition 9.1 that $\lambda \in \sigma_b(T)$ if and only if $\overline{\lambda} \in \sigma_b(T^*)$:

$$\sigma_b(T) = \sigma_b(T^*)^*.$$

Also observe that

$$\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) \subseteq \sigma(T).$$

Indeed, since $\mathcal{B} \subset \mathcal{W}$, Corollary 7.2 and Proposition 9.1 ensure that $\sigma_w(T) \subseteq \sigma_b(T)$. Moreover, $\sigma_b(T) \subseteq \sigma(T)$ by the very definition of $\sigma_b(T)$, but this can also be directly verified from Proposition 9.1 as follows. If $(\lambda I - T) \notin \mathcal{B}$, then either $(\lambda I - T) \notin \mathcal{F}$ or $(\lambda I - T) \in \mathcal{F}$ and $\operatorname{asc}(\lambda I - T) = \operatorname{dsc}(\lambda I - T) = \infty$. In the former case, λ lies in $\sigma_e(T) \subseteq \sigma(T)$. In the latter case, $\lambda \notin \rho(T)$ so that $\lambda \in \sigma(T)$. Indeed, if $\lambda \in \rho(T)$; that is, if $(\lambda I - T)$ is invertible, then $\operatorname{asc}(\lambda I - T) = 0$ because $\mathcal{N}(\lambda I - T) = \{0\}$ (and $\operatorname{dsc}(\lambda I - T) = 0$ because $\mathcal{R}(\lambda I - T) = \mathcal{H}$).

Corollary 9.2. $\sigma_b(T) = \sigma_e(T) \cup \sigma_{\mathrm{acc}}(T).$

Proof. According to Propositions 8.5 and 9.1,

$$\begin{aligned}\sigma_b(T) &= \big\{\lambda \in \mathbb{C}\colon\; (\lambda I - T) \notin \mathcal{F} \;\text{ or }\; 0 \notin \rho(\lambda I - T) \cup \sigma_{\mathrm{iso}}(\lambda I - T)\big\}\\ &= \big\{\lambda \in \mathbb{C}\colon\; \lambda \in \sigma_e(T) \;\text{ or }\; \lambda \notin \rho(T) \cup \sigma_{\mathrm{iso}}(T)\big\}\\ &= \sigma_e(T) \cup \big(\sigma(T)\backslash\sigma_{\mathrm{iso}}(T)\big) = \sigma_e(T) \cup \sigma_{\mathrm{acc}}(T).\end{aligned}$$

■

Remark 9.3. Take any $T \in \mathcal{B}[\mathcal{H}]$. Since every operator on a finite-dimensional space is a Browder operator, it follows by Proposition 9.1 that

$$\dim \mathcal{H} < \infty \quad\Longrightarrow\quad \sigma_b(T) = \varnothing.$$

The converse holds by Remark 7.3, since $\sigma_w(T) \subseteq \sigma_b(T)$:

$$\dim \mathcal{H} = \infty \quad\Longrightarrow\quad \sigma_b(T) \neq \varnothing,$$

and $\sigma_b(T)$ is compact because it is the intersection $\bigcap_{K \in \mathcal{B}_\infty[\mathcal{H}] \cap \{T\}'} \sigma(T + K)$ of compact sets. Moreover, since $\sigma_{\mathrm{acc}}(T) \subseteq \{0\}$ whenever T is compact, it follows by Remark 7.3, Corollary 9.2, and the preceding implication that, if $\dim \mathcal{H} = \infty$, then

$$T \text{ is compact} \quad\Longrightarrow\quad \sigma_e(T) = \sigma_w(T) = \sigma_b(T) = \{0\}.$$

Alternate characterizations of the Browder spectrum were given in the previous two results. Another one is given next, which says that $\sigma_b(T)$ is precisely the complement of the Riesz points $\pi_0(T) = \sigma_{\mathrm{iso}}(T) \cap \sigma_0(T)$ in $\sigma(T)$.

Corollary 9.4. $\sigma_b(T) = \sigma(T)\Big\backslash\Big(\sigma_{\rm iso}(T) \cap \sigma_0(T)\Big).$

Proof. By Proposition 5.1 and Corollary 9.2 we get

$$\begin{aligned}\sigma(T)\backslash\sigma_b(T) &= \sigma(T)\Big\backslash\Big(\sigma_e(T) \cup \sigma_{\rm acc}(T)\Big) = \Big(\sigma(T)\backslash\sigma_e(T)\Big) \cap \Big(\sigma(T)\backslash\sigma_{\rm acc}(T)\Big)\\ &= \bigcup_{k\in\mathbb{Z}} \sigma_k(T) \cap \sigma_{\rm iso}(T) = \sigma_0(T) \cap \sigma_{\rm iso}(T),\end{aligned}$$

because $\sigma_k(T)$ is open in $\mathbb{C}$ for each $0 \neq k \in \mathbb{Z}$, and so has no isolated point. ∎

Thus $\{\sigma_b(T), \pi_0(T)\}$ forms a partition of the spectrum $\sigma(T)$:

$$\sigma(T) = \sigma_b(T) \cup \pi_0(T) \quad \text{and} \quad \sigma_b(T) \cap \pi_0(T) = \varnothing.$$

Indeed, since $\pi_0(T) = \sigma_{\rm iso}(T) \cap \sigma_0(T)$, and since $\sigma_b(T)$ and $\pi_0(T)$ are both subsets of $\sigma(T)$, Corollary 9.4 says that $\sigma_b(T)$ is the complement of $\pi_0(T)$ in $\sigma(T)$, that is, $\sigma_b(T) = \sigma(T)\backslash\pi_0(T)$ and, consequently, $\pi_0(T)$ is the complement of $\sigma_b(T)$ in $\sigma(T)$, that is, $\pi_0(T) = \sigma(T)\backslash\sigma_b(T)$.

Corollary 9.5. $\sigma_b(T) = \sigma_w(T) \cup \sigma_{\rm acc}(T).$

Proof. Corollary 9.4 says that $\sigma_b(T) = \Big(\sigma(T)\backslash\sigma_{\rm iso}(T)\Big) \cup \Big(\sigma(T)\backslash\sigma_0(T)\Big)$, and therefore, by Corollary 7.4, $\sigma_b(T) = \sigma_{\rm acc}(T) \cup \sigma_w(T)$. ∎

Note that, according to Proposition 9.1 and Corollary 9.4, the set $\pi_0(T) = \sigma_{\rm iso}(T) \cap \sigma_0(T)$ of Riesz points of T is also given by

$$\pi_0(T) = \sigma(T)\backslash\sigma_b(T) = \Big\{\lambda \in \sigma(T)\colon\; (\lambda I - T) \in \mathcal{B}\Big\}.$$

Corollary 9.6. $\pi_0(T) = \sigma_{\rm iso}(T)\backslash\sigma_b(T) = \pi_{00}(T)\backslash\sigma_b(T).$

Proof. Since $\pi_0(T) \subseteq \pi_{00}(T) \subseteq \sigma_{\rm iso}(T) \subseteq \sigma(T)$ we get by Corollary 9.4 that

$$\pi_0(T) = \pi_0(T)\backslash\sigma_b(T) \subseteq \pi_{00}(T)\backslash\sigma_b(T) \subseteq \sigma_{\rm iso}(T)\backslash\sigma_b(T) = \sigma(T)\backslash\sigma_b(T) = \pi_0(T).$$ ∎

Remark 9.7. We had seen in Remark 7.7 that the set of Riesz points $\pi_0(T) = \sigma_{\rm iso}(T) \cap \sigma_0(T)$ and the set of isolated eigenvalues of finite multiplicity $\pi_{00}(T) = \sigma_{\rm iso}(T) \cap \sigma_{PF}(T)$ coincide whenever any of the equivalent assertions of Proposition 7.6 holds. According to Corollaries 7.5 and 9.6 the next four assertions (which include the identity in Remark 7.7) are pairwise equivalent.

(a) $\pi_0(T) = \pi_{00}(T)$.

(b) $\sigma_e(T) \cap \pi_{00}(T) = \varnothing$ (i.e., $\sigma_e(T) \cap \sigma_{\rm iso}(T) \cap \sigma_{PF}(T) = \varnothing$).

(c) $\sigma_w(T) \cap \pi_{00}(T) = \varnothing$ (i.e., $\sigma_w(T) \cap \sigma_{\rm iso}(T) \cap \sigma_{PF}(T) = \varnothing$).

(d) $\sigma_b(T) \cap \pi_{00}(T) = \varnothing$ (i.e., $\sigma_b(T) \cap \sigma_{\rm iso}(T) \cap \sigma_{PF}(T) = \varnothing$).

The equivalent assertions in the next proposition define an important class of operators. An operator for which any of those equivalent assertions holds is said to satisfy Browder's theorem. This will be discussed later in Section 10.

Proposition 9.8. *For any $T \in \mathcal{B}[\mathcal{H}]$ the assertions below are pairwise equivalent.*

(a) $\sigma(T)\backslash\sigma_w(T) = \pi_0(T)$.

(b) $\sigma_0(T) = \pi_0(T)$.

(c) $\sigma(T) = \sigma_w(T) \cup \pi_{00}(T)$.

(d) $\sigma_0(T) \subseteq \pi_{00}(T)$.

(e) $\sigma_0(T) \subseteq \sigma_{\mathrm{iso}}(T)$.

(f) $\sigma_{\mathrm{acc}}(T) \subseteq \sigma_w(T)$.

(g) $\sigma_w(T) = \sigma_b(T)$.

Proof. Assertion (a) implies (b) and (b) implies (c) by Corollary 7.4. In fact, since $\sigma(T)\backslash\sigma_w(T) = \sigma_0(T)$ (Corollary 7.4), it follows that (a) implies (b) and, if (b) holds, then (apply Corollary 7.4 again and recall that $\pi_0(T) \subseteq \pi_{00}(T)$)

$$\sigma(T) = \sigma_w(T) \cup \sigma_0(T) = \sigma_w(T) \cup \pi_0(T) \subseteq \sigma_w(T) \cup \pi_{00}(T) \subseteq \sigma(T)$$

so that (b) implies (c). Corollaries 7.4 and 7.5 ensure that (c) implies (a). Actually, recall that $\pi_{00}(T)\backslash\sigma_w(T) = \pi_0(T)$ (Corollary 7.5), $\pi_0(T) \subseteq \sigma_0(T)$, and $\sigma_0(T) = \sigma(T)\backslash\sigma_w(T)$ (Corollary 7.4). If (c) holds, then

$$\sigma(T)\backslash\sigma_w(T) = \pi_{00}(T)\backslash\sigma_w(T) = \pi_0(T) \subseteq \sigma_0(T) = \sigma(T)\backslash\sigma_w(T),$$

which implies (a). Thus (a), (b) and (c) are pairwise equivalent. Moreover, since $\pi_{00} = \sigma_{\mathrm{iso}}(T) \cap \sigma_{PF}(T)$ and $\sigma_0(T) \subseteq \sigma_{PF}(T)$, it follows that (d) is equivalent to (e). Furthermore, since $\pi_0(T) = \sigma_{\mathrm{iso}}(T) \cap \sigma_0(T)$, it also follows that (b) is equivalent to (e). Now recall that $\sigma_{\mathrm{acc}}(T)$ and $\sigma_{\mathrm{iso}}(T)$ are complement of each other in $\sigma(T)$, and $\sigma_w(T)$ and $\sigma_0(T)$ also are complement of each other in $\sigma(T)$ (Corollary 7.4). Therefore, (e) and (f) are equivalent. Similarly, since $\sigma_w(T)$ and $\sigma_0(T)$ are complement of each other in $\sigma(T)$ (Corollary 7.4), and since $\sigma_b(T)$ and $\pi_0(T)$ also are complement of each other in $\sigma(T)$ (Corollary 9.4), it follows that (g) and (b) are equivalent as well. ∎

10 Browder and Weyl Theorems

Take an arbitrary operator T in $\mathcal{B}[\mathcal{H}]$. Consider the set $\sigma_0(T)$ of all λ in $\sigma(T)$ for which $(\lambda I - T)$ is a Weyl operator. As we saw in Section 5, it is given by

$$\sigma_0(T) = \Big\{\lambda \in \sigma_P(T)\colon\ \mathcal{R}(\lambda I - T)^- = \mathcal{R}(\lambda I - T) \neq \mathcal{H} \text{ and } \\ \dim \mathcal{N}(\lambda I - T) = \dim \mathcal{N}(\overline{\lambda} I - T^*) < \infty\Big\}.$$

According to Corollaries 7.4 and 9.4 the Weyl and Browder spectra can be written in terms of $\sigma_0(T)$ as follows.

$$\sigma_w(T) = \sigma(T)\backslash\sigma_0(T) \qquad \text{and} \qquad \sigma_b(T) = \sigma(T)\Big\backslash\Big(\sigma_{\mathrm{iso}}(T) \cap \sigma_0(T)\Big).$$

Although $\sigma_0(T) \subseteq \sigma_{PF}(T)$ and $\sigma_{\rm acc}(T) \subseteq \sigma_b(T)$, in general $\sigma_0(T)$ may not consist of isolated points only or, equivalently, $\sigma_{\rm acc}(T)$ may not be included in $\sigma_w(T)$. An operator T is said to *satisfy Weyl's theorem* (or *Weyl's theorem holds* for T) if

$$\sigma_0(T) = \sigma_{\rm iso}(T) \cap \sigma_{PF}(T).$$

That is, since $\pi_{00}(T) = \sigma_{\rm iso}(T) \cap \sigma_{PF}(T)$, an operator T satisfies Weyl's theorem if $\sigma_0(T) = \pi_{00}(T)$, or any of the equivalent assertions in Proposition 7.6. An operator T is said to *satisfy Browder's theorem* (or *Browder's theorem holds* for T) if

$$\sigma_0(T) \subseteq \sigma_{\rm iso}(T) \cap \sigma_{PF}(T).$$

Thus an operator T satisfies Browder's theorem if $\sigma_0(T) \subseteq \pi_{00}(T)$, or any of the equivalent assertions in Proposition 9.8. In particular, T satisfies Browder's theorem if $\sigma_0(T) = \pi_0(T)$, where $\pi_0(T) = \sigma_{\rm iso}(T) \cap \sigma_0(T)$ or, equivalently, if

$$\sigma_0(T) \subseteq \sigma_{\rm iso}(T), \quad \text{or} \quad \sigma_{\rm acc}(T) \subseteq \sigma_w(T), \quad \text{or} \quad \sigma_w(T) = \sigma_b(T).$$

(These are the usual terminologies, although saying that T "satisfies Weyl's or Browder's *property*", rather than "satisfies Weyl's or Browder's *theorem*", would perhaps sound more appropriate.)

It is plain that, if T satisfies Weyl's theorem, then it also satisfies Browder's theorem. Note that, if $\sigma_w(T) = \sigma_{\rm acc}(T)$ or, equivalently, if $\sigma_0(T) = \sigma_{\rm iso}(T)$, then T satisfies Weyl's theorem. (Reason: $\sigma_0(T) = \sigma_{\rm iso}(T)$ implies $\sigma_0(T) = \sigma_{\rm iso}(T) \cap \sigma_{PF}(T)$ because $\sigma_0(T) \subseteq \sigma_{PF}(T)$.) In fact (see Remark 9.7),

$$T \text{ satisfies Browder's theorem but not Weyl's} \implies \sigma_w(T) \cap \sigma_{\rm iso}(T) \cap \sigma_{PF}(T) \neq \varnothing.$$

Remark 10.1. Consider the equivalent assertions of Proposition 7.6 and of Proposition 9.8. If Browder's theorem holds and $\pi_0(T) = \pi_{00}(T)$ then Weyl's theorem holds (i.e., if $\sigma_0(T) = \pi_0(T)$ and $\pi_0(T) = \pi_{00}(T)$, then $\sigma_0(T) = \pi_{00}(T)$ tautologically). Conversely, if Weyl's theorem holds, then $\pi_0(T) = \pi_{00}(T)$ (cf. Remark 7.7) and Browder's theorem holds trivially. Summing up,

$$\text{Weyl's theorem holds} \quad \iff \quad \text{Browder's theorem holds and } \pi_0(T) = \pi_{00}(T).$$

In other words, Weyl's theorem holds if and only if Browder's theorem and any of the equivalent assertions of Remark 9.7 hold.

Further necessary and sufficient conditions for an operator to satisfy Weyl's theorem can be found in [12]. According to Remark 7.3 every operator T on a finite-dimensional space satisfies Weyl's theorem with $\sigma_0(T) = \pi_{00}(T) = \sigma(T)$ (this extends to finite-rank but not to compact operators — see examples in [11]) and, on the other hand, every operator T without eigenvalues ($\sigma_P(T) = \varnothing$) also satisfies Weyl's theorem with $\sigma_0(T) = \pi_{00}(T) = \varnothing$. These are the trivial cases. Weyl proved in [30] that Weyl's theorem holds for self-adjoint operators. This was extended to normal operators in [26], to hyponormal operators in [5], and to seminormal operators in [2]. Recall that $T \in \mathcal{B}[\mathcal{H}]$ is hyponormal if $O \leq T^*T - TT^*$ and cohyponormal if T^* is hyponormal, so that T is normal if it is both hyponormal and cohyponormal. If T is either hyponormal or cohyponormal, then it is seminormal.

A subspace $\mathcal{M}$ of $\mathcal{H}$ is invariant for an operator $T \in \mathcal{B}[\mathcal{H}]$ (or T-invariant) if $T(\mathcal{M}) \subseteq \mathcal{M}$, and reducing if it is invariant for both T and T^*. A part of an operator is a restriction of it to an invariant subspace, and a direct summand is a restriction of it to a reducing subspace. An operator is isoloid if every isolated point of its spectrum is an eigenvalue (i.e., T is isoloid if $\sigma_{\rm iso}(T) \subseteq \sigma_P(T)$), and dominant if $\mathcal{R}(\lambda I - T) \subseteq \mathcal{R}(\overline{\lambda} I - T^*)$. The main theorem from [2], namely, *if each finite-dimensional eigenspace of an operator $T \in \mathcal{B}[\mathcal{H}]$ is reducing and every direct summand of it is isoloid, then T satisfies Weyl's theorem,* includes many of the previous results along this line, and has also been frequently applied to yield further results; mainly through the following corollary [11]: *If $T \in \mathcal{B}[\mathcal{H}]$ is dominant and every direct summand of it is isoloid, then T satisfies Weyl's theorem.* Since every hyponormal operator is dominant, every part (and, in particular, every direct summand) of a hyponormal operator is again hyponormal, and every hyponormal operator is isoloid, it follows by the above italicized result that every hyponormal operator satisfies Weyl's theorem.

Weyl's theorem has been extended to classes of nondominant operators that properly include the hyponormal operators. In particular, it was extended to paranormal operators [29] and, beyond, to totally hereditarily normaloid operators [9]. Recall that a normaloid is an operator whose spectral radius coincides with the norm, an operator T is paranormal if $\|Tx\|^2 \leq \|T^2x\|\,\|x\|$ for all vectors x, and a totally hereditarily normaloid operator is one whose all parts are normaloid, as well as the inverse of all invertible parts — these classes are related by proper inclusion [10]: Hyponormal $\subset$ Paranormal $\subset$ Totally Hereditarily Normaloid $\subset$ Normaloid.

Let T and S be arbitrary operators acting on Hilbert spaces. First we consider their direct sum. It is readily verified that the spectrum of a direct sum coincides with the union of the spectra of the summands, $\sigma(T \oplus S) = \sigma(T) \cup \sigma(S)$. For the Weyl spectrum only inclusion is ensured: the Weyl spectrum of a direct sum is included in the union of the Weyl spectra of the summands,

$$\sigma_w(T \oplus S) \subseteq \sigma_w(T) \cup \sigma_w(S),$$

but equality does not hold in general. However, it does if the essential and Weyl spectra coincide for one of the direct summands [15],

$$\sigma_e(T) = \sigma_w(T) \quad \Longrightarrow \quad \sigma_w(T \oplus S) = \sigma_w(T) \cup \sigma_w(S),$$

and also if $\sigma_w(T) \cap \sigma_w(S)$ has empty interior [22],

$$\Big(\sigma_w(T) \cap \sigma_w(S)\Big)^\circ = \varnothing \quad \Longrightarrow \quad \sigma_w(T \oplus S) = \sigma_w(T) \cup \sigma_w(S).$$

In general, Weyl's theorem does not transfer from T and S to their direct sum $T \oplus S$. The above identity involving the Weyl spectrum of a direct sum plays an important role in establishing sufficient conditions for a direct sum to satisfy Weyl's theorem, as it was recently investigated in [21] and [11]. As for the problem of transferring Browder's theorem from T and S to their direct sum $T \oplus S$, the following necessary and sufficient condition was proved in [15].

Proposition 10.2. *If both operators T and S satisfy Browder's theorem, then the direct sum $T \oplus S$ satisfies Browder's theorem if and only if*

$$\sigma_w(T \oplus S) = \sigma_w(T) \cup \sigma_w(S).$$

Now consider the tensor product $T \otimes S$ of a pair of Hilbert space operators T and S (see [19] for an expository paper containing the essential properties of tensor products needed here). It is known from [4] that the spectrum of a tensor product coincides with the product of the spectra of the factors, $\sigma(T \otimes S) = \sigma(T) \cdot \sigma(S)$. For the Weyl spectrum it was proved in [16] that the inclusion

$$\sigma_w(T \otimes S) \subseteq \sigma_w(T) \cdot \sigma(S) \cup \sigma(T) \cdot \sigma_w(S)$$

holds, but it remains as an open question whether the equality holds. That is, it is not know if there exist a pair of operators T and S for which the above inclusion may be proper. Sufficient conditions ensuring that the equality holds were recently investigated in [20]. For instance, if

$$\sigma_e(T)\backslash\{0\} = \sigma_w(T)\backslash\{0\} \quad \text{and} \quad \sigma_e(S)\backslash\{0\} = \sigma_w(S)\backslash\{0\}$$

(which holds, in particular, for compact operators T and S), then

$$\sigma_w(T \otimes S) = \sigma_w(T) \cdot \sigma(S) \cup \sigma(T) \cdot \sigma_w(S).$$

Also, if the tensor product satisfies Browder's theorem, then the equality holds:

$$\sigma_w(T \otimes S) = \sigma_b(T \otimes S) \quad \Longrightarrow \quad \sigma_w(T \otimes S) = \sigma_w(T) \cdot \sigma(S) \cup \sigma(T) \cdot \sigma_w(S).$$

Again, Weyl's theorem does not transfer from T and S to their tensor product $T \otimes S$ in general. The above identity involving the Weyl spectrum of a tensor product plays a crucial role in establishing sufficient conditions for a tensor product to satisfy Weyl's theorem, as it was recently investigated in [27] and [20]. As for the problem of transferring Browder's theorem from T and S to their tensor product $T \otimes S$, the following necessary and sufficient condition was proved in [20].

Proposition 10.3. *If both operators T and S satisfy Browder's theorem, then the tensor product $T \otimes S$ satisfies Browder's theorem if and only if*

$$\sigma_w(T \otimes S) = \sigma_w(T) \cdot \sigma(S) \cup \sigma(T) \cdot \sigma_w(S).$$

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