Metrizability of totally ordered groups of infinite rank and their completions*

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Abstract

In [4], Ochsenius and Schikhof ask the following question. Given a totally ordered group G with a cofinal sequence, if every element of its Dedekind completion $G^{\#}$ is the supremum of a sequence in G, does it follow that $G^{\#}$ is metrizable? We answer their question by studying topological properties of a family of totally ordered groups, Γ_α, and their completions $\Gamma_\alpha^{\#}$. Furthermore we obtain for this family conditions both necessary and sufficient for the metrizability of $\Gamma_\alpha^{\#}$.

Introduction

Nowadays, Normed Hilbert Spaces is one of the principal lines of development in non-Archimedean analysis of infinite rank (see [5]). In that theory G-modules, introduced by H. Ochsenius and W. Schikhof in [4] play a fundamental role as the natural range of the norms of vectors. Prominent among them is $G^{\#}$, the Dedekind completion of G.

In this infinite rank theory there is a strong connection between the properties of the ordered value group, the valued field and the normed space. The next proposition states this fact with regard to absolutely convex subsets of the field K and the existence of sequences with given properties in G and $G^{\#}$.

Proposition 1.4.4 [4] Let K be a valued field with value group G. The following are equivalent.

(α) Each absolutely convex subset of K is countably generated as a B_K-module.

(β) G has a cofinal sequence. For each $s \in G^{\#}$ there are $g_1, g_2, \ldots \in G, g_n < s$ for

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all n, such that $\sup_{G^\#}\{t \in G^\# : t < s\} = \sup_{G^\#}\{g_1, g_2, \ldots\}$.
(γ) G has a coinitial sequence. For each $s \in G^\#$ there are $g_1, g_2, \ldots \in G, g_n > s$ for all n, such that $\inf_{G^\#}\{t \in G^\# : t > s\} = \inf_{G^\#}\{g_1, g_2, \ldots\}$.
(δ) The interval topology on $G^\#$ satisfies the first axiom of countability. $G^\#$ has a cofinal sequence.

These statements suggest that topological properties of ordered groups and their completions will have a strong bearing in the structure of fields with infinite rank valuations. In particular, Ochsenius and Schikhof ask in that paper if the properties described in Proposition 1.4.4. are necessary and/or sufficient to ensure the metrizability of $G^\#$.

In this paper we will prove that this is not the case, that metrizability of $G^\#$ is independent of them. For this we will refer to a canonical family of totally ordered groups which have any prescribed ordinal α as their rank, they are the so-called Γ_α groups (for details see [6]). In fact the construction can easily be generalized to groups with arbitrary rank I (for instance $I = \mathbb{Q}$ or $I = \mathbb{R}$).

The structure of this paper is as follows. In the Preliminaries we describe the groups Γ_α as well as their Dedekind completions $\Gamma_\alpha^\#$. In section 2 it is shown that totally ordered multiplicative groups are in fact topological groups, and that every Γ_α is metrizable. In section 3 we prove that a G-module is continuous (see [5]) if and only if it is a topological G-module. We deal with separability conditions for Γ_α and $\Gamma_\alpha^\#$ in section 4. With all this work done, we can derive conditions both necessary and sufficient for the metrizability of $\Gamma_\alpha^\#$, thus giving an answer to the question posed in [4].

1 Preliminaries

Let G be a nontrivial totally ordered multiplicatively written group with unit 1. We denote the (Dedekind) completion of G by $G^\#$. We can extend the multiplication of G to a map $G \times G^\# \to G^\#$ by

$$gx = \sup_{G^\#}\{gh \in G : h \le x\} = \inf_{G^\#}\{gh \in G : h \ge x\}$$

for each $g \in G$ and $x \in G^\#$. This defines an action of g on $G^\#$ which is increasing in both variables and such that any orbit Gx is cofinal in $G^\#$. That is, $G^\#$ becomes a G-module (see [4]).

It is also possible to extend the multiplication of G to a binary operation on $G^\#$. In [4] this is done in two different ways, and in both cases $G^\#$ becomes a commutative unitary semigroup. It is well known that no extension that makes $G^\#$ a group can be defined.

A totally ordered group G is **quasidiscrete** if $\min\{g \in G : g > 1\}$ exists and it is **quasidense** if $\inf\{g \in G : g > 1\} = 1_G$ (See [5], Definition 1.2.1.).

In this paper we will work with the class of the groups Γ_α, which were defined in [6] as follows.

Let $\alpha \neq 0$ be an ordinal. For each $\beta < \alpha$, let G_β be a totally ordered group of rank 1 (that is to say each group G_β is isomorphic to a multiplicative subgroup of $(0, \infty)$).

The group Γ_α associated to the family $\{G_\beta\}_{\beta<\alpha}$ is given by

$$\begin{aligned}\Gamma_\alpha \ := \ &\{f : \alpha \to \bigcup_{\beta<\alpha} G_\beta \ : \ f(\beta) \in G_\beta \text{ and} \\ &\text{supp}(f) = \{\beta < \alpha : f(\beta) \neq 1_{G_\beta}\} \text{ is finite}\}\end{aligned}$$

with componentwise multiplication and antilexicographical ordering.
If $f \in \Gamma_\alpha$, then the **degree** of f is $\deg(f) := \max \text{supp}(f)$.
Every element $b \in G_\beta$ is represented by the element $\chi_{(\beta,b)} \in \Gamma_\alpha$ as

$$\chi_{(\beta,b)}(\gamma) := \begin{cases} b & \text{if} \quad \gamma = \beta \\ 1_{G_\gamma} & \text{if} \quad \gamma \neq \beta \end{cases}$$

The convex subgroups of Γ_α are easily described. For each $\beta < \alpha$, there exist two convex subgroups associated, $H_\beta := \{f \in \Gamma_\alpha : \deg(f) \leq \beta\}$ and $H^*_\beta := \{f \in \Gamma_\alpha : \deg(f) < \beta\}$. Furthermore, H_β always is a principal convex subgroup. On the other hand, if $\beta = \delta + 1$ is a successor ordinal, $H^*_{\delta+1} = H_\delta$, and if β is a limit ordinal, H^*_β is a limit convex subgroup (union of a chain of principal convex subgroups). Note that $H^*_0 = \{1_{\Gamma_\alpha}\}$, because 0 is the first limit ordinal. Moreover, if H is a convex subgroup of Γ_α then there exists an ordinal $\beta < \alpha$ such that either $H = H_\beta$ or $H = H^*_\beta$ (See [6] Proposition 2.1). We denote by s_β and t_β (respectively s^*_β and t^*_β), the supremum and infimum of H_β (respectively H^*_β) in $\Gamma^\#_\alpha$.
The next theorem gives a complete description of the elements of $\Gamma^\#_\alpha$ (see [7]).

Theorem 1.1 *Let $\{G_\beta\}_{\beta<\alpha}$ be an arbitrary family of totally ordered groups of rank 1. For each $\beta < \alpha$, let G'_β be the (Dedekind) completion of G_β. Let Γ_α the group associated with the family $\{G_\beta\}_{\beta<\alpha}$. Then, each element $x \in \Gamma^\#_\alpha$ satisfies one of the following sentences.*

1. *$x = gs$, with $g \in \Gamma_\alpha$ and s the supremum of some convex subgroup (s may be 1_{Γ_α}).*
2. *$x = gt$, for some $g \in \Gamma_\alpha$ and t the infimum of a convex subgroup H, such that G/H is quasidense.*
3. *$x = g\chi_{(\beta,b)}t^*_\beta$, for some $g \in \Gamma_\alpha$ and $\beta < \alpha$ such that $G_\beta \neq G'_\beta$ and $b \notin G_\beta$.*

Remark. The above construction can be generalized to any totally ordered set I and any family $\{G_i\}_{i\in I}$ of totally ordered groups of rank 1 written multiplicatively. The group Γ_I associated to the family $\{G_i\}_{i\in I}$ is formed by all $f : I \to \bigcup_{i\in I} G_i$ such that $f(i) \in G_i$ and $\text{supp}(f) = \{i \in I : f(i) \neq 1_{G_i}\}$ is finite with componentwise multiplication and antilexicographical ordering. The rank of Γ_I is the order-type of I.

2 Topological ordered groups

We shall start by showing that, for totally ordered groups as well as for their completion, the order (or interval) topology has "nice properties". For instance it is known that in that topology compact subsets are Dedekind complete subsets, in particular, closed bounded intervals are compact (so complete) subsets in that topology. We will now prove that totally ordered groups are, in fact, ordered topological groups.

Proposition 2.1 *Every quasidiscrete totally ordered group provided with the order topology is metrizable.*

Proof.
It is enough to prove that the order topology in a quasidiscrete group is precisely the discrete topology. Note first each element in a quasidiscrete group has an immediate successor and predecessor. Indeed, let $g_0 = \min\{g \in G : g > 1\}$ and let $g \in G$, then ${gg_0}^{-1} < g < gg_0$. Now, if there were $g' \in G$ such that $g < g' < gg_0$, then we would have that $1 < g'g^{-1} < g_0$, a contradiction. Hence, ${gg_0}^{-1}$ and gg_0 are the predecessor and successor of g, respectively. Therefore the open interval $({gg_0}^{-1}, gg_0)$ is a singleton, the order topology is the discrete topology and the group has the trivial metric. ∎

For quasidense groups, we have two previous lemmas. It is clear that singletons are closed in the order topology, the next lemma shows that they are not open.

Lemma 2.1 *If G is a quasidense totally ordered group, then each open interval contains an infinite number of elements of G, hence, singleton are not open in quasidense groups.*

Proof.
If $a < b$, there exists $h > 1$ such that $ah = b$. The quasidensity of G implies that there are infinitely many elements k such that $1 < k < h$. Since $a < ak < b$, this also the case for the interval (a, b).

Lemma 2.2 *If G is a quasidense totally ordered group, then for each $h > 1$ in G, there exists $l \in G$ such that $1 < l^2 < h$.*

Proof.
Let $x \in G$ with $1 < x < h$. Then there exists $y \in G$ such that $xy = h$. It is clear that $y > 1$, (if not $h \leq x$), therefore the statement is true for any $l \in G$ with $1 < l < \min\{x, y\}$, since $l^2 < ly < xy = h$. ∎

Proposition 2.2 *Every totally ordered group endowed with the order topology is a topological group.*

Proof.
Let $(G, \cdot)$ be a non trivial totally ordered group. By [3] p.105, it is enough to prove that for each $x, y \in G$, and for any open interval W containing xy^{-1}, there are neighbourhoods U of x and V of y such that $UV^{-1} \subseteq W$.
It is enough to suppose G is quasidense. The "open intervals" (a, b) with $a, b \in G$ and $a < b$ are the basic open subsets in the order topology. Let $x, y \in G$ and let $W = (a, b)$ a neighbourhood of xy^{-1}. Then $a < xy^{-1} < b$ and, since G is a group, there are two elements $h_1, h_2 \in G$ greater than 1_G such that $ah_1 = xy^{-1}$ y $xy^{-1}h_2 = b$.
Let $l \in G$ such that $1_G < l^2 < \min\{h_1, h_2\}$ (Lemma 2.2) and let $U = (l^{-1}x, lx)$ and $V = (l^{-1}y, ly)$ be neighbourhoods of x and y respectively.

Let $z \in UV^{-1}$. Then, there are $u \in U$ and $v \in V$ such that $z = uv^{-1}$. We have

$$\begin{array}{rcccccc} & & l^{-1}x & < & u & < & lx \\ & & l^{-1}y^{-1} & < & v^{-1} & < & ly^{-1} \\ \hline a = {h_1}^{-1}xy^{-1} & < & l^{-2}xy^{-1} & < & uv^{-1} & < & l^2xy^{-1} & < h_2xy^{-1} = b \end{array}$$

Hence, $UV^{-1} \subseteq W$ and we are done. ∎

Remark. From the fact that a topological group is metrizable if and only if it is first countable (see [1]), it follows immediately

1. Each Γ_α is metrizable because the identity 1_{Γ_α} has a countable base of neighbourhoods and therefore Γ_α is first countable.

2. For an arbitrary totally ordered set I, Γ_I is metrizable if and only if either I has a least element 0 or I has a coinitial sequence.

3. The condition given in [4], Proposition 1.4.4 (δ) implies that G is metrizable, but the converse is not true. For example, Γ_{ω_1} is metrizable but it has not a cofinal sequence (see [6] Proposition 2.4).

4. There exist totally ordered groups which are non metrizable as the following example shows.

Example 2.3 Let α be an ordinal, we write $\overleftarrow{\alpha}$ to represent α with the dual order. Then the rank of the group $\Gamma_{\overleftarrow{\alpha}}$ is the inverse order-type of Γ_α. Therefore if $\mathrm{cof}(\alpha) > \omega_0$, the group $\Gamma_{\overleftarrow{\alpha}}$ is nonmetrizable, because $1_{\Gamma_{\overleftarrow{\alpha}}}$ does not have a countable base of neighbourhoods.

3 Topological G-modules

We say that a G-module X is a topological G-module if the action of G over X is a continuous map $G \times X \to X$, where G and X are provided with the order topology. We now will show that there does not exist any extension of the multiplication of G that is order compatible and such that $G^\#$ is a topological semigroup. Moreover, we will prove that a G-module X is a topological G-module if and only if it is a continuous G-module (that is for each $W \subseteq G$ such that $\inf_G W$ exists, $(\inf_G W)r = \inf_X(Wr)$ for all $r \in X$, see [5] section 1.6).

Proposition 3.1 *No extension of the multiplication of G to $G^\#$ that is order compatible can be continuous in the order topology.*

Proof.
Let $\diamond : G^\# \times G^\# \to G^\#$ a extension of the multiplication from G to $G^\#$ and let s and t be respectively, the supremum and infimum, of some nontrivial convex subgroup H. Let U and V be neighbourhoods of s and t respectively. Then the set $U \diamond V = \{x \diamond y \in G^\# : x \in U \wedge y \in V\}$ contains some elements which are less than t and elements which are greater than s, therefore there always exists a neighbourhood of $s \diamond t$ that does not contain the set $U \diamond V$. ∎

Proposition 3.2 *A totally ordered set X is a topological G-module if and only if X is a continuous G-module. In particular, $G^\#$ is a topological G-module.*

Proof.
By [5] Proposition 1.6.4, if G is quasidiscrete, each G-module X is continuous and by Proposition 2.1 above the order topology on a quasidiscrete group is discrete, hence the action of G over X is a continuous operation. Therefore let us suppose now that G is quasidense.
($\Rightarrow$) We will prove that if X is not a continuous G-module, then the action of G over X is not continuous. Indeed, if X is not continuous, by [5] Proposition 1.6.8, there exists an element $r \in X$ such that $\inf_X\{gr : g \in G \wedge g > 1\} := r' > r$. Hence, for every neighbourhood (u, r') of $1r = r$ there exists $g > 1$ such that $gr \geq r'$ which implies the action is not continuous.
($\Leftarrow$) We will prove that if X is a continuous G-module, then the action of G over X is a continuous operation. Let (a, b) be a neighbourhood of an element gr of X. Then $g^{-1}a < r < g^{-1}b$. Since X is (left and right at each $r \in X$) continuous, there exists $h > 1$ in G such that $g^{-1}a < h^{-1}r < r < hr < g^{-1}b$. By lemma 2.2, there exists $l \in G$ such that $1 < l^2 < h$. Then,

$$\begin{array}{ccccccc} & l^{-1}r & < & r & < & lr & \\ & l^{-1}g & < & g & < & lg & \\ \hline a < h^{-1}gr < & l^{-2}gr & < & gr & < & l^2gr & < hgr < b \end{array}$$

and we are done. ∎

4 Topological properties of $\Gamma_\alpha^\#$.

In this section we study the properties of separability and second countability of the Dedekind completion of totally ordered groups.
Recall the convex hull of a subset $A \subseteq G$ is $\mathrm{conv}_{G^\#}A = \{x \in G^\# : \exists\, g_1, g_2 \in A\, (g_1 \leq x \leq g_2)\}$.

Proposition 4.1 *For any totally ordered group G, the completion $G^\#$ is separable if and only if G is separable.*

Proof.
By construction, each element of the completion is the supremum of a set of elements of G, hence open intervals of $G^\#$ are the union of a chain of open intervals with extremes in G. Therefore, if A is a dense and countable subset of G and U is an open set of $G^\#$, then $U \cap G$ is open in G, hence $U \cap A \neq \emptyset$.
On the other hand, let $A = \{a_i\}_{i<\omega_0}$ be a dense and countable subset of $G^\#$. By density of G in $G^\#$, $a_i < a_j$ implies there exists an element $g_{ij} \in G$ such that $a_i \leq g_{ij} < a_j$. If $a_i \in G$, we choose $g_{ij} = a_i$ for each j. We claim the family $\{g_{ij}\}$ is a dense and countable subset of G. Let (a, b) be a nonempty open interval G. Then $\mathrm{conv}_{G^\#}(a, b) \cap A \neq \emptyset$ and the only problem could be if this set is a singleton, $\{a_i\}$. But in that case, $\mathrm{conv}_{G^\#}(a, a_i)$ is clearly empty, that is to say $a_i \in G$. Hence the family $\{g_{ij}\}$ intersects each open interval of G. ∎

The next theorem give us a family of examples of metrizable groups which are not separable.

Theorem 4.1 *If G_β is uncountable for some $\beta > 0$, then Γ_α is not separable and hence it is not second countable.*

Proof.
The open intervals $(f^{-1}\chi_{(\beta,r_1)}, f\chi_{(\beta,r_1)})$ and $(f^{-1}\chi_{(\beta,r_2)}, f\chi_{(\beta,r_2)})$ are disjoint for all $f > 1 \in \Gamma_\alpha$, with $\deg(f) < \beta$ and $r_1 \neq r_2$ in G_β. Therefore we have a non-countable family of disjoint open sets. In fact, if $r_1 < r_2$ then $g < h$ whenever $g \in (f^{-1}\chi_{(\beta,r_1)}, f\chi_{(\beta,r_1)})$ and $h \in (f^{-1}\chi_{(\beta,r_2)}, f\chi_{(\beta,r_2)})$. Hence, Γ_α does not contain a subset both countable and dense. ∎

Proposition 4.2 *$\Gamma_\alpha^{\#}$ is second countable (and separable) if and only if Γ_α is countable.*

Proof.
($\Rightarrow$) We will prove Γ_α is not countable, then $\Gamma_\alpha^{\#}$ is not second countable. By the above Theorem, we only have to consider the case when α is not countable. In that case $\{s_\beta, s_{\beta+1}\}_{\beta<\alpha}$ is an uncountable family of disjoint open sets, hence $\Gamma_\alpha^{\#}$ cannot have a countable base of open sets.

($\Leftarrow$) Suppose that Γ_α is countable. Then α is a countable ordinal, each group G_β is countable and also the rank of Γ_α is countable. Therefore S_α, the set of the suprema and infima of convex subgroups is countable. We claim that a base of the order topology of $\Gamma_\alpha^{\#}$ is $\mathcal{B} := \{(fu, f'u') : f, f' \in \Gamma_\alpha \ \wedge \ u, u' \in S_\alpha\}$.
It is clear $\mathcal{B}$ is a countable set. By Theorem 1.1 (see [7] for details), the only elements of $\Gamma_\alpha^{\#}$ which are not of the form fu with $f \in \Gamma_\alpha, u \in S_\alpha$ appear when some G_β is a countable non complete group (of rank 1).
Consider an interval which has one or both of its extremes of the form $f\chi_{(\beta,r_\beta)}t_\beta{}^*$ with $f \in \Gamma_\alpha$ and $r_\beta \in G'_\beta$, the completion of G_β. It is easy to check that such an interval contains an interval $(f\chi_{(\beta,q_\beta)}t_\beta{}^*, f'\chi_{(\beta,q'_\beta)}t_\beta{}^*)$, with $q_\beta,\ q'_\beta \in G_\beta$ as we claimed.
So, if the interval $(f\chi_{(\beta,r_\beta)}t_\beta{}^*, f'\chi_{(\beta,r'_\beta)}t_\beta{}^*)$ is non empty, then $f\chi_{(\beta,r_\beta)}t_\beta{}^* < f'\chi_{(\beta,r'_\beta)}t_\beta{}^*$ and there exists $\gamma \leq \beta$ such that $f(\gamma)\chi_{(\beta,r_\beta)} < f'(\gamma)\chi_{(\beta,r'_\beta)}$. The following two cases are the only ones for consideration.

- $\gamma > \beta$. Then $(f\chi_{(\beta,r_\beta)}t_\beta{}^*, f'\chi_{(\beta,r'_\beta)}t_\beta{}^*) \supseteq (f\chi_{(\beta,q_\beta)}t_\beta{}^*, f'\chi_{(\beta,q'_\beta)}t_\beta{}^*)$ with $r_\beta < q_\beta < q'_\beta < r'_\beta$, con $q_\beta,\ q'_\beta \in G_\beta$.

- $\gamma = \beta$. Then we choose $q_\beta,\ q'_\beta \in G_\beta$ such that $q_\beta > r_\beta$ and $q'_\beta < r'_\beta$ with $f\chi_{(\beta,r_\beta)} < f\chi_{(\beta,q_\beta)} < f\chi_{(\beta,q'_\beta)} < f\chi_{(\beta,r'_\beta)}$.

∎

5 Metrizability of $\Gamma_\alpha^\#$

In this section we show that condition (δ) of the Proposition 1.4.4 of [4] is not a guarantee for the metrizability of the completion.

Proposition 5.1 *If Γ_α is countable, then $\Gamma_\alpha^\#$ is metrizable.*

Proof. It is easy to see that $\Gamma_\alpha^\#$ is T_1 , regular and second countable, then by [3] Theorem 17 p.125, it is metrizable and separable. ∎

Proposition 5.2 *If $\alpha < \omega_1$, and G_β is uncountable for some $\beta < \alpha$, then $\Gamma_\alpha^\#$ is nonmetrizable.*

Proof.
Recall that in metric spaces the concepts of second countability, separability and Lindelöf are equivalent (see for example [2] Theorem 5.6 p. 187). Therefore, by Proposition 4.2, $\Gamma_\alpha^\#$ is not second countable.
To prove that $\Gamma_\alpha^\#$ is Lindelöf we will use Theorem 7.2, p.241 of [2], which establish a space Y is a countable union of relatively compacts open sets U_i such that $\overline{U}_i \subset U_{i+1}$ if and only if Y is a Lindelöf locally compact space.
In fact, note that ${\Gamma_\alpha}^\#$ is a countable union of a chain of relatively compact open subsets $\Gamma_\alpha^\# = \bigcup_{\beta<\alpha} \operatorname{conv}_{\Gamma_\alpha^\#} H_\beta$, where $\overline{\operatorname{conv}_{\Gamma_\alpha^\#} H_\beta} = [t_\beta, s_\beta] \subseteq (t_{\beta+1}, s_{\beta+1})$ for each $\beta < \alpha$ is compact. Hence $\Gamma_\alpha^\#$ is a Lindelöf space and the assertion is proved. ∎

Remark. It is easy to construct examples of groups with non metrizable completions. For example, for a group Γ_{ω_0}, if one of G_β is isomorphic to the infinite interval $(0, \infty)$ its completion $\Gamma_{\omega_0}^\#$ is non metrizable. For higher cardinalities we always have non metrizability. If $\alpha > \omega_1$ that is clear, because the group Γ_α is not first countable, for instance the element s_{ω_1} has not a countable base of neighbourhoods (see [6]).

Proposition 5.3 *If $\alpha = \omega_1$, then $\Gamma_\alpha^\#$ is non metrizable.*

Proof.
By [2] Theorem 4.1, p.233, a countably compact space is metrizable if and only if it is second countable. By Proposition 4.2, $\Gamma_{\omega_1}^\#$ is not second countable, then it is enough to prove that every open countable cover has a finite subcover.
Suppose that $\Gamma_{\omega_1}^\#$ is not countably compact and let $\{U_n\}_{n<\omega_0}$ be a countably open cover of $\Gamma_{\omega_1}^\#$ which have not a finite subcover. Then, for every $n < \omega_0$, there exists $x_n \in \Gamma_{\omega_1}^\#$ such that $x_n \notin \bigcup_{i=0}^{n} U_i$. Because $\{x_n\}_{n<\omega_0}$ is countable, it is bounded and there exists $\beta_0 < \omega_0$ such that $\{x_n\}_{n<\omega_0} \subseteq [t_{\beta_0}, s_{\beta_0}]$. But $[t_{\beta_0}, s_{\beta_0}]$ is compact and $\{U_n\}_{n<\omega_0}$ is a cover of $[t_{\beta_0}, s_{\beta_0}]$ without finite subcover, a contradiction. ∎

Therefore we have obtained a complete answer to the question posed by Ochsenius and Schikhof in [4] after the Proposition 1.4.4.

Theorem 5.1 *$\Gamma_\alpha^\#$ is metrizable if and only if Γ_α is countable.*

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