Evolution Equations and Functions of Hypergeometric Type over Fields of Positive Characteristic

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Abstract

We consider a class of partial differential equations with Carlitz derivatives over a local field of positive characteristic, for which an analog of the Cauchy problem is well-posed. Equations of such type correspond to quasi-holonomic modules over the ring of differential operators with Carlitz derivatives. The above class of equations includes some equations of hypergeometric type. Building on the work of Thakur, we develop his notion of the hypergeometric function of the first kind (whose parameters belonged initially to $\mathbb{Z}$) in such a way that it becomes fully an object of the function field arithmetic, with the variable, parameters and values from the field of positive characteristic.

1 Introduction

Let K be the field of formal Laurent series $t = \sum\limits_{j=N}^{\infty} \xi_j x^j$ with coefficients ξ_j from the Galois field $\mathbb{F}_q$, $\xi_N \neq 0$ if $t \neq 0$, $q = p^{\upsilon}$, $\upsilon \in \mathbf{Z}_+$, where p is a prime number. It is well known that any non-discrete locally compact field of characteristic p is isomorphic to such K. The absolute value on K is given by $|t| = q^{-N}$, $|0| = 0$. This absolute value can be extended in a unique way onto the completion $\overline{K}_c$ of an algebraic closure of K. See [18] for a detailed description of the extension procedure (valid for any non-Archimedean valued field).

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An important feature of analysis over K and $\overline{K}_c$ initiated by Carlitz [3] and developed by Wagner, Goss, Thakur, the author, and many others (see surveys in [7, 21, 16]) is the availability of many non-trivial $\mathbb{F}_q$-linear functions, that is such functions f defined on $\mathbb{F}_q$-subspaces $K_0 \subset K$, with values from $\overline{K}_c$, that

$$f(t_1+t_2)=f(t_1)+f(t_2), \quad f(\alpha t)=\alpha f(t),$$

for any $t,t_1,t_2 \in K_0$, $\alpha \in \mathbb{F}_q$. Such are, for example, polynomials and power series of the form $\sum a_k t^{q^k}$. Within this class, there are analogs of the exponential and logarithm, the Bessel and hypergeometric functions, the polylogarithms and zeta function.

The basic ingredients of the calculus of $\mathbb{F}_q$-linear functions [9] are the Frobenius operator $\tau u = u^q$, the difference operator

$$\Delta u(t) = u(xt) - xu(t) \tag{1.1}$$

introduced in [3], and the nonlinear ($\mathbb{F}_q$-linear) operator $d=\tau^{-1}\Delta$ called *the Carlitz derivative*. The latter appears, as an analog of the classical derivative, in the theory of ordinary differential equations for $\mathbb{F}_q$-linear functions, which has been developed both in the traditional analytic direction (the Cauchy problem [10, 13], regular singularity [11], new special functions defined via differential equations [14]) and within various algebraic frameworks (analogs of the canonical commutation relations of quantum mechanics [8, 9], umbral calculus [12], an analog of the Weyl algebra [10, 15, 2]). Note that in our situation the meaning of a polynomial coefficient in a differential equation is not a usual multiplication by a polynomial, but the action of a polynomial in the operator τ (similarly, for holomorphic coefficients). Thus, the operator $\Delta = \tau d$ can be seen as the counterpart of $t\dfrac{d}{dt}$.

As an example, consider Thakur's hypergeometric function [19, 20, 21] (in this paper we deal only with what Thakur calls "the first analog of the hypergeometric function for a finite place of $\mathbb{F}_q(x)$"). For $n \in \mathbb{Z}_+$, $\alpha \in \mathbb{Z}$, denote $D_n = [n][n-1]^q \cdots [1]^{q^{n-1}}$, $[n]=x^{q^n}-x$, $L_n=[n][n-1]\cdots[1]$ ($n \ge 1$), $D_0=L_0=1$,

$$(\alpha)_n = \begin{cases} D_{n+\alpha-1}^{q^{-(\alpha-1)}}, & \text{if } \alpha \ge 1; \\ (-1)^{n-\alpha} L_{-\alpha-n}^{-q^n}, & \text{if } \alpha \le 0, n \le -\alpha; \\ 0, & \text{if } \alpha \le 0, n > -\alpha. \end{cases} \tag{1.2}$$

For $\alpha_i, \beta_i \in \mathbb{Z}$, such that the series below makes sense, Thakur's hypergeometric function is defined as

$${}_rF_s(\alpha_1,\dots,\alpha_r;\beta_1,\dots,\beta_s;z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n D_n} z^{q^n} \tag{1.3}$$

Thakur has carried out a thorough investigation of the function (1.3) and obtained analogs of many classical properties. In particular, he found a differential equation for the function (1.3) (see [19]). Using the commutation relation $d\tau - \tau d = [1]^{1/q} I$, where I is the identity operator, we can write the equation from [19] in the form

$$\left\{ \prod_{i=1}^{r} (\Delta - [-\alpha_i]) - \left(\prod_{j=1}^{s} (\Delta - [-\beta_j]) \right) d \right\} {}_rF_s = 0 \tag{1.4}$$

convenient for us in a sequel. For $r=2, s=1$, this can be written in a familiar-looking form: the function $u = {}_2F_1(\alpha,\beta;\gamma;z)$, $\alpha,\beta,\gamma\in\mathbb{Z}$, satisfies the equation

$$\tau(1-\tau)d^2u + \{([-1]^q + [-\alpha] + [-\beta])\,\tau - [-\gamma]\}\,du - [-\alpha][-\beta]u = 0. \tag{1.5}$$

It must be remembered that everywhere in this theory we have to deal with nonlinear equations: the operators τ and d are only $\mathbb{F}_q$-linear, not K-, nor $\overline{K}_c$-linear.

A natural next step in developing analysis over K is to try to consider partial differential equations with Carlitz derivatives. However, here we encounter a serious difficulty noticed in [15]: the Carlitz derivatives with respect to different variables do not commute. Considering the Carlitz rings of "differential operators" in the above sense in [15], the author found a class of partial differential operators (acting on an appropriate class of functions of several variables), which nevertheless possess reasonable properties. Operators from this class contain the derivative d in only one distinguished variable, and the linear operator Δ in every other variable. Such operators d and Δ (in different variables) do not commute either but satisfy a simple commutation relation.

In this paper we pursue this idea further, introducing a class of partial differential equations of the above type, for which an analog of the Cauchy problem (the initial value problem for a differential equation) with respect to the distinguished variable is well-posed. In this regard, such equations may be seen as function field analogs of classical evolution equations of mathematical physics. On the other hand, it is easy to notice an analogy between our new equations and general hypergeometric equations of classical analysis (see, for example, [1, 5]; note also a different non-Archimedean generalization of the hypergeometric function [6] where the functions, defined on a non-Archimedean field, are complex-valued). In this wider framework, it becomes clear how to extend the definition (1.3) of Thakur's hypergeometric function to the case of the parameters from $\overline{K}_c$, so that the resulting function is locally analytic with respect to the parameters and coincides with Thakur's one (up to a change of variable), as the parameters are of the form $[-n]$, $n\in\mathbb{N}$. Thus, now Thakur's hypergeometric function becomes fully an object of the function field arithmetic, with the variable, parameters, and values belonging to $\overline{K}_c$.

The structure of the paper is as follows. In Section 2, we prove the well-posedness of the Cauchy problem for the above class of partial differential equations. We show (Section 3) that these equations generate quasi-holonomic modules (in the sense of [15]). Section 4 is devoted to the general hypergeometric function. Analogs of the contiguous relations for the latter are given in Section 5.

2 Cauchy Problem

Denote by $\mathcal{F}_{n+1}$ $(n\ge 1)$ the set of all germs of functions of the form

$$u = \sum_{i_1=0}^{\infty}\dots\sum_{i_n=0}^{\infty}\sum_{m=0}^{\min(i_1,\dots,i_n)} c_{m,i_1,\dots,i_n} s_1^{q^{i_1}}\dots s_n^{q^{i_n}}\frac{z^{q^m}}{D_m} \tag{2.1}$$

where $c_{m,i_1,\dots,i_n}\in\overline{K}_c$ are such that all the series are convergent on some neighbourhoods of the origin. Let $\hat{\mathcal{F}}_{n+1}$ be the set of polynomials from $\mathcal{F}_{n+1}$.

Below d will denote the Carlitz derivative in the variable z, while Δ_j will mean the difference operator (1.1) in the variable s_j. In the action of each operator d, Δ_j on a function from $\mathcal{F}_{n+1}$ (acting in a single variable) other variables are treated as scalars. Obviously, linear operators Δ_j commute with multiplications by scalars: $\Delta_j \lambda = \lambda \Delta_j$, $\lambda \in \overline{K}_c$, while $d\lambda = \lambda^{1/q} d$. We have also the following commutation relations:

$$d\tau - \tau d = [1]^{1/q} I, \quad d\Delta_j - \Delta_j d = [1]^{1/q} d, \quad \Delta_j \tau - \tau \Delta_j = [1]\tau \quad (j = 1, \dots, n). \tag{2.2}$$

The Carlitz ring $\mathfrak{A}_{n+1}$ (see [15]) is generated by τ, d, Δ_j $(j = 1, \dots, n)$ and scalars $\lambda \in \overline{K}_c$, and the action of any operator from $\mathfrak{A}_{n+1}$ on $\mathcal{F}_{n+1}$ is well-defined.

Let us consider equations of the form

$$\{P(\Delta_1, \dots, \Delta_n) + Q(\Delta_1, \dots, \Delta_n) d\} \, u = 0 \tag{2.3}$$

where P, Q are non-zero polynomials with coefficients from $\overline{K}_c$. We look for a solution $u \in \mathcal{F}_{n+1}$ of the form (2.1) satisfying an “initial condition”

$$\lim_{z \to 0} z^{-1} u(z, s_1, \dots, s_n) = u_0(s_1, \dots, s_n) \tag{2.4}$$

where $u_0(s_1, \dots, s_n)$ is an $\mathbb{F}_q$-linear holomorphic function on a neighbourhood of the origin. The condition (2.4) (similar to the initial conditions for “ordinary” differential equations with Carlitz derivatives [10]) means actually that the coefficients $c_{0,i_1,\dots,i_n}$ of the solution (2.1) are prescribed for any $i_1, \dots, i_n$.

Below we use the notation $[\infty] = -x$. Then $[n] \to [\infty]$, as $n \to \infty$.

Theorem 1. *Suppose that*

$$Q([i_1], \dots, [i_n]) \neq 0 \quad \text{for all } i_1, \dots, i_n = 0, 1, \dots, \infty. \tag{2.5}$$

Then the Cauchy problem (2.3)-(2.4) has a unique solution $u \in \mathcal{F}_{n+1}$.

Proof. It is easy to see that

$$\Delta_j \left(s^{q^{i_j}} \right) = \begin{cases} [i_j] s^{q^{i_j}}, & \text{if } i_j \neq 0; \\ 0, & \text{if } i_j = 0. \end{cases}$$

The identity $D_m = [m] D_{m-1}^q$ implies the relation

$$d \left(\frac{z^{q^m}}{D_m} \right) = \begin{cases} \frac{z^{q^{m-1}}}{D_{m-1}}, & \text{if } m \neq 0; \\ 0, & \text{if } m = 0. \end{cases}$$

Therefore for a function (2.1) we get

$$\begin{aligned} du &= \sum_{i_1=1}^{\infty} \dots \sum_{i_n=1}^{\infty} \sum_{m=1}^{\min(i_1,\dots,i_n)} c_{m,i_1,\dots,i_n}^{1/q} s_1^{q^{i_1-1}} \dots s_n^{q^{i_n-1}} \frac{z^{q^{m-1}}}{D_{m-1}} \\ &= \sum_{j_1=0}^{\infty} \dots \sum_{j_n=0}^{\infty} \sum_{\nu=0}^{\min(j_1,\dots,j_n)} c_{\nu+1,j_1+1,\dots,j_n+1}^{1/q} s_1^{q^{j_1}} \dots s_n^{q^{j_n}} \frac{z^{q^\nu}}{D_\nu}. \end{aligned}$$

Next,

$$P(\Delta_1,\dots,\Delta_n)u=\sum_{i_1=0}^{\infty}\dots\sum_{i_n=0}^{\infty}\sum_{m=0}^{\min(i_1,\dots,i_n)}c_{m,i_1,\dots,i_n}P([i_1],\dots,[i_n])s_1^{q^{i_1}}\dots s_n^{q^{i_n}}\frac{z^{q^m}}{D_m}.$$

Writing a similar formula for $Q(\Delta_1,\dots,\Delta_n)u$ and substituting all this into (2.3) we find that

$$\sum_{i_1=0}^{\infty}\dots\sum_{i_n=0}^{\infty}\sum_{m=0}^{\min(i_1,\dots,i_n)}\Big\{c_{m,i_1,\dots,i_n}P([i_1],\dots,[i_n]) + c^{1/q}_{m+1,i_1+1,\dots,i_n+1}Q([i_1],\dots,[i_n])\Big\}s_1^{q^{i_1}}\dots s_n^{q^{i_n}}\frac{z^{q^m}}{D_m}=0$$

for arbitrary values of the variables. Hence, we come to the recursion

$$c_{m+1,i_1+1,\dots,i_n+1}=-c^q_{m,i_1,\dots,i_n}\left\{\frac{P([i_1],\dots,[i_n])}{Q([i_1],\dots,[i_n])}\right\}^q,\quad m\le\min(i_1,\dots,i_n). \tag{2.6}$$

Since all the elements $c_{0,i_1,\dots,i_n}$ $(i_1,\dots,i_n=0,1,2,\dots)$ are given, from (2.6) we find all the coefficients of (2.1).

The set $\{[i],\ i=0,1,\dots,\infty\}$ is compact in K. Therefore the condition (2.5) implies the inequality

$$|Q([i_1],\dots,[i_n])|\ge\mu>0 \tag{2.7}$$

for all $i_1,\dots,i_n=0,1,2,\dots,\infty$. Note also that $\|[i]\|=q^{-1}$ for any i, and

$$|c_{0,i_1,\dots,i_n}|\le Cr^{q^{i_1}+\dots+q^{i_n}} \tag{2.8}$$

for some positive constants C and r, since the series for the initial condition

$$u_0(s_1,\dots,s_n)=\sum_{i_1=0}^{\infty}\dots\sum_{i_n=0}^{\infty}c_{0,i_1,\dots,i_n}s_1^{q^{i_1}}\dots s_n^{q^{i_n}}$$

converges near the origin.

By (2.7) and (2.8),

$$|c_{1,i_1+1,\dots,i_n+1}|\le C_1^q r^{q^{i_1+1}+\dots+q^{i_n+1}}$$

(where $C_1>0$ does not depend on $i_1,\dots,i_n$), thus

$$|c_{2,i_1+2,\dots,i_n+2}|\le C_1^{q^2+q}r^{q^{i_1+2}+\dots+q^{i_n+2}},$$

and, by induction

$$|c_{l,i_1+l,\dots,i_n+l}|\le C_1^{q^l+q^{l-1}+\dots+q}r^{q^{i_1+l}+\dots+q^{i_n+l}}$$

for any $l\ge0$. This means that for any $j_1,\dots,j_n\ge l$,

$$|c_{l,j_1,\dots,j_n}|\le C_2^{q^l}r^{q^{j_1}+\dots+q^{j_n}},\quad C_2>0,$$

which, together with the equality

$$|D_m| = q^{-\frac{q^m-1}{q-1}},$$

implies the convergence of the series in (2.1) near the origin. ■

Remark. It is easy to generalize Theorem 1 to the case of systems of equations, where P and Q are matrices whose elements are polynomials of $\Delta_1, \dots, \Delta_n$. In this case, instead of (2.5) we have to require the invertibility of $Q([i_1], \dots, [i_n])$ for all $i_1, \dots, i_n = 0, 1, \dots, \infty$. In an obvious way, this generalization covers also the case of a scalar equation of a higher order in d.

Below we will consider in detail some specific examples of the equation (2.3). However, before that we consider a holonomic property of such equations, valid in the most general situation, even without the assumption (2.5).

3 Quasi-Holonomic Modules

The class $\mathcal{F}_{n+1}$ of functions, among which we looked for a solution of (2.3), is basic in the function field counterpart [15] of the theory of holonomic modules [4]. Let us recall its principal notions.

Every operator $a \in \mathfrak{A}_{n+1}$ can be written in a unique way as a finite sum

$$A = \sum a_{l,\mu,i_1,\dots,i_n} \tau^l d^\mu \Delta_1^{i_1} \dots \Delta_n^{i_n}, \quad a_{l,\mu,i_1,\dots,i_n} \in \overline{K}_c. \tag{3.1}$$

We introduce a filtration in $\mathfrak{A}_{n+1}$ denoting by Γ_ν, $\nu \in \mathbb{Z}_+$, the $\overline{K}_c$-vector space of operators (3.1) with $\max\{l + \mu + i_1 + \dots + i_n\} \le \nu$ where the maximum is taken over all the non-zero terms contained in (3.1). Then $\mathfrak{A}_{n+1}$ becomes a filtered ring.

Suppose M is a left module over $\mathfrak{A}_{n+1}$ with a filtration $\{\mathfrak{M}_j\}$, that is

$$\mathfrak{M}_0 \subset \mathfrak{M}_1 \subset \dots \subset M, \quad M = \bigcup_{j \ge 0} \mathfrak{M}_j,$$

each $\mathfrak{M}_j$ is a finite-dimensional vector space over $\overline{K}_c$, and $\Gamma_\nu \mathfrak{M}_j \subset \mathfrak{M}_{\nu+j}$ for any $\nu, j \in \mathbb{Z}_+$. The filtration is called good if the corresponding graded module is finitely generated.

For a good filtration there exist a polynomial $\chi \in \mathbb{Q}[t]$ and a number $N \in \mathbb{N}$, such that

$$\dim \mathfrak{M}_s = \sum_{i=0}^{s} \dim(\mathfrak{M}_i/\mathfrak{M}_{i-1}) = \chi(s) \text{ for } s \ge N$$

(dim means the dimension over $\overline{K}_c$). The number $d(M) = \deg \chi$, called the (Gelfand-Kirillov or Bernstein) *dimension* of M, does not depend on the choice of a good filtration. In general, $d(M) \le n + 2$. In some cases $d(M)$ can be evaluated explicitly. For example, $d(\hat{\mathcal{F}}_{n+1}) = n + 1$, while $d(\mathfrak{A}_{n+1}) = n + 2$, if $\mathfrak{A}_{n+1}$ is considered as a left module over itself. In contrast to the classical Bernstein theory (see [4]), in the positive characteristic situation $d(M)$ can be arbitrarily small [15]. A module M is called *quasi-holonomic* if $d(M) = n + 1$ (in the survey papers [16, 17] the term "holonomic" is used). As in the classical case, modules associated with basic special functions on K possess this property [15]. In general, the holonomic property is a kind of a "quality certificate" to distinguish objects (equations, functions etc) with a reasonable behavior.

If I is a non-zero left ideal in $\mathfrak{A}_{n+1}$, then the left $\mathfrak{A}_{n+1}$-module $\mathfrak{A}_{n+1}/I$ can be endowed with a good filtration, and

$$d\left(\mathfrak{A}_{n+1}/I\right) \leq n+1. \tag{3.2}$$

Returning to the equation (2.3) denote by R the operator in the left-hand side:

$$R = P(\Delta_1, \dots, \Delta_n) + Q(\Delta_1, \dots, \Delta_n)d$$

where P, Q are non-zero polynomials. Let $I = \mathfrak{A}_{n+1}R$.

Theorem 2. *The module $M = \mathfrak{A}_{n+1}/I$ is quasi-holonomic.*

Proof. Due to (3.2), we have to show only that $d(M) \geq n+1$. First we prove two lemmas.

Lemma 1. *An operator (3.1) is linear if and only if $a_{l,\mu,i_1,\dots,i_n} = 0$ for $l \neq \mu$.*

Proof. Let $\sigma \in \overline{K}_c$. Suppose that $A\sigma = \sigma A$, that is

$$\sum \sigma a_{l,\mu,i_1,\dots,i_n} \tau^l d^\mu \Delta_1^{i_1} \dots \Delta_n^{i_n} = \sum a_{l,\mu,i_1,\dots,i_n} \sigma^{q^{l-\mu}} \tau^l d^\mu \Delta_1^{i_1} \dots \Delta_n^{i_n}.$$

By the uniqueness of the representation (3.1) [15], we find that $\sigma^{q^{l-\mu}} = \sigma$, whenever $a_{l,\mu,i_1,\dots,i_n} \neq 0$. Since σ is arbitrary, that is possible if and only if $l = \mu$. ■

Lemma 2. *The ideal I does not contain non-zero linear operators.*

Proof. Using the commutation relations (2.2) we can rewrite the representation (3.1) of an arbitrary operator $A \in \mathfrak{A}_{n+1}$ in the form

$$A = \sum a'_{l,\mu,i_1,\dots,i_n} \Delta_1^{i_1} \dots \Delta_n^{i_n} \tau^l d^\mu, \quad a'_{l,\mu,i_1,\dots,i_n} \in \overline{K}_c. \tag{3.3}$$

Just as in (3.1), the coefficients $a'_{l,\mu,i_1,\dots,i_n}$ are determined in the unique way, and Lemma 1 remains valid for the representation (3.3).

Suppose that an operator $A \in \mathfrak{A}_{n+1}$ is such that AR is linear. Let us write (3.3) in the form

$$A = \sum_{l=0}^{N} \sum_{\mu=0}^{N} \alpha_{l\mu} \tau^l d^\mu$$

where $\alpha_{l\mu}$ are the appropriate elements of the commutative $\overline{K}_c$-algebra $\mathcal{D}$ (without zero divisors) generated by the linear operators $\Delta_1, \dots, \Delta_n$. We have also $R = \gamma + \delta d$, $\gamma, \delta \in \mathcal{D}$, $\gamma \neq 0$, $\delta \neq 0$. In this new notation,

$$AR = \left(\sum_{l=0}^{N} \sum_{\mu=0}^{N} \alpha_{l\mu} \tau^l d^\mu \right) (\gamma + \delta d).$$

As an element $\gamma \in \mathcal{D}$ is permuted with powers of τ and d, in additional terms (appearing in accordance with (2.2)) the powers of τ and d respectively are the same, while the degrees of elements from $\mathcal{D}$ (as polynomials of $\Delta_1, \ldots, \Delta_n$) decrease. Therefore

$$AR = \sum_{l=0}^{N} \sum_{\mu=0}^{N} \alpha_{l\mu} \gamma' \tau^l d^\mu + \sum_{l=0}^{N} \sum_{\mu=0}^{N} \alpha_{l\mu} \delta' \tau^l d^{\mu+1}$$

where $\gamma', \delta' \in \mathcal{D}$, $\gamma' \neq 0, \delta' \neq 0$, whence

$$AR = \sum_{l=0}^{N} \sum_{\nu=1}^{N} \left(\alpha_{l\nu}\gamma' + \alpha_{l,\nu-1}\delta'\right) \tau^l d^\nu + \sum_{l=0}^{N} \left(\alpha_{l0}\gamma'\tau^l + \alpha_{lN}\delta'\tau^l d^{N+1}\right).$$

By Lemma 1,

$$\alpha_{l0} = 0, \quad l = 1, \ldots, N; \tag{3.4}$$

$$\alpha_{lN} = 0, \quad l = 0, 1, \ldots, N. \tag{3.5}$$

Considering terms with $l < N, \nu = N$, we find that

$$\alpha_{lN}\gamma' + \alpha_{l,N-1}\delta' = 0,$$

and (3.5) yields $\alpha_{l,N-1} = 0$, $0 \leq l \leq N-1$. Repeating the reasoning we obtain that $\alpha_{l\nu} = 0$ for $l \leq \nu$.

On the other hand, for $l \geq 2, \nu = 1$ we get

$$\alpha_{l1}\gamma' + \alpha_{l0}\delta' = 0,$$

and, by (3.4), $\alpha_{l1} = 0$. Repeating we come to the conclusion that $\alpha_{l\nu} = 0$ for $l > \nu$, so that $A = 0$. ■

Proof of Theorem 2 (continued). Let us consider the induced filtration in M (see [4]). The subspace $\mathfrak{M}_\nu$ is generated by images in M of the elements (3.1) with $\max(l + \mu + i_1 + \cdots + i_n) \leq \nu$; those two elements whose difference belongs to I are identified. Let us consider elements with $l = \mu$.

Elements of the form $\tau^l d^l \Delta_1^{i_1} \cdots \Delta_n^{i_n}$ with different collections of parameters $(l, i_1, \ldots, i_n)$ are linearly independent in $\mathfrak{A}_{n+1}$. If some linear combination of their images equals zero in M, then the corresponding linear combination of the elements themselves must belong to I, which (by Lemma 2) is possible only if it is equal to zero. Therefore the images of the above elements are linearly independent, so that

$$\begin{aligned} \dim \mathfrak{M}_\nu \geq \operatorname{card} &\left\{(l, i_1, \ldots, i_n) \in \mathbb{Z}_+^{n+1} :\ 2l + i_1 + \cdots + i_n \leq \nu\right\} \\ &\geq \operatorname{card}\left\{(l, i_1, \ldots, i_n) \in \mathbb{Z}_+^{n+1} :\ l + i_1 + \cdots + i_n \leq [\nu/2]\right\} \end{aligned}$$

(in contrast to the rest of the paper, here $[\cdot]$ means the integer part of a real number).

Evaluating the number of non-negative integral solutions of the above inequality as in the proof of Lemma 1 from [15] we find that

$$\dim \mathfrak{M}_\nu \geq \binom{[\nu/2] + n + 1}{n+1} \geq c_1 [\nu/2]^{n+1} \geq c_2 \nu^{n+1}$$

for large values of ν (c_1, c_2 are positive constants independent of ν). Thus, $d(M) \geq n + 1$, as desired. ■

4 Hypergeometric Equations

Let $n \geq \max(r, s)$, $r, s \in \mathbb{N}$,

$$P(t_1, \ldots, t_n) = \prod_{i=1}^{r} (t_i - a_i),$$
$$Q(t_1, \ldots, t_n) = \prod_{j=1}^{s} (t_j - b_j),$$

where $a_i, b_j \in \overline{K}_c$, and the elements b_j do not coincide with any of the elements $[\nu]$, $\nu = 0, 1, \ldots, \infty$. Then the condition (2.5) is satisfied, and the Cauchy problem for the equation (2.3) is well-posed.

Let us specify the initial condition (in terms of prescribing the values of $c_{0,i_1,\ldots,i_n}$) as follows:

$$c_{0,0,\ldots,0} = 1, \quad c_{0,i_1,\ldots,i_n} = 0$$

for all other values of $i_1, \ldots, i_n$. Then $c_{m,i_1,\ldots,i_n} = 0$ for all sets of indices $(m, i_1, \ldots, i_n)$ except those with $m = i_1 = \ldots = i_n$. Denote $\sigma_m = c_{m,m,\ldots,m}$. By (2.6), we find that

$$\sigma_1 = \left\{ \frac{(-1)^r \prod\limits_{i=1}^{r} a_i}{(-1)^s \prod\limits_{j=1}^{s} b_j} \right\}^q,$$

$$\sigma_2 = \left\{ \frac{(-1)^r \prod\limits_{i=1}^{r} a_i}{(-1)^s \prod\limits_{j=1}^{s} b_j} \right\}^{q^2} \left\{ \frac{\prod\limits_{i=1}^{r} ([1] - a_i)}{\prod\limits_{j=1}^{s} ([1] - b_j)} \right\}^q,$$

and, by induction, after rearranging the factors we get

$$\sigma_m = \frac{\prod\limits_{i=1}^{r} ([0] - a_i)^{q^m} ([1] - a_i)^{q^{m-1}} \cdots ([m-1] - a_i)^q}{\prod\limits_{j=1}^{s} ([0] - b_j)^{q^m} ([1] - b_j)^{q^{m-1}} \cdots ([m-1] - b_j)^q}, \quad m = 1, 2, \ldots. \tag{4.1}$$

For $a \in \overline{K}_c$, denote $\langle a \rangle_0 = 1$,

$$\langle a \rangle_m = ([0] - a)^{q^m} ([1] - a)^{q^{m-1}} \cdots ([m-1] - a)^q, \quad m \geq 1 \tag{4.2}$$

(of course, $[0] = 0$, but we maintain the symbol $[0]$ to have an orderly notation). The Pochhammer type symbol $\langle \cdot \rangle_m$ satisfies the recurrence

$$\langle a \rangle_{m+1} = ([m] - a)^q \langle a \rangle_m^q. \tag{4.3}$$

If $a = [-\alpha]$, $\alpha \in \mathbb{Z}$, then

$$([m] - a)^q = \left(x^{q^m} - x^{q^{-\alpha}} \right)^q = \left(x^{q^{m+\alpha}} - x \right)^{q^{-(\alpha-1)}} = [m + \alpha]^{q^{-(\alpha-1)}},$$

so that in this case the recurrence (4.3) coincides with the recursive relation [19] for the Pochhammer-Thakur symbols (1.2). Our normalization $\langle a \rangle_0 = 1$ is different from (1.2) and resembles the classical one.

In the above situation, it follows from (4.1) that the solution $u(z; t_1, \dots, t_n)$ of the Cauchy problem (2.3)-(2.4) is given by the formula

$$u(z; t_1, \dots, t_n) = {}_rF_s(a_1, \dots, a_r; b_1, \dots, b_s; t_1 \cdots t_n z) \tag{4.4}$$

where ${}_rF_s$ is the new hypergeometric function

$${}_rF_s(a_1, \dots, a_r; b_1, \dots, b_s; z) = \sum_{m=0}^{\infty} \frac{\langle a_1\rangle_m \cdots \langle a_r\rangle_m}{\langle b_1\rangle_m \cdots \langle b_s\rangle_m} \frac{z^{q^m}}{D_m}. \tag{4.5}$$

The series (4.5) has a positive radius of convergence since $b_1, \dots, b_s$ do not coincide with any of the elements $[\nu]$, $\nu = 0, 1, \dots, \infty$ (such parameters will be called *admissible*).

If $a_i = [-\alpha_i]$, $b_j = [-\beta_j]$ $(i = 1, \dots, r;\ j = 1, \dots, s)$, $\alpha_i \in \mathbb{Z}$, $\beta_j \in \mathbb{N}$, then the function (4.5) coincides with Thakur's hypergeometric function (1.3) up to a change of variable $z \Rightarrow \rho z$ where ρ depends on all the parameters but does not depend on z.

Let h_m be the coefficients of the power series (4.4), that is

$$h_m = \frac{\langle a_1\rangle_m \cdots \langle a_r\rangle_m}{\langle b_1\rangle_m \cdots \langle b_s\rangle_m D_m}, \quad m = 0, 1, 2, \dots .$$

Since $\dfrac{D_{m+1}}{D_m^q} = [m+1] = \left(x^{q^m} - x^{q^{-1}}\right)^q = ([m] - [-1])^q$, we find that

$$\frac{h_{m+1}}{h_m^q} = \left\{ \frac{([m] - a_1) \cdots ([m] - a_r)}{([m] - b_1) \cdots ([m] - b_s)([m] - [-1])} \right\}^q . \tag{4.6}$$

The identity (4.6) means that the ratio $\dfrac{h_{m+1}}{h_m^q}$ is the q-th power of a rational function of $[m]$, which is a clear analog of the basic property of the classical hypergeometric function. Note that any rational function of $[m]$ may appear in (4.6), except those for which (4.6) does not make sense. Thakur's hypergeometric function corresponds to the case of rational functions with zeroes and poles of the form $[\nu], \nu \in \mathbb{Z}$.

It follows from (4.4) that $\Delta_i u = \Delta u$ where $\Delta = \tau d$ is the difference operator (1.1) in the variable z. Therefore ${}_rF_s$ satisfies the equation

$$\left\{ \prod_{i=1}^{r} (\Delta - a_i) - \left(\prod_{j=1}^{s} (\Delta - b_j) \right) d \right\} {}_rF_s = 0.$$

In particular, for the Gauss-like hypergeometric function ${}_2F_1$ we have

$$\{(\Delta - a)(\Delta - b) - (\Delta - c)d\}{}_2F_1 = 0.$$

Substituting $\Delta = \tau d$ and using the commutation relations (2.2) we can rewrite this equation in the form

$$\left\{ \tau(1 - \tau)d^2 - \left(c - ([1]^{1/q} + a + b)\tau \right) d - ab \right\} {}_2F_1 = 0,$$

which corresponds to (1.5).

If $b \neq [\nu]$, $\nu = 0, 1, \ldots, \infty$, then, in particular, $|b + x| \geq \mu > 0$, whence

$$|b - [\nu]| = \left|(b + x) - x^{q^\nu}\right| = |b + x| \geq \mu$$

for large values of ν. This means that

$$|b - [\nu]| \geq \mu_1 > 0, \quad \nu = 0, 1, 2, \ldots,$$

so that

$$|\langle b\rangle_\nu| \geq \mu_1^{q^\nu + q^{\nu-1} + \cdots + q} = \mu_1^{\frac{q^{\nu+1}-1}{q-1} - 1} = \mu_1^{-(q-1)^{-1} - 1} \left(\mu_1^{\frac{q}{q-1}}\right)^{q^\nu}.$$

Therefore, if $|z|$ is small enough, then the series (4.5) converges uniformly with respect to the parameters $a_i \in \overline{K}_c$ and $b_j \in \overline{K}_c \setminus \{[\nu], \nu = 0, 1, \ldots, \infty\}$ on any compact set. Thus, the function ${}_rF_s$ is a locally analytic function of its parameters.

5 Contiguous Relations

Among many identities for classical hypergeometric functions, a special role belongs to relations between contiguous functions, that is the hypergeometric functions ${}_2F_1$ whose parameters differ by ± 1 [1]. For Thakur's hypergeometric functions, analogs of the contiguous relations were found in [19, 20, 21]. Here we present analogs of the contiguous relations for our more general situation. Of 15 possible relations, we give, just as Thakur did, only two, which is sufficient to demonstrate specific features of the function field case; other relations can be obtained in a similar way. Note that the specializations of our identities for the case of parameters like $[-\alpha]$, $\alpha \in \mathbb{Z}$, are slightly different from those in [19, 20, 21], due to a different normalization of our Pochhammer-type symbols.

Our first task is to find an appropriate counterpart of the shift by 1 for parameters from $\overline{K}_c$. Denote

$$T_1(a) = (a - [1])^{1/q}, \quad a \in \overline{K}_c. \tag{5.1}$$

If $a = [-\alpha]$, $\alpha \in \mathbb{Z}$, then

$$T_1([-\alpha]) = ([-\alpha] - [1])^{1/q} = x^{q^{-\alpha-1}} - x = [-\alpha - 1],$$

so that the transformation T_1 indeed extends the unit shift of integers. The inverse transformation is given by

$$T_{-1}(a) = a^q + [1], \quad a \in \overline{K}_c. \tag{5.2}$$

Theorem 3. *The following identities for the Pochhammer-type symbol and the Gauss-type hypergeometric function are valid for every $m \in \mathbb{N}$ and any admissible parameters from $\overline{K}_c$:*

$$\langle T_1(a)\rangle_m = a^{-q^m}(a - [m])\langle a\rangle_m, \quad a \neq 0; \tag{5.3}$$

$$\langle a\rangle_{m+1} = -a^{q^{m+1}}\langle T_1(a)\rangle_m^q; \tag{5.4}$$

$$\langle T_{-1}(a)\rangle_m = -([1] + a^q)^{q^m}\langle a\rangle_{m-1}^q; \tag{5.5}$$

$$\langle T_{-1}(a)\rangle_m = -\frac{([1] + a^q)^{q^m}}{([m-1] - a)^q}\langle a\rangle_m; \tag{5.6}$$

$$
{}_2F_1(T_1(a),b;c;az) - {}_2F_1(a,T_1(b);c;bz) = (a-b){}_2F_1(a,b;c;z); \tag{5.7}
$$

$$
\begin{aligned}
{}_2F_1(a,b;c;z) - {}_2F_1(a,b;c;z)^q + (c^q-b^q){}_2F_1(a,b;T_1(c);c^{-1}z)^q \\
- (a^q+[1]){}_2F_1(T_{-1}(a),b;c;(a^q+[1])^{-1}z) = 0.
\end{aligned} \tag{5.8}
$$

Proof. Substituting (5.1) into (4.2) and using the fact that $[\nu]+[1]^{1/q} = x^{q^\nu} - x^{1/q} = [\nu+1]^{1/q}$, we get

$$
\langle T_1(a)\rangle_m = ([1]-a)^{q^{m-1}}([2]-a)^{q^{m-2}}\cdots([m]-a), \tag{5.9}
$$

which implies (5.3). If we raise both sides of (5.9) to the power q and compare the resulting identity with (4.3), we come to (5.4). The proofs of (5.5) and (5.6) are similar, based on the identity $[\nu]-[1] = [\nu-1]^q$.

Using (5.3) we find that

$$
{}_2F_1(T_1(a),b;c;az) = \sum_{m=0}^{\infty}(a-[m])\frac{\langle a\rangle_m\langle b\rangle_m}{\langle c\rangle_m D_m}z^{q^m},
$$

$$
{}_2F_1(a,T_1(b);c;bz) = \sum_{m=0}^{\infty}(b-[m])\frac{\langle a\rangle_m\langle b\rangle_m}{\langle c\rangle_m D_m}z^{q^m},
$$

which implies (5.7). Similarly, if we write down all the terms involved in (5.8) and use the identities (4.3), (5.3), (5.4), and (5.6), after rather lengthy but quite elementary calculations we verify the required identity (5.8). ∎

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