Complete Spaces of p-adic Measures

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Abstract

Let $\mathbb{K}$ be a complete non-Archimedean valued field and let $C(X,E)$ be the space of all continuous functions from a zero-dimensional Hausdorff topological space X to a non-Archimedean Hausdorff locally convex space E. We will denote by $C_b(X,E)$ (resp. by $C_{rc}(X,E)$) the space of all $f \in C(X,E)$ for which $f(X)$ is a bounded (resp. relatively compact) subset of E. The dual space of $C_{rc}(X,E)$, under the topology t_u of uniform convergence, is a space $M(X,E')$ of finitely-additive E'-valued measures on the algebra $K(X)$ of all clopen , i.e. both closed and open, subsets of X. Some subspaces of $M(X,E')$ turn out to be the duals of $C(X,E)$ or of $C_b(X,E)$ under certain locally convex topologies.
In this paper we continue with the investigation of certain subspaces of $M(X,E')$. Among other results we show that, if E is a polar Fréchet space, then :
1. The space $\mathcal{M}_{\theta_o}(X,E')$, of all $m \in M(X,E')$ for which the support of the corresponding measure m^{β_o}, on the Banaschewski compactification of X, is contained in the θ_o-repletion of X, is complete under the topology of uniform convergence on the family $\mathcal{E}$ of all equicontinuous subsets B of $C(X,E)$ for which $B(x)$ is a compactoid subset of E for all $x \in X$.
2. The space $M_{bs}(X,E')$, of all the so called strongly-separable members of $M(X,E')$ is complete under the topology of uniform convergence on the family of all uniformly bounded members of $\mathcal{E}$.
3. The space $M_s(X,E')$ of all $m \in M(X,E')$, for which ms is separable for all $s \in E$, is complete under the topology of uniform convergence on the family of all $B \in \mathcal{E}$ for which the set $B(X)$ is compactoid.

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1 Introduction

Let $\mathbb{K}$ be a complete non-Archimedean valued field and let $C(X,E)$ be the space of all continuous functions from a zero-dimensional Hausdorff topological space X to a non-Archimedean Hausdorff locally convex space E. We will denote by $C_b(X,E)$ (resp. by $C_{rc}(X,E)$) the space of all $f \in C(X,E)$ for which $f(X)$ is a bounded (resp. relatively compact) subset of E. The dual space of $C_{rc}(X,E)$, under the topology t_u of uniform convergence, is a space $M(X,E')$ of finitely-additive E'-valued measures on the algebra $K(X)$ of all clopen , i.e. both closed and open, subsets of X. Some subspaces of $M(X,E'$ turn out to be the duals of $C(X,E)$ or of $C_b(X,E)$ under certain locally convex topologies.
In this paper we continue with the investigation of certain subspaces of $M(X,E')$. Among other results we show that, if E is a polar Fréchet space, then :
1. The space $\mathcal{M}_{\theta_o}(X,E')$, of all $m \in M(X,E')$ for which the support of the corresponding measure m^{β_o}, on the Banaschewski compactification of X, is contained in the θ_o-repletion of X, is complete under the topology of uniform convergence on the family $\mathcal{E}$ of all equicontinuous subsets B of $C(X,E)$ for which $B(x)$ is a compactoid subset of E for all $x \in X$.
2. The space $M_{bs}(X,E')$, of all the so called strongly-separable members of $M(X,E')$, is complete under the topology of uniform convergence on the family of all uniformly bounded members of $\mathcal{E}$.
3. The space $M_s(X,E')$ of all $m \in M(X,E')$, for which ms is separable for all $s \in E$, is complete under the topology of uniform convergence on the family of all $B \in \mathcal{E}$ for which the set $B(X)$ is compactoid.

2 Preliminaries

Throughout this paper, $\mathbb{K}$ will be a complete non-Archimedean valued field, whose valuation is non-trivial. By a seminorm, on a vector space over $\mathbb{K}$, we will mean a non-Archimedean seminorm. Similarly, by a locally convex space we will mean a non-Archimedean locally convex space over $\mathbb{K}$ (see [22]). Unless it is stated explicitly otherwise, X will be a Hausdorff zero-dimensional topological space , E a Hausdorff locally convex space and $cs(E)$ the set of all continuous seminorms on E. The space of all $\mathbb{K}$-valued linear maps on E is denoted by $E^\star$, while E' denotes the topological dual of E. A seminorm p, on a vector space G over $\mathbb{K}$, is called polar if $p = \sup\{|f| : f \in G^\star, |f| \le p\}$. A locally convex space G is called polar if its topology is generated by a family of polar seminorms. A subset A of G is called absolutely convex if $\lambda x + \mu y \in A$ whenever $x, y \in A$ and $\lambda, \mu \in \mathbb{K}$, with $|\lambda|, |\mu| \le 1$. We will denote by $\beta_o X$ the Banaschewski compactification of X (see [5]) and by $\upsilon_o X$ the $\mathbf{N}$-repletion of X, where $\mathbf{N}$ is the set of natural numbers. By $\theta_o X$ we denote the θ_o-completion of X (see [1]). We will let $C(X,E)$ denote the space of all continuous E-valued functions on X and $C_b(X,E)$ (resp. $C_{rc}(X,E)$) the space of all $f \in C(X,E)$ for which $f(X)$ is a bounded (resp. relatively compact) subset of E. In case $E = \mathbb{K}$, we will simply write $C(X), C_b(X)$ and $C_{rc}(X)$ respectively. For $A \subset X$, we denote by χ_A the $\mathbb{K}$-valued characteristic function of A. Also, for $X \subset Y \subset \beta_o X$, we denote by $\bar{B}^Y$ the closure of B in Y. If $f \in E^X, p$ a seminorm on E and $A \subset X$, we define

$$\|f\|_p = \sup_{x \in X} p(f(x)), \quad \|f\|_{A,p} = \sup_{x \in A} p(f(x)).$$

For a locally convex space F, we denote by F^c the c-dual of F, i.e the dual space F' equipped with the topology of uniform convergence on the compactoid subsets of F.

Let $\Omega = \Omega(X)$ be the family of all compact subsets of $\beta_o X \setminus X$. By Ω_u we will denote the family of all $Q \in \Omega$ with the following property: There exists a clopen partition $(A_i)_{i \in I}$ of X such that Q is disjoint from each $\overline{A_i}^{\beta_o X}$. Also Ω_1 is the family of all zero set members of Ω, i.e all sets in Ω of the form $\{x \in \beta_o X : h(x) = 0\}$, for some $h \in C(\beta_o X)$.

For $H \in \Omega$ let C_H be the space of all $h \in C_{rc}(X)$ for which the continuous extension h^{β_o} to all of $\beta_o X$ vanishes on H. For $p \in cs(E)$, let $\beta_{H,p}$ be the locally convex topology on $C_b(X, E)$ generated by the seminorms $f \mapsto \|hf\|_p, \ \ h \in C_H$. For $H \in \Omega, \beta_H$ is the locally convex topology on $C_b(X, E)$ generated by the seminorms $f \mapsto \|hf\|_p, \ \ h \in C_H, p \in cs(E)$. The inductive limit of the topologies $\beta_H, H \in \Omega$, is the topology β.

For d a continuous ultra-pseudometric on X , we denote by X_d the corresponding ultrametric space and by $\pi_d : X \to X_d$ the quotient map. Let

$$T_d : C_b(X_d, E) \to C_b(X, E)$$

be the induced linear map. The topology β_e is defined to be the finest of all locally convex topologies τ on $C_b(X, E)$ for which each

$$T_d : (C_b(X_d, E), \beta) \to (C_b(X, E), \tau)$$

is continuous (see [13]).

Let now $K(X)$ be the algebra of all clopen subsets of X. We denote by $M(X, E')$ the space of all finitely-additive E'-valued measures m on $K(X)$ for which the set $m(K(X))$ is an equicontinuous subset of E'. For each such m, there exists a $p \in cs(E)$ such that $\|m\|_p = m_p(X) < \infty$, where, for $A \in K(X)$,

$$m_p(A) = \sup\{|m(B)s|/p(s) : p(s) \neq 0, \quad A \supset B \in K(X)\}.$$

The space of all $m \in M(X, E')$ for which $m_p(X) < \infty$ is denoted by $M_p(X, E')$. In case $E = \mathbb{K}$, we denote by $M(X)$ the space of all finitely-additive bounded $\mathbb{K}$-valued measures on $K(X)$. An element m of $M(X)$ is called τ-additive if $m(V_\delta) \to 0$ for each decreasing net (V_δ) of clopen subsets of X with $\bigcap V_\delta = \emptyset$. In this case we write $V_\delta \downarrow \emptyset$. We denote by $M_\tau(X)$ the space of all τ-additive members of $M(X)$. Analogously, we denote by $M_\sigma(X)$ the space of all σ-additive m, i.e. those m with $m(V_n) \to 0$ when $V_n \downarrow \emptyset$. For an $m \in M(X, E')$ and $s \in E$, we denote by ms the element of $M(X)$ defined by $(ms)(V) = m(V)s$.

Next we recall the definition of the integral of an $f \in E^X$ with respect to an $m \in M(X, E')$. For a non-empty clopen subset A of X, let $\mathcal{D}_\mathcal{A}$ be the family of all $\alpha = \{A_1, A_2, \ldots, A_n; x_1, x_2, \ldots, x_n\}$, where $\{A_1, \ldots, A_n\}$ is a clopen partition of A and $x_k \in A_k$. We make $\mathcal{D}_\mathcal{A}$ into a directed set by defining $\alpha_1 \geq \alpha_2$ iff the partition of

A in α_1 is a refinement of the one in α_2. For an $\alpha = \{A_1, A_2, \ldots, A_n; x_1, x_2, \ldots, x_n\} \in \mathcal{D}_{\mathcal{A}}$ and $m \in M(X, E')$, we define

$$\omega_\alpha(f, m) = \sum_{k=1}^{n} m(A_k) f(x_k).$$

If the limit $\lim \omega_\alpha(f, m)$ exists in $\mathbb{K}$, we will say that f is m-integrable over A and denote this limit by $\int_A f\, dm$. We define the integral over the empty set to be 0. For $A = X$, we write simply $\int f\, dm$. It is easy to see that if f is m-integrable over X, then it is integrable over every clopen subset A of X and $\int_A f\, dm = \int \chi_A f\, dm$. If τ_u is the topology of uniform convergence, then every $m \in M(X, E')$ defines a τ_u-continuous linear functional ϕ_m on $C_{rc}(X, E)$, $\phi_m(f) = \int f\, dm$. Also every $\phi \in (C_{rc}(X, E), \tau_u)'$ is given in this way by some $m \in M(X, E')$.
For all unexplained terms on locally convex spaces, we refer to [21] and [22].

3 The Space $L(X, E')$

For $x \in X$ and $x' \in E'$, we will denote by $\delta_{x,x'}$ the linear functional on $C(X, E)$ defined by $\delta_{x,x'}(f) = x'(f(x))$. Let $L(X, E')$ be the linear subspace of $C(X, E)^\star$ spanned by the set $\{\delta_{x,x'} : x \in X, x' \in E'\}$. Also $C_{co}(X, E)$ is the subspace of $C(X, E)$ consisting of all f for which the set $f(X)$ is a compactoid subset of E. We will consider the following families of subsets of $C(X, E)$:
1. $\mathcal{E} = \mathcal{E}(X, E)$ is the family of all equicontinuous subsets B of $C(X, E)$ for which the set $B(x) = \{f(x) : f \in B\}$ is compactoid for each $x \in X$.
2. $\mathcal{E}_b = \mathcal{E}_b(X, E)$ is the family of all uniformly bounded members of $\mathcal{E}$.
3. $\mathcal{E}_{co} = \mathcal{E}_{co}(X, E)$ is the family of all $B \in \mathcal{E}$ for which the set $B(X)$ is compactoid.
Let e, e_b, e_{co} be the locally convex topologies on $L(X, E')$ which are the topologies of uniform convergence on the members of $\mathcal{E}$, $\mathcal{E}_b$, $\mathcal{E}_{co}$, respectively. For $B \in \mathcal{E}$, the seminorm p_B on $L(X, E')$, defined by $p_B(u) = \sup_{f \in B} |u(f)|$, is polar. Thus each of the topologies e, e_b, e_{co} is polar.
Recall that a locally convex space F is said to be c-complete if every closed compactoid subset of F is complete.

Theorem 3.1. *Assume that E is polar and c-complete. Then, the dual spaces of $L(X, E')$, under the topologies e, e_b and e_{co}, coincide with the spaces $C(X, E)$, $C_b(X, E)$ and $C_{co}(X, E)$, respectively.*

Proof: 1. For $f \in C(X, E)$, the set $\{f\}$ is in $\mathcal{E}$. It follows from this that $C(X, E)$ is a subspace of the dual space of $G_e = (L(X, E'), e)$ (considering each element of $C(X, E)$ as a linear functional on G_e). On the other hand, let $\phi \in G_e'$. There exists a $B \in \mathcal{E}$ such that

$$\{u \in G_e : p_B(u) \le 1\} \subset \{u : |\phi(u)| \le 1\}.$$

For $x \in X$, we consider the linear form $\phi_x(x') =< \phi, \delta_{x,x'} >$, $x' \in E'$. If x' is in the polar $B(x)^o$ of $B(x)$ in E', then $\delta_{x,x'} \in B^o$ and so $|\phi_x(x')| \le 1$. As $B(x)$ is compactoid, it follows that ϕ_x is continuous on the c-dual space E^c of E. Since E is polar and c-complete, there exists a unique element $f(x) \in E$ such that $\phi_x(x') =$

$x'(f(x))$ for all $x' \in E'$ (by [18], Theorem 4.7). Thus we get a map $f : X \to E$. This map is continuous. Indeed, let p be a polar continuous seminorm on E. By the equicontinuity of B, given $x \in X$, there exists a neighborhood Z of x such that $p(g(x) - g(y)) \leq 1$ for all $g \in B$ and all $y \in Z$. Let $x' \in E'$, $|x'| \leq p$. If $g \in B$ and $y \in Z$, then

$$| < g, \delta_{x,x'} - \delta_{y,x'} > | = |x'(g(x) - g(y))| \leq p(g(x) - g(y)) \leq 1.$$

Thus $\delta_{x,x'} - \delta_{y,x'} \in B^o$ and so

$$|x'(f(x) - f(y)| = | < \phi, \delta_{x,x'} - \delta_{y,x'} > | \leq 1.$$

Since p is polar, it follows that $p(f(x) - f(y)) \leq 1$ for all $y \in Z$, which proves that f is continuous at x. Now, for $u = \sum_{k=1}^{n} \delta_{x_k, x'_k}$, we have

$$< f, u >= \sum_{k=1}^{n} x'_k(f(x_k)) =< \phi, u >$$

and so $\phi = f$ (as linear functionals on G_e). This completes the proof for e.
2. Let $G_b = (L(X, E'), e_b)$. Since e_b is coarser than e, it follows that $G'_b \subset G'_e = C(X, E$. Let $f \in C(X, E)$ be in G'_b and let $B \in \mathcal{E}_b$ be such that $| < f, u > | \leq 1$ if $u \in B^o$. We will show that $f(X)$ is bounded in E. Since E is polar, it suffices to prove that $f(X)$ is weakly bounded. So let $x' \in E'$. As $B(X)$ is a bounded subset of E, there exists a $\lambda \in \mathbb{K}$ such that $|x'(s)| \leq |\lambda|$ for all $s \in B(X)$. Now $\lambda^{-1}\delta_{x,x'} \in B^o$, for all $x \in X$, and so $\sup_{x \in X} |x'(f(x) \leq |\lambda|$. Thus $f(X)$ is weakly bounded and hence $f \in C_b(X, E)$. Conversely, if $f \in C_b(X, E)$, then $\{f\} \in \mathcal{E}_b$, from which it follows that $f \in G'_b$.
3. If $G_{c_o} = (L(X, E'), e_{c_o})$, then the proof of the equality $G'_{c_o} = C_{c_o}(X, E)$ is analogous to the one used for e_b using the fact that, if D is a compactoid subset of the polar space E, then the bipolar B^{oo} is also compactoid.

Let $\sigma = \sigma(C(X, E), L(X, E')$. If E is polar, then, on each member B of $\mathcal{E}$, the weak topology σ coincides with the topology of simple convergence since, for each $x \in X$, $B(x)$ is compactoid.

Theorem 3.2. *Assume that E is polar and consider the dual pair*

$$< C(X, E), L(X, E') > .$$

Let $B \subset C(X, E)$. If B is a member of one of the families $\mathcal{E}$, $\mathcal{E}_b$, $\mathcal{E}_{c_o}$, then the bipolar B^{oo} is also a member of the same family.

Proof : By [21], Proposition 4.10, we have that $B^{oo} = \left(\overline{co(B)}^{\sigma}\right)^e$, where $D = \overline{co(B)}^{\sigma}$ is the σ-closure of the absolutely convex hull $co(B)$ of B and D^e is the edged hull of D. Let $x \in X$, $\epsilon > 0$ and $p \in cs(E)$. Since B is equicontinuous, there exists a neighborhood Z of x such that $p(f(y) - f(x)) \leq \epsilon$ for all $f \in B$ and all $y \in Z$. Let now $f \in \overline{co(B)}^{\sigma}$ and $y \in Z$. There exists a net (f_δ) in $co(B)$ which is σ-convergent to f. The set $M = [B(y)]^{oo}$ is also compactoid since E is polar. The map

$$\omega : (C(X, E), \sigma) \to (E, \sigma(E, E'),$$

$g \mapsto g(y)$, is continuous. Thus $f_\delta(y) \to f(y)$ weakly in E. As M is weakly closed, we have that $f(y) \in M$. On compactoid subsets of E, the weak topology and the original topology coincide (by [21], Theorem 5.12). Thus $f_\delta(y) \to f(y)$ in E. Now, for $y \in Z$, we have that $f_\delta(y) - f_\delta(x) \to f(y) - f(x)$ and hence $p(f(y) - f(x)) \leq \epsilon$. This proves that D is equicontinuous. If $x \in X$, then $D(x) \subset [B(x)]^{oo}$ and so $D(x)$ is compactoid, which proves that $D \in \mathcal{E}$. Finally, if $|\lambda| > 1$, then $B^{oo} \subset \lambda D$, from which it follows that $B^{oo} \in \mathcal{E}$. If $B \in \mathcal{E}_b$, then $B(X)$ is bounded and hence $[B(X)]^{oo}$ is bounded. Since $B^{oo}(X) \subset [B(X)]^{oo}$, it follows that $B^{oo} \in \mathcal{E}_b$. The case of a $B \in \mathcal{E}_{co}$ is analogous taking into account the fact that, if $A \subset E$ is compactoid, then A^{oo} is also compactoid.

Theorem 3.3. *Assume that E is polar and let $B \subset C(X, E)$. Then B is equicontinuous with respect to one of the topologies e, e_b, e_{co}, iff B is a member of $\mathcal{E}$, $\mathcal{E}_b$, or $\mathcal{E}_{co}$, respectively.*

Proof : It follows from the preceding Theorem and from the fact that, if B is a member of $\mathcal{E}$, $\mathcal{E}_b$, or $\mathcal{E}_{co}$, then every subset of B is also a member of the same family.

4 The Space $\mathcal{M}_{\theta_o}(X, E')$ as a Completion

We will denote by $M_{\theta_o}(X)$ the space of all $\mu \in M(X)$ for which the support $supp(\mu^{\beta_o})$, of the corresponding measure μ^{β_o} on $\beta_o X$ is contained in $\theta_o X$. Also, by $\mathcal{M}_{\theta_o}(X, E')$ we will denote the space of all $m \in M(X, E')$ for which $supp(m^{\beta_o}) \subset \theta_o X$. By Ω_{θ_o} we will denote the family of all compact subsets of $\beta_o X$ which are disjoint from $\theta_o X$.

Theorem 4.1. *For an $m \in M(X, E'$, the following are equivalent :*

1. *$m \in \mathcal{M}_{\theta_o}(X, E')$.*
2. *If (V_δ) is a net of clopen subsets of X with $\overline{V_\delta}^{\beta_o X} \downarrow H \in \Omega_{\theta_o}$, then there exists a δ_o such that $m(V_\delta) = 0$ for each $\delta \geq \delta_o$.*
3. *If $\overline{V_\delta}^{\beta_o X} \downarrow H \in \Omega_{\theta_o}$, then there exists a δ such that $m(V) = 0$ for each clopen subset V of V_δ.*
4. *If $(V_i)_{i \in I}$ is a clopen partition of X, then there exists a finite subset J of I such that $m(V) = 0$ for each clopen subset V of $\bigcup_{i \notin J} V_i$.*

Proof : (1) $\Rightarrow$ (2). Since

$$supp(m^{\beta_o}) \subset \theta_o X \subset \beta_o X \setminus H = \bigcup_\delta \overline{V_\delta^c}^{\beta_o X},$$

there exists a δ_o such that $supp(m^{\beta_o}) \subset \overline{V_\delta^c}^{\beta_o X}$. If now $\delta \geq \delta_o$, then $m(V_\delta) = m^{\beta_o}(\overline{V_\delta}^{\beta_o X}) = 0$.
(2) $\Rightarrow$ (3) . Suppose that, for each δ, there exists a clopen subset V of V_δ with $m(V) \neq 0$. Let now δ be given and let A be a clopen subset of V_δ such that $m(A) \neq 0$. For each γ in the index set, let $Z_\gamma = V_\gamma \cap A$, $W_\gamma = V_\gamma \setminus Z_\gamma$. The net (W_γ) is decreasing

and $\bigcap \overline{W_\gamma}^{\beta_o X} \subset H$. By our hypothesis (2), there exists $\gamma \geq \delta$ such that $m(W_\gamma) = 0$. If $B = A \cup W_\gamma$, then $V_\gamma \subset B \subset V_\delta$ and $m(B) = m(A) + m(W_\gamma) = m(A) \neq 0$. Let now $\mathcal{F}$ be the family of all clopen subsets A of X with the following property: there are γ, δ, in the index set, with $V_\gamma \subset A \subset V_\delta$ and $m(A) \neq 0$. Then $\mathcal{F}$ is downwards directed and $\bigcap_{F \in \mathcal{F}} \overline{F}^{\beta_o X} = H \in \Omega_{\theta_o}$. Since $m(F) \neq 0$ for each $F \in \mathcal{F}$, we got a contradiction.
(3) $\Rightarrow$ (4). For each finite subset J of I, set $W_J = \bigcup_{i \notin J} V_i$. Then $\overline{W_J}^{\beta_o X} \downarrow H \in \Omega_{\theta_o}$. By our hypothesis (3), there exists a finite subset J of I such that $m(V) = 0$ for each clopen set V contained in W_J.
(4) $\Rightarrow$ (1). Let $z \notin \theta_o X$. There exists a clopen partition $(V_i)_{i \in I}$ of X such that $z \notin \bigcup_{i \in I} \overline{V_i}^{\beta_o X}$. By (4), there exists a finite subset J of I such that $m(V) = 0$ for each clopen set V contained in $\bigcup_{1 \notin J} V_i$. Now $supp(m^{\beta_o})$ is contained in $\bigcup_{i \in J} \overline{Vi}^{\beta_o X}$ and so $z \notin supp(m^{\beta_o})$. This clearly completes the proof.

Theorem 4.2. *If $m \in \mathcal{M}_{\theta_o}(X, E')$, then every $f \in C(X, E)$ is m-integrable.*

Proof : Let $f \in C(X, E)$ and $\epsilon > 0$. There exists a $p \in cs(E)$ such that $m_p(X) \leq 1$. Let $(V_i)_{i \in I}$ be the clopen partition of X corresponding to the equivalence relation $x \sim y$ iff $p(f(x) - f(y)) \leq \epsilon$. In view of the preceding Theorem, there exists a finite subset J of I such that $m(V) = 0$ for each clopen set V contained in $D = \bigcup_{i \notin J} V_i$. Consider the finite clopen partition $\mathcal{A} = \{V_i : i \in J\} \bigcup \{D\}$. If $A \in \mathcal{A}$, then for all clopen subsets V of A and all $x, y \in A$, we have

$$|m(V)[f(x) - f(y)]| \leq p(f(x) - f(y)) \cdot m_p(A) \leq \epsilon.$$

This proves that f is m-integrable by [14], Theorem 7.1.

Next we will assume that E is polar and c-complete and we will look at the completion $\hat{G}_e$ of the space

$$G_e = (L(X, E'), e).$$

Since G_e is Hausdorff and polar, its completion coincides with the space of all linear functionals ϕ on $G'_e = C(X, E)$ which are $\sigma(C(X, E), G_e)$-continuous on e-equicontinuous subsets of $C(X, E)$, i.e. on the members of $\mathcal{E}$ (by [16]). As we remarked in section 2, on members of $\mathcal{E}$, the weak topology coincides with the topology of simple convergence. The topology of $\hat{G}_e$ coincides with the topology of uniform convergence on the members of $\mathcal{E}$.

Theorem 4.3. *Let E be polar and c-complete. If $m \in \mathcal{M}_{\theta_o}(X, E')$, then the map*

$$\phi_m : C(X, E) \to \mathbb{K},\ \phi_m(f) = \int f\, dm,$$

belongs to $\hat{G}_e$.

Proof : Let $p \in cs(E)$ be such that $m_p(X) \leq 1$ and let $B \in \mathcal{E}$. Define d on $X \times Y$ by

$$d(x, y) = \sup_{f \in B} p(f(x) - f(y)).$$

Then d is a continuous ultrapseudometric on X. Let $\epsilon > 0$ and let $(V_i)_{i \in I}$ be the clopen partition of X corresponding to the equivalence relation $x \sim y$ iff $d(x, y) \leq \epsilon$.

Let (f_γ) be a net in B which converges pointwise to some $f \in B$. Since $m \in \mathcal{M}_{\theta_o}(X, E')$, there exists a finite subset J of I such that $m(V) = 0$ for each clopen subset of $W = \bigcup_{i \notin J} V_i$. Consider the finite clopen partition $\mathcal{A} = \{V_i : i \in J\} \bigcup \{W\}$ of X. If $g \in B$ and if x, y are in the same $A \in \mathcal{A}$, then

$$|m(V)[g(x) - g(y)]| \leq p(g(x) - g(y)) \cdot m_p(A) \leq \epsilon.$$

If $x_i \in V_i$, $i \in I$, we have that

$$\left| \int g\, dm - \sum_{i \in J} m(V_i) g(x_i \right| \leq \epsilon.$$

Since $f_\gamma(x) \to f(x)$, for all x, there exists a γ_o such that $p(f_\gamma(x_i) - f(x_i)) \leq \epsilon$ for all $\gamma \geq \gamma_o$ and all $i \in J$. If now $\gamma \geq \gamma_o$, then

$$\left| \int f_\gamma\, dm - \sum_{i \in J} m(V_i) f_\gamma(x_i) \right| \leq \epsilon \quad \text{and} \quad \left| f\, dm - \sum_{i \in J} m(V_i) f(x_i) \right| \leq \epsilon.$$

Since

$$\left| \sum_{i \in J} m(V_i)[f_\gamma(x_i) - f(x_i)] \right| \leq \max_{i \in J} p(f_\gamma(x_i) - f(x_i)) \cdot m_p(X) \leq \epsilon,$$

it follows that $\int f_\gamma\, dm \to \int f\, dm$, which shows that $\phi_m \in \hat{G}_e$.

Theorem 4.4. *Let E be polar and c-complete and let $\phi \in \hat{G}_e$. Then, for each $s \in E$, there exists a $\mu_s \in M_{\theta_o}(X)$ such that $\phi(gs) = \int g\, d\mu_s$ for each $g \in C(X)$.*

Proof : Let $s \in E$ and consider the linear functional

$$\phi_s : C(X) \to \mathbb{K},\ \phi_s(g) = \phi(gs).$$

Let $\mathcal{A}$ be an equicontinuous pointwise bounded subset of $C(X)$ and let (g_γ) be a net in $\mathcal{A}$ which converges pointwise to a $g \in \mathcal{A}$. The set $B = \{gs : g \in \mathcal{A}\}$ is in $\mathcal{E}$. If $f_\gamma = g_\gamma s$, $f = gs$, then $f_\gamma \to f$ pointwise. Since $\phi \in \hat{G}_e$, we have that $\phi(f_\gamma) \to \phi(f)$, i.e. $\phi_s(g_\gamma) \to \phi_s(g)$. In view of Theorem 8.9 in [15], there exists a $\mu_s \in M_{\theta_o}(X)$ such that $\phi(gs) = \int g\, d\mu_s$ for each $g \in C(X)$. Hence the result follows.

Theorem 4.5. *Let E be a polar Fréchet space and let $\phi \in \hat{G}_e$. Then, there exists a $p \in cs(E)$ and an $m \in M(X, E')$ such that:*

1. *for each $s \in E$, we have that $ms \in M_{\theta_o}(X)$ and $\phi(gs) = \int g\, d(ms)$ for each $g \in C(X)$.*
2. $$\{f \in C(X, E) : \|f\|_p \leq 1\} \subset \{f : |\phi(f)| \leq 1\}.$$

Proof : In view of the preceding Theorem, for each $s \in E$, there exists a $\mu_s \in M_{\theta_o}(X)$ such that $\phi(gs) = \int g\, d\mu_s$ for each $g \in C(X)$. Let (p_n) be an increasing sequence of continuous seminorms on E generating its topology, and let

$$D = \{f \in C(X, E) : |\phi(f)| \leq 1\}.$$

We claim that, there exists an n and $\epsilon > 0$ such that

$$\{f \in C(X,E) : \|f\|_{p_n} \leq \epsilon\} \subset D.$$

Assume the contrary and let $0 < |\lambda| < 1$. For each n, there exists an f_n in $C(X,E)$ with $\|f_n\|_{p_n} \leq |\lambda|^n$ and $|\phi(f_n)| > 1$. Then $f_n \to 0$ uniformly. Indeed, let k be given and let $\epsilon > 0$. Choose $n_o \geq k$ such that $\lambda|^n < \epsilon$ for $n \geq n_o$. Now, for $n \geq n_o$, we have that $\|f_n\|_{p_k} \leq \|f_n\|_{p_n} \leq |\lambda|^n < \epsilon$ and so $f_n \to 0$ uniformly. This, together with the fact that each f_n is continuous, implies that the set $B = \{f_1, f_2, \ldots\}$ is in $\mathcal{E}$ and $f_n \to f$ pointwise. Since $\phi \in \hat{G}_e$, we should have that $\phi(f_n) \to 0$, a contradiction. This proves (2). Now $\phi|_{C_{rc}(X,E)}$ is continuous with respect to the topology of uniform convergence. Hence, there exists an $m \in M(X,E')$ such that $\phi(f) = \int f\,dm$ for all $f \in C_{rc}(X,E)$. In particular, taking $f = \chi_V s$, where $V \in K(X)$ and $s \in E$, we have that

$$m(V)s = \phi(f) = \int \chi_V \, d\mu_s = \mu_s(V)$$

and thus $ms = \mu_s \in M_{\theta_o}(X)$. Hence the Theorem follows.

Theorem 4.6. *Let E be a polar Fréchet space. Then the map*

$$m \mapsto \phi_m, \quad \phi_m(f) = \int f\,dm,$$

from $\mathcal{M}_{\theta_o}(X,E')$ to $\hat{G}_e$, is an algebraic isomorphism. Thus $\hat{G}_e = \mathcal{M}_{\theta_o}(X,E')$.

Proof : Let $\phi \in \hat{G}_e$. By the preceding Theorem, there exists an $m \in M(X,E')$ such that $ms \in M_{\theta_o}(X)$, for all $s \in E$, and $\phi(gs) = \int g\,d(ms)$ for each $g \in C(X)$. We will show that $m \in \mathcal{M}_{\theta_o}(X,E')$. To this end, consider a clopen partition $(V_i)_{i\in I}$ of X.
Claim : The set J of all $i \in I$, for which there exists a clopen subset A of V_i with $m(A) \neq 0$, is finite. Indeed, let $\{i_1, i_2, \ldots\}$ be an infinite sequence of distinct elements of J. For each k, there exists a clopen set $B_k \subset V_{i_k}$ such that $m(B_k) \neq 0$. Choose $s_k \in E$ with $|m(B_k)s_k| > 1$. The set $V = (\bigcup_{k=1}^{\infty} B_k)^c$ is clopen. The function $g_n = \sum_{k=n}^{\infty} \chi_{B_k} s_k$ is continuous. It is easy to see that $B = \{g_1, g_2, \ldots\} \in \mathcal{E}$ and $g_n \to 0$ pointwise. Since $\phi \in \hat{G}_e$, we must have that $\phi(g_n) \to 0$. Let k_o be such that $|\phi(g_n)| < 1$ if $n \geq k_o$. If $n \geq k_o$, then $\chi_{B_n} s_n = g_n - g_{n+1}$ and thus $|m(B_n)s_n| = |\phi(g_n) - \phi(g_{n+1})| < 1$, a contradiction. This proves that J is finite, $J = \{i_1, i_2, \ldots, i_n\}$. Let $V = \bigcup_{i\notin J} V_i$ and let A be a clopen subset of V. Then $A = \bigcup_{i\notin J} V_i \cap A$. If $s \in E$, then $ms \in M_{\theta_o}(X) \subset M_s(X)$ and so $(ms)(A) = \sum_{i\notin J} m(V_i \cap A)s = 0$, i.e. $m(A)s = 0$ for all $s \in E$, which means that $m(A) = 0$. By Theorem 3.1, we have that $m \in \mathcal{M}_{\theta_o}(X,E')$. It remains to prove that $\phi(f) = \int f\,dm$ for all $f \in C(X,E)$. There exists a $p \in cs(E)$ such that

$$\{f \in C(X,E) : \|f\|_p \leq 1\} \subset \{f : |\phi(f)| \leq 1\}.$$

If $|\lambda| > 1$, then $m_p(X) \leq |\lambda|$. Let now $f \in C(X,E)$, α a non-zero element of $\mathbb{K}$ and let $(V_i)_{i\in I}$ be the clopen partition of X corresponding to the equivalence relation $x \sim y$ iff $p(f(x) - f(y)) \leq |\alpha|$. There exists a finite subset J of I such that $\bigcup_{i\in J} V_i$ is a support set for m. If $\imath \in J$ and $x_i \in V_i$, then for each $x \in V_i$ and each clopen subset B of V_i, we have $|m(B)[f(x) - f(x_i)]| \leq |\lambda\alpha|$. Thus

$$\left| f\,dm - \sum_{i\in J} m(V_i) f(x_i) \right| \leq |\lambda\alpha|.$$

For $S \subset I$ finite, let

$$g_S = \sum_{i \in S} \chi_{V_i} f(x_i), \quad g = \sum_{i \in I} \chi_{V_i} f(x_i).$$

It is easy to see that the set

$$B = \{g_S : S \subset I, \, S \, \text{finite}\} \cup \{g\}$$

is in $\mathcal{E}$ and that $g_S \to g$ pointwise. Thus

$$\phi(g) = \lim_S \phi(g_S) = \lim_S \sum_{i \in S} m(V_i) f(x_i = \sum_{i \in I} m(V_i) f(x_i) = \sum_{i \in J} m(V_i) f(x_i).$$

Since $\|g - f\|_p \leq |\alpha|$, it follows that $|\phi(g) - \phi(f)| \leq |\alpha|$ and so

$$\left| \int f \, dm - \phi(f) \right| \leq \max \left\{ \left| \int f \, dm - \phi(g) \right|, \, |\phi(g) - \phi(f)| \right\} \leq |\lambda \alpha|.$$

As α was arbitrary, we conclude that $\int f \, dm = \phi(f)$ and the result follows from this and from Theorem 3.3.

5 The Space $M_{bs}(X, E')$ as a Completion

Let $m \in M(X, E')$. For a bounded subset S of E and $V \in K(X)$, we define

$$|m|_S(V) = \sup\{|m(A)s| : s \in S, A \in K(X), A \subset V\}.$$

Definition 5.1. *An element m of $M(X, E')$ is said to be :*

1. *Strongly σ-additive if, for each sequence (V_n) of clopen subsets of X which decreases to the empty set, we have that $m(V_n) \to 0$ in the strong dual E'_b of E.*

2. *Strongly τ-additive if $m(V_\delta) \to 0$ in E'_b when $V_\delta \downarrow \emptyset$.*

3. *Strongly separable if it is strongly σ-additive and, for each continuous ultrapseudometric d on X and each bounded subset S of E, there exists a d-closed, d-separable subset D of X such that $m(V)s = 0$ for each $s \in S$ and each d-clopen set V which is disjoint from D*

We will denote by $M_{bs}(X, E')$ the space of all strongly separable members of $M(X, E')$.

Theorem 5.2. *Let $m \in M(X, E')$. Then :*

1. *m is strongly τ-additive iff, for each net $V_\delta \downarrow \emptyset$ and each bounded subset S of E we have that $|m|_S(V_\delta) \to 0$.*

2. *m is strongly σ-additive iff $|m|_S(V_n) \to 0$ for each bounded subset S of E and each sequence $V_n \downarrow \emptyset$.*

Proof : 1). It is clear that the condition is sufficient. Conversely, assume that m is strongly τ-additive and that there exist a bounded subset S of E, an $\epsilon > 0$ and a net $V_\delta \downarrow \emptyset$, $\delta \in \Delta$ such that $|m|_S(V_\delta) > \epsilon$ for all δ. Let $\delta \in \Delta$. There exist a clopen subset A of V_δ and an $s_o \in S$ such that $|m(A)s_o| > \epsilon$. For each element $\gamma \in \Delta$, let $Z_\gamma = V_\gamma \cap A$, $W_\gamma = V_\gamma \setminus Z_\gamma$. Then $W_\gamma \downarrow \emptyset$. By our hypothesis, there exists a $\gamma \geq \delta$ such that $|m(W_\gamma)s| < \epsilon$ for all $s \in E$. Let $B = A \cup W_\gamma$. Then $V_\gamma \subset B \subset V_\delta$ and $m(B) = m(A) + m(W_\gamma)$, which implies that $|m(B)s_o| = |m(A)s_o| > \epsilon$. Consider now the family $\mathcal{F}$ of all clopen subset A of X with the following property: There are $\gamma \geq \delta$ in Δ , with $V_\gamma \subset A \subset V_\delta$ and $\sup_{s\in E} |m(A)s| > \epsilon$. Then $\mathcal{F} \downarrow \emptyset$. Since $\sup_{s\in S} |m(A)s| > \epsilon$, for all $A \in \mathcal{F}$, we arrived at a contradiction.
2. Assume that m is strongly σ-additive and that there exist a sequence $V_n \downarrow \emptyset$, a bounded subset S of E and an $\epsilon > 0$ such that $|m|_S(V_n > \epsilon$ for all n. As in the proof of (1), we get a sequence $n_1 < n_2 < \ldots$ of positive integers, a sequence (s_k) in S and a sequence (A_k) of clopen sets such that $V_{n_{k+1}} \subset A_k \subset V_{n_k}$ and $|m(A_k)s_k| > \epsilon$, for all k, which is a contradiction. This clearly completes the proof.

Theorem 5.3. *Let (X, d) be an ultrametric space and let H be a uniformly τ-additive subset of the dual space $M_\tau(X)$ of $(C_b(X), \beta)$. Then the support of H, i.e. the set*

$$supp(H) = \overline{\bigcup_{m\in H} supp(m)},$$

is separable.

Proof : For each finite subset Y of X each $\epsilon > 0$, let $N(Y, \epsilon) = \{x : d(x, Y) \leq \epsilon\}$. Then $N(Y, \epsilon)$ is clopen and the family

$$\{X \setminus N(Y, \epsilon) : Y \quad \text{finite subset of } X\}$$

is downwards directed to the empty set. Since H is uniformly τ-additive, given $\epsilon_1 > 0$, there exists a finite subset Y of X such that $\sup_{m\in H} |m|(X \setminus N(Y, \epsilon)) < \epsilon_1$. For positive integers n, k, choose a finite subset $Y_{n,k}$ of X such that

$$\sup_{m\in H} |m|(X \setminus N(Y_{n,k}, 1/k)) < 1/n.$$

Let

$$D_n = \bigcup_k [X \setminus N(Y_{n,k}, 1/k)],\ M = \bigcup_n X \setminus D_n, \quad F = \overline{M}.$$

Then $X \setminus F \subset \bigcap D_n$. Let now $x \in X \setminus F$ and $m \in H$. For each n, choose a k such that $x \notin N(Y_{n,k}, 1/k)$ and so $N_m(x) \leq |m|(X \setminus N(Y_{n,k}, 1/k)) < 1/n$, which proves that $N_m(x) = 0$. If B is a clopen subset of X disjoint from F, then (by [22]) we have $|m|(B) = \sup_{x\in B} N_m(x) = 0$ and so $supp(m) \subset F$. It follows that $supp(H) \subset F$. Finally, $supp(H)$ is separable. In fact, let $\epsilon > 0$ and $x \in F$. There exists $y \in M$ such that $d(x, y) < \epsilon$. Let n be such that $y \notin D_n$. Choose $k > 1/\epsilon$. Since $y \in N(Y_{n,k}, 1/k)$, there exists a $z \in Y_{n,k}$, with $d(y, z) \leq 1/k < \epsilon$, and so $d(x, z) < \epsilon$. The set $Y = \bigcup_{n,k} Y_{n,k}$ is countable and $F \subset \overline{Y}$. Since $\bar{Y}$ is separable, the same is true for the subset $supp(H)$. This completes the proof.

Theorem 5.4. *For an element m of $M(X, E')$, the following are equivalent :*

1. *$ms \in M_s(X)$, for each $s \in E$, and, for each clopen partition $(V_i)_{i\in I}$ of X, each bounded subset S of E and each $\epsilon > 0$, there exists a finite subset J of I such that $|m|_S(V_i) < \epsilon$ for all $i \notin J$.*

2. *If $(V_i)_{i\in I}$ is a clopen partition of X, S a bounded subset of E and $\epsilon > 0$, then there exists a finite subset J of I such that $|m|_S(\bigcup_{i\notin J} V_i) \le \epsilon$.*

3. *If (V_δ) is a net of clopen subsets of X, with $\overline{V_\delta}^{\beta_o X} \downarrow Z \in \Omega_u$, and if S is a bounded subset of E, then $|m|_S(V_\delta) \to 0$.*

4. *If $\overline{V_\delta}^{\beta_o X} \downarrow Z \in \Omega_u$, then $m(V_\delta) \to 0$ in the strong dual of E.*

5. *If $(V_i)_{i\in I}$ is a clopen partition of X, then $m(X) = \sum_{i\in I} m(V_i)$, where the convergence of the sum is in the strong dual of E.*

6. *$m \in M_{bs}(X, E')$.*

Proof : (1) $\Rightarrow$ (2). Let J be a finite subset of I such that $|m|_S(V_i) < \epsilon$ if $i \notin J$. Let A be a clopen subset of $D = \bigcup_{i\notin J} V_i$. Then $A = \bigcup_{i\notin J} A \cap V_i$. If $s \in S$, then $ms \in M_s(X)$ and so

$$m(A)s = \sum_{i\notin J} m(V_i \cap A)s$$

(by [12], Theorem 6.9). But, for $i \notin J$, we have $|m(V_i \cap A)s| \le |m|_S(V_i) < \epsilon$. Thus $|m(A)s| \le \epsilon$, which proves that $|m|_S(D) \le \epsilon$.
(2) $\Rightarrow$ (3). There exists a clopen partition $(V_i)_{i\in I}$ of X such that

$$Z \subset \beta_o X \setminus \bigcup_{i\in I} \overline{V_i}^{\beta_o X}.$$

Let S be a bounded subset of E and $\epsilon > 0$. There exists a finite subset J of I such that $|m|_S\left(\bigcup_{i\notin J} V_i\right) < \epsilon$. There is a δ such that $\bigcup_{i\in J} \overline{V_i}^{\beta_o X} \subset \overline{V_\delta^c}^{\beta_o X}$, and so

$$|m|_S(V_\delta) \le |m|_S\left(\bigcup_{i\notin J} V_i\right) < \epsilon.$$

(3) $\Rightarrow$ (4). It is trivial.
(4) $\Rightarrow$ (5). Let $(V_i)_{i\in I}$ be a clopen partition of X. For each finite subset J of I, let $W_J = \bigcup_{i\in J} V_i$ and $D_J = X \setminus W_J$. Then $m(X) - \sum_{i\in J} m(V_i) = m(D_J)$. Since $D_J \downarrow Z \in \Omega_u$, our hypothesis (4) implies that $m(D_J) \to 0$ in the strong dual of E.
(5) $\Rightarrow$ (1). Let $(V_i)_{i\in I}$ be a clopen partition of X. Then $m(X) = \sum_{i\in I} m(V_i)$ in E'_b and hence $(ms)(X) = \sum_{i\in I}(ms)(V_i)$, which proves that $ms \in M(X)$ (by [12], Theorem 6.9). Let now S be a bounded subset of E. For each i, there exists a clopen subset A_i of V_i and an $s_i \in S$ such that $|m(A_i)s_i| \ge |m|_S(V_i)/2$. The set $A = \left(\bigcup_{i\in I} B_i\right)^c$ is clopen. By our hypothesis (5), we have that

$$m(X) = m(A) + \sum_{i\in I} m(A_i)$$

in E'_b. Given $\epsilon > 0$, there exists a finite subset J_o of I such that

$$\sup_{s\in S}\left| m\left(\bigcup_{i\notin J} A_i\right) s\right| < \epsilon/2$$

for each finite subset J of I containing J_o. It follows from this that, for each $i \notin J_o$, we have that $|m(A_i)s_i)| < \epsilon/2$ and hence $|m|_S(V_i) < \epsilon$.
(3) ⇒ (6). Let $V_n \downarrow \emptyset$. Then $\overline{V_n}^{\beta_o X} \downarrow Z \in \Omega_1 \subset \Omega_u$, Hence $|m|_S(V_n) \to 0$, which proves that m is strongly σ-additive. Let now d be a continuous ultrapseudometric on X and S a bounded subset of E. If $\overline{V_\delta}^{\beta_o X} \downarrow Z \in \Omega_u$,then

$$\sup_{ms\in S} |ms|(V_\delta) = |m|_S(V_\delta) \to 0.$$

Also, if $\|m\|_p \leq 1$, then

$$\sup_{s\in S} \|ms\| \leq m_p(X) \cdot \sup_{s\in S} p(s) < \infty.$$

It follows that the set $F = \{ms : s \in S\}$ is a β_e-equicontinuous subset of the dual space $M_s(X)$ of $(C_b(X), \beta_e)$ (by [12], Theorems 6.13 and 6.14). Hence the set $\Phi = T_d^\star(F)$ is a β-equicontinuous subset of the dual space $M_\tau(X_d)$ of $(C_b(X_d, \beta)$, which implies that the set $D = supp(\Phi)$ is separable, by Theorem 4.3. Now the set $A = \pi_d^{-1}(D)$ is d-closed and d-separable. If V is a d-clopen subset of $X \setminus A$, then $\pi_d(V)$ is a clopen subset of X_d which is disjoint from D. If $s \in S$, then $ms \in F$ and so $\mu_s = T_d^\star(ms) \in \Phi$. Thus $m(V)s = \mu_s(\pi_d(V)) = 0$, which completes the proof of the implication (3) → (6).
(6) ⇒ (5). Let $(V_i)_{i\in I}$ be a clopen partition of X. Define

$$d : X \times X \to \mathbf{R}, \quad d(x,y) = \sup_{i\in I} |\chi_{V_i}(x) - \chi_{V_i}(y)|.$$

Then d is a continuous ultrapseudometric on X. Let S be a bounded subset of E and let A be a d-closed, d-separable subset of X such that $m(V)s = 0$ for each $s \in S$ and each d-clopen set V disjoint from A. As A is d-separable, there exists a sequence (i_n) in I such that $A \subset B = \bigcup_n V_{i_n}$. Now B is d-clopen. Since m is strongly σ-additive, we have that

$$m(X)s = m(B^c)s + \sum_{k=1}^{\infty} m(V_{i_k})s = \sum_{k=1}^{\infty} m(V_{i_k})s = \sum_{i\in I} m(V_i)s$$

uniformly for $s \in S$. Thus $m(X) = \sum_{i\in I} m(V_i)$ in E'_b and the result follows.

Theorem 5.5. *Let $m \in M_{bs}(X, E')$. Then :*

1. *Every $f \in C_b(X, E)$ is m-integrable.*
2. *If E is polar and c-complete, then the map*

$$u_m : C_b(X, E) \to \mathbb{K}, \quad u_m(f) = \int f\, dm$$

is a member of the completion $\hat{G}_b$ of the space $G_b = (L(X, E'), e_b)$.

Proof : (1). Let $p \in cs(E)$ be such that $m_p(X) \leq 1$ and let $\epsilon > 0$. Let $(V_i)_{i\in I}$ be the clopen partition of X corresponding to the equivalence relation $x \sim y$ iff $p(f(x) - f(y)) \leq \epsilon$. If $S = f(X)$, then there exists a finite subset J of I such that $|m|_S(D) \leq \epsilon$, where $D = \bigcup_{i\notin J} V_i$. Consider the finite clopen partition $\mathcal{F} = \{V_i : i \in J\} \cup \{D\}$ of X. If $A \in \mathcal{F}$, $x, y \in A$ and V a clopen subset of A, then $|m(V)[f(x) - f(y)]| \leq \epsilon$. In view of [14], Theorem 7.1, it follows that f is m-integrable.
(2. Assume that E is polar and c-complete. Then $G'_b = C_b(X,E)$. We need to show that $u_m \in \hat{G}_b$. Let $B \in \mathcal{E}_b$. The set $S = B(X)$ is bounded. Define d on $X \times X$ by

$$d(x,y) = \sup_{f\in B} p(f(x) - f(y))$$

and let $(V_i)_{i\in I}$ be the clopen partition of X corresponding to the equivalence relation $x \sim y$ iff $d(x,y) \leq \epsilon$, where ϵ is a given positive number. Let (f_γ) be a net in B converging pointwise to some $f \in B$. There exists a finite subset J of I such that $|m|_S(\bigcup_{\imath\notin J} V_i) < \epsilon$. Let $x_i \in V_i$, $i \in J$. As in the proof of Theorem 3.3, it follows that

$$\left|\int g\,dm - \sum_{i\in J} m(V_i)g(x_i)\right| \leq \epsilon$$

for all $g \in B$. Let γ_o be such that $p(f_\gamma(x_i) - f(x_i)) < \epsilon$ for all $i \in J$ and all $\gamma \geq \gamma_o$. As in the proof of Theorem 3.3, it follows that $|\int f_\gamma\,dm - \int f\,dm| \leq \epsilon$ for all $\gamma \geq \gamma_o$. This proves that $u_m \in \hat{G}_b$ and the result follows.

Theorem 5.6. *Let E be a polar Fréchet space. Then the map*

$$m \mapsto u_m, \quad u_m(f) = \int f\,dm,$$

from $M_{bs}(X,E')$ to $\hat{G}_b$, is an algebraic isomorphism. Thus the completion of G_b is the space $M_{bs}(X,E')$ equipped with the topology of uniform convergence on the members of $\mathcal{E}_b$.

Proof : It only remains to show that every element of $\hat{G}_b$ is of the form u_m for some $m \in M_{bs}(X,E')$. So, let $u \in \hat{G}_b$. If $\mathcal{A}$ is a uniformly bounded equicontinuous subset of $C_b(X)$ and $s \in E$, then the set $B = \mathcal{A}s = \{gs : g \in \mathcal{A}\}$ is a member of $\mathcal{E}_b$. Let

$$u_s : C_b(X) \to \mathbb{K}, \quad u_s(g) = u(gs)$$

and let (g_γ) be a net in $\mathcal{A}$ which converges pointwise to some $g \in \mathcal{A}$. If $f_\gamma = g_\gamma s$, $f = gs$, then $f_\gamma \to f$ pointwise and so $u_s(g_\gamma) = u(f_\gamma) \to u(f) = u_s(g)$. In view of [15], Theorem 7.6, there exists a $\mu_s \in M_s(X)$ such that $u_s(g) = \int g\,d\mu_s$ for all $g \in C_b(X)$. Using an argument analogous to the one used in the proof of Theorem , we get that there exists a $p \in cs(E)$ such that $|u(f)| \leq 1$ if $\|f\|_p \leq 1$. Also there exists an $m \in M_p(X,E')$ such that $ms = \mu_s$ for all $s \in E$.
Claim I. If $g \in C_b(X,E)$ is of he form $g = \sum_{i\in I} \chi_{V_i} s_i$, where $(V_i)_{i\in I}$ is a clopen partition of X, then $u(g) = \sum_{i\in I} m(V_i)s_i$. Indeed, for $J \subset I$ finite, let $h_J = \sum_{i\in J} \chi_{V_i} s_i$. Then $B = \{h_J : J \quad \text{finite}\}$ is in $\mathcal{E}_b$ and $h_J \to g$ pointwise, which implies that

$$u(g) = \lim u(h_J) = \lim_J \sum_{i\in J} m(V_i)s_i = \sum_{i\in I} m(V_i)s_i.$$

Claim II. $m \in M_{bs}(X, E')$. In fact, let $(A_i)_{i\in I}$ be a clopen partition of X and let S be a bounded subset of E. For each $i \in I$, there exist a clopen subset B_i of A_i and an $s_i \in S$ such that $|m(B_i)s_i| \geq |m|_S(A_i)/2$. By claim I,

$$u(\sum_{i\in I} \chi_{B_i} s_i) = \sum_{i\in I} m(B_i)s_i.$$

Thus, given $\epsilon > 0$, there exists a finite subset J of I such that $|m(B_i)s_i| < \epsilon/2$ if $\imath \notin J$. But then, for $\imath \notin J$, we have that $|m|_S(B_i) < \epsilon$. This, together with the fact that $ms \in M_s(X)$ for all $s \in E$, implies that $m \in M_{bs}(X, E')$.
Claim III. If g is as in claim I, then $u(g) = \int g\,dm$. In fact, let $S = g(X)$ and $\epsilon > 0$. Since $m \in M_{bs}(X, E')$ and $u(g) = \sum_{i\in I} m(V_i)s_i$, there exists a finite subset J of I such that $|m|_S(V^c) < \epsilon$ and $|u(g) - \sum_{i\in J} m(V_i)s_i| < \epsilon$, where $V = \bigcup_{i\in J} V_i$. If $x \in V^c$ and A a clopen subset of V^c, then $|m(A)g(x)| < \epsilon$. This implies that $|\int_{V^c} g\,dm| \leq \epsilon$. Also, $\int_V g\,dm = \sum_{i\in J} m(V_i)s_i$. Thus

$$\left|u(g) - \int g\,dm\right| \leq \max\left\{\left|u(g) - \int_V g\,dm\right|, \quad \left|\int_{V^c} g\,dm\right|\right\} \leq \epsilon$$

and hence $u(g) = \int g\,dm$ since $\epsilon > 0$ was arbitrary.
Claim IV. $u(f) = \int f\,dm$ for all $f \in C_b(X, E)$. Indeed, let $\epsilon > 0$ and choose a $\lambda \in \mathbb{K}$ with $|\lambda| \cdot m_p(X) \leq \epsilon$, $0 < |\lambda| < \epsilon$. Let $(V_i)_{i\in I}$ be the clopen partition of X corresponding to the equivalence relation $x \sim y$ iff $p(f(x) - f(y)) \leq |\lambda|$. Let $x_i \in V_i$, $g = \sum_{i\in I} \chi_{V_i} f(x_i$. Then $\|f - g\|_p \leq |\lambda|$ and hence $|u(f - g)| \leq |\lambda|$. Also

$$\left|u(f) - \int f\,dm\right| \leq \max\left\{|u(f-g)|, \quad \left|\int (g - f)\,dm\right|\right\} \leq \epsilon.$$

Thus $u(f) = \int f\,dm$ since $\epsilon > 0$ was arbitrary. This completes the proof.

6 $M_s(X, E')$ as a Completion

We denote by $M_s(X, E')$ the space of all $m \in M(X, E')$ for which $ms \in M_s(X)$ for all $s \in E$.

Theorem 6.1. *Assume that E is polar and let $m \in M(X, E')$ be such that, for each $s \in E$ and each $g \in C_b(X)$, the function gs is m-integrable. Then every $f \in C_{co}(X, E)$ is m-integrable.*

Proof : Let p be a polar continuous seminorm on E such that $\|m\|_p \leq 1$ and let $f \in C_{co}(X, E)$.
Claim : For each $\epsilon > 0$, there are $g_1, g_2, \ldots, g_n$ in $C_b(X)$ and $s_1, s_2, \ldots, s_n$ in E such that $\|f - h\|_p \leq \epsilon$, where $h = \sum_{k=1}^n g_k s_k$. In fact, the set $Z = f(X)$ is compactoid in E. Since E is polar, it has the approximation property. Thus, there exists a continuous linear map $\phi : E \to E$, of finite rank, such that $p(s - \phi(s)) \leq \epsilon$ for all $s \in Z$. Let $x'_1, x'_2, \ldots, x'_n \in E'$ and $s_1, s_2, \ldots, s_n \in E$ be such that $\phi(s) = \sum_{k=1}^n x'_k(s)s_k$, for all $s \in E$. If $g_k = x'_k \circ f$ and $h = \sum_{k=1}^n g_k s_k$, then $\|f - h\|_p \leq \epsilon$, which proves our claim.
In view of our hypothesis, h is m-integrable and so (by [14], Theorem 7.1) there exists a clopen partition $\{A_1, A_2 \ldots, A_N\}$ of X such that, if $x, y \in A_k$, then $|m(B)[h(x) -$

$h(y)]\| \leq \epsilon$ for all clopen subsets B of A_k. For B a clopen subset of A_k and $x \in A_k$, we have

$$|m(B)[f(x) - h(x)]| \leq m_p(X) \cdot p(f(x) - h(x)) \leq \epsilon.$$

Thus, for $B \subset A_k$ and $x, y \in A_k$, we have $|m(B)[f(x) - f(y)]| \leq \epsilon$. In view of [14], Theorem 7.1, it follows that f is m-integrable.

Theorem 6.2. *Assume that E is polar and c-complete and let $G_{co} = (L(X, E'), e_{co})$. If $m \in M_s(X, E')$, then the map*

$$v_m : C_{co}(X, E) \to \mathbb{K},\ v_m(f) = \int f\, dm,$$

is a member of the completion $\hat{G_{co}}$ of G_{co}.

Proof : For each $s \in E$, we have that $ms \in M_s(X)$ and thus every $g \in C_b(X)$ is (ms)-integrable. In view of the preceding Theorem, every $f \in C_{co}(X, E)$ is m-integrable. Let $B \in \mathcal{E}_{co}$. We may assume that B is absolutely convex. The set $Z = B(X)$ is compactoid. Let $p \in cs(E)$ be polar and such that $\|m\|_p \leq 1$ and let $\epsilon > 0$. There are $x'_1, x'_2, \ldots, x'_n \in E'$ and $s_1, s_2, \ldots, s_n \in E$ such that

$$p(s - \sum_{k=1}^{n} x'_k(s) s_k) < \epsilon$$

for all $s \in Z$. Let now (f_γ) be a net in B, which converges pointwise to the zero function, and let $h^k_\gamma = x'_k \circ f_\gamma$. The set $\mathcal{A}_k = \{x'_k \circ g : g \in B\}$ is uniformly bounded and equicontinuous. Moreover, $h^k_\gamma \to 0$ pointwise. Since $ms_k \in M_s(X)$, it follows that $\int h^k_\gamma\, d(ms_k) \to 0$, by [15], Theorem 7.6. If $g_\gamma = \sum_{k=1}^{n} h^k_\gamma s_k$, then $\int g_\gamma\, dm \to 0$. Also

$$\left| \int (f_\gamma - g_\gamma)\, dm \right| \leq \|f_\gamma - h_\gamma\|_p \cdot m_p(X) \leq \epsilon.$$

Thus, there exists γ_o such that $|\int f_\gamma\, dm| \leq \epsilon$ for all $\gamma \geq \gamma_o$. This proves that $v_m \in \hat{G_{co}}$.

Theorem 6.3. *Let E be polar and c-complete and let $v \in \hat{G_{co}}$. Then :*

1. *For each $s \in E$, there exists a $\mu_s \in M_s(X)$ such that $v(gs) = \int g\, d\mu_s$ for all $g \in C_b(X)$.*

2. *v is sequentially continuous with respect to the topology of uniform convergence on $C_{co}(X, E)$.*

Proof : (1). If $\mathcal{A}$ is a uniformly bounded equicontinuous subset of $C_b(X)$, then, for each $s \in E$, the set $\mathcal{A}s = \{gs : g \in \mathcal{A}\}$ is in $\mathcal{E}_{co}$. As in the proof of Theorem 4.4, there exists a $\mu_s \in M_s(X)$ such that $v(gs) = \int g\, d\mu_s$ for all $g \in C_b(X)$.
(2) Let (f_n) be a sequence in $C_{c_o}(X, E)$ which is uniformly convergent to the zero function. If $p \in cs(E)$ and $V = \{s \in E : p(s) \leq 1\}$, then there exists a k such that $f_n(X) \subset V$ for all $n > k$. Since the set $\bigcup_{n=1}^{k} f_n(X)$ is compactoid, it follows that the set $\bigcup_{n=1}^{\infty} f_n(X)$ is compactoid. Hence the set $B = \{f_n : n \in \mathbf{N}\}$ is in $\mathcal{E}_{co}$. Also $f_n \to 0$ pointwise and hence $v(f_n) \to 0$. Thus the result follows.

Theorem 6.4. *Let E be a polar Fréchet space. Then the map $m \mapsto v_m$, from $M_s(X, E')$ to $\hat{G}_{co}$, is an algebraic isomorphism. Therefore the completion of G_{co} is the space $M_s(X, E')$ equipped with the topology of uniform convergence on the members of $\mathcal{E}_{co}$.*

Proof : Let $v \in \hat{G_{co}}$. Since E is metrizable, the topology of uniform convergence on $C_{co}(X, E)$ is metrizable. By the preceding Theorem, there exists a continuous seminorm p on E such that

$$\{f \in C_{co}(X, E) : \|f\|_p \leq 1\} \subset \{f : |v(f)| \leq 1\}.$$

Now $v|_{C_{rc}(X,E)}$ is continuous with respect to the topology of uniform convergence and hence there exists a $m \in M(X, E')$ such that $\int f \, dm = v(f)$ for all $f \in C_{rc}(X, E)$. It is easy to see that $ms = \mu_s$, for all $s \in E$, and so $m \in M_s(X, E')$. As we have seen in the proof of Theorem 5.1, the space F spanned by the functions gs, $s \in E$ and $g \in C_b(X)$, is dense in $C_{co}(X, E)$, with respect to the topology τ_u of uniform convergence. Since both v and v_m are τ_u-continuous and they coincide on F, it follows that $v = v_m$ on $C_{co}(X, E)$. This clearly completes the proof.

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