On Ideals of the Algebra of p-adic Bounded Analytic Functions on a Disk

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Abstract

Let K be an algebraically closed field, complete for a non-trivial ultrametric absolute value. We denote by A the K- Banach algebra of bounded analytic functions in the unit disk $\{x \in K \mid |x| < 1\}$. We study some properties of ideals of A. We show that maximal ideals of infinite codimension are not of finite type and that A is not a Bezout ring.

1 Introduction and Results

Definitions and notation: Let K be an algebraically closed field complete with respect to a non-trivial ultrametric absolute value $|\ .\ |$.

Given $a \in K$ and $r,\ s \in]0, +\infty[$ $(r < s)$, we put $d(a, r) = \{x \in K \mid\ |x - a| \leq r\}$, $d(a, r^-) = \{x \in K \mid |x - a| < r\}$ and $\Gamma(a, r, s) = \{x \in K \mid r < |x - a| < s\}$.

We denote by A the K-algebra of bounded power series converging inside $d(0, 1^-)$.

Given $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $r \in]0, 1]$, we put $|f|(r) = \sup_{n \in \mathbb{N}} |a_n| r^n$ and $\|f\| = |f|(1)$. The multiplicative norm $\|\ .\ \|$ defined on A makes A a K-Banach algebra, [1, 2].

One of the main differences between p-adic and complex analytic functions consists in the existence of sequences of zeroes for some elements of A. This is recalled in Theorem A, [1] (theorem 25.5) and [7].

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Theorem A: *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of $d(0,1^-)$ such that $|a_n| \leq |a_{n+1}|$, $\forall n \in \mathbb{N}$, and $\lim_{n\to+\infty} |a_n| = 1$. Let $(q_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ and $B \in]1,+\infty[$. There exists $f \in A$ satisfying*

1. $f(0) = 1$,
2. $\sup\{|f(x)| \mid x \in d(0,|a_n|)\} \leq B \prod_{j=0}^{n} \left|\frac{a_n}{a_j}\right|^{q_j}$, $\forall n \in \mathbb{N}$,
3. *a_n is a zero of f of order $s_n \geq q_n$, $\forall n \in \mathbb{N}$.*

Moreover, if K is spherically complete, for every sequence $(a_n)_{n\in\mathbb{N}}$ of $d(0,1^-)$ such that $\lim_{n\to+\infty} |a_n| = 1$ and for every sequence of positive integers $(s_n)_{n\in\mathbb{N}}$, there exist functions $f \in A$ admitting each a_n as a zero of order s_n and having no other zero.

If K is not spherically complete, there exist sequences $(a_n)_{n\in\mathbb{N}}$ of $d(0,1^-)$ such that $\lim_{n\to+\infty} |a_n| = 1$ and sequences of positive integers $(s_n)_{n\in\mathbb{N}}$ such that no function $f \in A$ admits each a_n as a zero of order s_n and has no other zero.

Theorem B: *Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of $d(0,1^-)$ such that $0 < |\alpha_n| < |\alpha_{n+1}|$, $\forall n \in \mathbb{N}$, and $\lim_{n\to+\infty} |\alpha_n| = 1$. If the ideal I of the $f \in A$ such that $\lim_{n\to+\infty} f(\alpha_n) = 0$ is not null, it is not of finite type.*

Remark and definition: In a complex Banach algebra, every maximal ideal has codimension 1, [5], [4]. This is not the same on an ultrametric field. The maximal ideals of codimension 1 are easily characterized by the points of $d(0,1^-)$ e.g. a maximal ideal of codimension 1 of A is of the form $(x-a)A$, where $|a| < 1$. But there also exist maximal ideals of infinite codimension. They are called *non-trivial maximal ideals of A*, [1, 2].

Recall that a ring is called *a Bezout ring* if it has no divisor of zero and if any ideal of finite type is principal.

Theorem C: *Non-trivial maximal ideals of A are not of finite type.*

Theorem D: *A is not a Bezout ring.*

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2 The Proofs

Definitions and notation: Let D be a closed bounded subset of K. We denote by $R(D)$ the K-algebra of rational functions without pole in D. It is provided with the K-algebra norm of uniform convergence on D that we denote by $\| \,.\, \|_D$. We then denote by $H(D)$ the completion of $R(D)$ for the topology of uniform convergence on D: $H(D)$ is a Banach K-algebra whose elements are called the *analytic elements on D*, [1, 6]. It is known that if $f \in A$ then $f \in H(d(0,r))$, $\forall r \in]0,1[$, [1] (Th. 13.3).

For $a \in K$ and $r > 0$, we call *circular filter of center a and diameter r on K* the filter $\mathcal{F}$ which admits as a generating system the family of sets $\Gamma(\alpha, r', r'')$ with

$\alpha \in d(a,r), r' < r < r''$, i.e. $\mathcal{F}$ is the filter which admits for base the family of sets of the form $\bigcap_{i=1}^{q} \Gamma(\alpha_i, r'_i, r''_i))$ with $\alpha_i \in d(a,r), r'_i < r < r''_i \ \ (1 \le i \le q\ ,\ q \in \mathbb{N})$.

We call *circular filter with no center, of diameter r of canonical base* $(D_n)_{n\in\mathbb{N}}$ a filter admitting for base a sequence $(D_n)_{n\in\mathbb{N}}$ where each D_n is a disk $d(a_n, r_n)$, such that $\bigcap_{n=1}^{\infty} d(a_n, r_n) = \emptyset$ and $\lim_{n\to\infty} r_n = r$ [1], [2], [3]

Finally the filter of neighborhoods of a point $a \in K$ is called *circular filter of center a and diameter* 0 or *Cauchy circular filter of limit a*.

A circular filter is said to be *large* if it has diameter different from 0. If $\mathcal{F}$ is a large circular filter secant to some disk $d(0,r)$, then for any $f \in H(d(0,r))$, the limit $\lim_{\mathcal{F}} |f(x)|$ exists and is strictly positive if $f \neq 0$, [1].

A sequence $(u_n)_{n\in\mathbb{N}}$ in L is said to be *an increasing distances sequence* (resp. *a decreasing distances sequence*) if the sequence $|u_{n+1} - u_n|$ is strictly increasing (resp. decreasing) and has a limit $\ell \in \mathbb{R}_+^*$.

The sequence $(u_n)_{n\in\mathbb{N}}$ will be said to be *a monotonous distances sequence* if it is either an increasing distances sequence or a decreasing distances sequence.

A sequence $(u_n)_{n\in\mathbb{N}}$ in L will be said to be *an equal distances sequence* if $|u_n - u_m| = |u_m - u_q|$ whenever $n, m, q \in \mathbb{N}$ such that $n \neq m \neq q \neq n$.

Lemma 1: *Let* $(\alpha_n)_{n\in\mathbb{N}}$ *be a sequence of* $d(0,1^-)$ *without any cluster point and let* $f \in A,\ f \neq 0$, *such that* $\lim_{n\to+\infty} f(\alpha_n) = 0$. *Then* $\lim_{n\to+\infty} |\alpha_n| = 1$.

Proof. Suppose the lemma is false. Then there exists a disk $d(0,s) \subset d(0,1^-)$ containing a subsequence of $(\alpha_n)_{n\in\mathbb{N}}$ and by Theorem 3.1, [1], we can extract a subsequence which is either a monotonous distances sequence or an equal distances sequence. Therefore, by Proposition 3.15, [1], there exists a unique large circular filter $\mathcal{F}$ secant with $d(0,s)$ and less thin than this subsequence. Since, by Lemma 12.5 [1] $|f(x)|$ has a limit $\varphi_{\mathcal{F}}(fs)$ along $\mathcal{F}$ we then have $\lim_{\mathcal{F}} f(x) = 0$. On the other hand, the restriction of f to $d(0,s)$ belongs to $H(d(0,s))$. Now, by Proposition 40.1 in [1], $\varphi_{\mathcal{F}}$ is an absolute value on $H(d(0,s))$, so $\lim_{\mathcal{F}} f(x) = 0$ implies $f = 0$.

Lemma 2 is immediate:

Lemma 2: *Let* $f \in A$. *Then* $|f(x) - f(y)| \le \|f\| |x - y|$.

Corollary: *Let* $(\alpha_n)_{n\in\mathbb{N}}$ $(\beta_n)_{n\in\mathbb{N}}$ *be sequences of* $d(0,1^-)$ *such that* $\lim_{n\to+\infty} |\alpha_n| = 1$ *and* $\lim_{n\to+\infty} \alpha_n - \beta_n = 0$. *The ideal of the* $f \in A$ *such that* $\lim_{n\to+\infty} f(\alpha_n) = 0$ *is equal to the ideal of the* $f \in A$ *such that* $\lim_{n\to+\infty} f(\beta_n) = 0$.

Lemma 3 is given in [9] as (3.1):

Lemma 3: *Let* $f_1, ..., f_q \in A$ *satisfying* $\inf_{x\in D}(\max(|f_1(x)|, ..., |f_q(x)|)) > 0$. *Then there exist* $g_1, ..., g_q \in A$ *such that* $\sum_{j=1}^{q} g_j f_j = 1$.

Proof of Theorem B. Suppose $I \neq \{0\}$ and suppose that there exist $f_1, \cdots, f_q \in I$ such that $I = \sum_{j=1}^{q} f_j A$.

Since the zeroes of each f_j are isolated, we can obviously find a sequence $(\beta_n)_{n\in\mathbb{N}}$ in $d(0, 1^-)$ such that $|\alpha_n| = |\beta_n|$ $\forall n \in \mathbb{N}$, $f_j(\beta_n) \neq 0$ $\forall j = 1, ..., q$ $\forall n \in \mathbb{N}$ and $\lim_{n\to+\infty} f(\beta_n) = 0$. Then by the Corollary of Lemma 2, I is the ideal of the $f \in A$ such that $\lim_{n\to+\infty} f(\beta_n) = 0$. Thus, without loss of generality, we may assume that $f_j(\alpha_n) \neq 0$ $\forall j = 1, ..., q$ $\forall n \in \mathbb{N}$.

Now, since $\lim_{n\to+\infty} \max_{1\le j\le q}(|f_j(\alpha_n)|) = 0$, we can extract a subsequence $(\alpha_{\tau(m)})_{m\in\mathbb{N}}$ such that

$$\max_{1\le j\le q}(|f_j(\alpha_{\tau(m)})|) < \max_{1\le j\le q}(|f_j(\alpha_{\tau(m-1)})|) \ \forall m \in \mathbb{N}.$$

Then, for at least one of the index k (among $1, ..., q$) the equality $\max_{1\le j\le q}(|f_j(\alpha_{\tau(m)})|) = |f_k(\alpha_{\tau(m)}|$ holds for infinitely many integers m. Thus we can extract a new sequence $(\alpha_{\tau(\phi(m))})_{m\in\mathbb{N}}$ such that $\max_{1\le j\le q}(|f_j(\alpha_{\tau(\phi(m))})|) = |f_k(\alpha_{\tau(\phi(m))}|$ $\forall m \in \mathbb{N}$.

Set $t(m) = \tau(\phi(m))$. Thus, we have $\max_{1\le j\le q}(|f_j(\alpha_{t(m))})|) = |f_k(\alpha_{t(m))}|$ $\forall m \in \mathbb{N}$. For convenience, we may suppose $k = 1$ and set $M = \|f_1\|$. For each $m \in \mathbb{N}$, set $r_m = |\alpha_{t(m)}|$, let $(\gamma_j)_{1\le j\le u(m)}$ be the finite sequence of the zeroes of f_1 in $d(0, r_m)$ and let s_j be the order of γ_j $(1 \le j \le u(m)$.

Now, consider $\psi_m = \dfrac{f_1}{\prod_{j=1}^{u(m)}(1 - \frac{x}{\gamma_j})^{s_j}}$. Since ψ_m has no zero in $d(0, r_m)$, by Theorem 23.6 [1], we know that $|\psi_m(x)| = |\psi_m(0)| = |f_1(0)|$, $\forall x \in d(0, |r_m|)$.

Next, since $\prod_{j=1}^{u(m)}(1 - \frac{x}{\gamma_j})^{s_j}$ has no zeroes in $\Gamma(0, r_m, 1)$ and has all its zeroes in $d(0, r_m)$, we know that $\left|\prod_{j=1}^{u(m)}(1 - \frac{x}{\gamma_j})^{s_j}\right| \ge \prod_{j=1}^{u(m)}(\frac{|x|}{|\gamma_j|})^{s_j}$ $\forall x \in \Gamma(0, r_m, 1)$, hence $\|\psi_m\| \le M$.

By induction, we can clearly define a sequence $(\lambda_m)_{m\in\mathbb{N}}$ in K such that $\sqrt{|f_1(\alpha_{t(m)})|} \le |\lambda_m| < \sqrt{|f_1(\alpha_{t(m-1)})|}$, $\forall m \ge 1$ and satisfying further for each $m \in \mathbb{N}$ $|\lambda_m \psi_m(\alpha_{t(m)})| \neq |\lambda_j \psi_j(\alpha_{t(m)})|$ $\forall j \neq m$. Since $\lim_{m\to+\infty} |\lambda_m| = 0$ and since $\|\psi_m\| \le M$, the series $h = \sum_{m=0}^{+\infty} \lambda_m \psi_m$ converges in A. Then, since the $|\lambda_j \psi_j(\alpha_{t(m)})|$ are all distinct, we have $|h(\alpha_{t(m)})| = \max_{j\in\mathbb{N}} |\lambda_j \psi_j(\alpha_{t(m)})| \ge |\lambda_m \psi_m(\alpha_{t(m)})| \ge |\lambda_m f_1(0)|$ (because $|\psi_m(x)| = |f_1(0)|$ $\forall x \in d(0, r_m)$), hence $|h(\alpha_{t(m)})| \ge \sqrt{|f_1(\alpha_{t(m)})|}$ i.e. $|h(\alpha_{t(m)})| \ge \max_{1\le j\le q} \sqrt{|f_j(\alpha_{t(m)})|}$. Consequently $\lim_{n\to+\infty} \dfrac{|h(\alpha_{t(m)})|}{\max_{1\le j\le q} |f_j(\alpha_{t(m)})|} = +\infty$ and therefore h does not belong to I.

But now, we notice that for each $n > t(m)$, we have

$$|h(\alpha_n)| = |\sum_{n=0}^{\infty} \lambda_m \psi_m(\alpha_n)| \le \sup_{m\in\mathbb{N}} |\lambda_m||f_1(\alpha_n)|,$$

hence $\lim_{n\to+\infty} h(\alpha_n) = 0$ and hence, h belongs to I, a contradiction that finishes the proof.

Proof of Theorem C. Let $\mathcal{M}$ be a non-trivial maximal ideal of A and let us suppose that $\mathcal{M} = \sum_{j=1}^{q} f_j A$. By Lemma 3 there exists a sequence $(\beta_s)_{s\in\mathbb{N}}$ in $d(0, 1^-)$ such that $\lim_{s\to\infty} |f_j(\beta_s)| = 0$, for any $j = 1, .., q$ because if such a sequence does not exist, then $\sum_{j=1}^{q} f_j A = A$.

If the sequence $(\beta_s)_{s\in\mathbb{N}}$ has a cluster point $a \in d(0,1^-)$, then $f_j(a) = 0$ for any $j = 1,..,q$, hence $f(a) = 0\ \forall f \in \mathcal{M}$ and it follows that $\mathcal{M}$ is the ideal of the $f \in A$ such that $f(a) = 0$. By Corollary 13.4 [1] we know that such functions factorize in the form $(x-a)g$, with $g \in A$, hence $\mathcal{M} = (x-a)A$ a contradiction. Hence the sequence $(\beta_s)_{s\in\mathbb{N}}$ has no cluster point. Then, by Lemma 1, we can extract a subsequence $(\alpha_n)_{n\in\mathbb{N}}$, where $\alpha_n = \beta_{\sigma(n)}$, $\forall n \in \mathbb{N}$, such that $0 < |\alpha_n| < |\alpha_{n+1}|$, $\lim_{n\to+\infty} |\alpha_n| = 1$. We then have $\lim_{n\to\infty} f_j(\alpha_n) = 0$, for any $j = 1,..,q$ and hence $\lim_{n\to\infty} f(\alpha_n) = 0$, for any $f \in \mathcal{M}$. But since $\mathcal{M}$ is maximal, $\mathcal{M}$ is the ideal of the $f \in A$ such that $\lim_{n\to\infty} f(\alpha_n) = 0$ and so $\mathcal{M}$ is not of finite type by Theorem B.

Proof of Theorem D. Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence of K such that the sequence $(|\frac{a_n}{a_{n+1}}|)$ is strictly increasing. Let $f(x) = \sum_{n=0}^{+\infty} a_n x^n$, and for any $n \in \mathbb{N}$, set $r_n = |\frac{a_n}{a_{n+1}}|$. Since the sequence $(a_n)_{n\in\mathbb{N}}$ is bounded, we know that f belongs to A. Then, by Theorem 23.15 ([1]), we know that f admits a unique zero $\alpha_n \in C(0, r_n)$, of order 1, for any $n \in \mathbb{N}$ and does not admit any other zero.

Let (β_n) be a sequence of $d(0,1^-)$ such that $\beta_n \in C(0, r_n)$, $0 < |\alpha_n - \beta_n| < r_n$, $\lim_{n\to+\infty}(\beta_n - \alpha_n) = 0$. For any $\rho > 0$, we set $D_\rho = d(0,1^-) \setminus \bigcup_{n=0}^{+\infty} d(\alpha_n, \rho^-)$. We then know that the meromorphic product $u(x) = \prod_{n=0}^{+\infty} \frac{x-\beta_n}{x-\alpha_n}$ converges in $H(D_\rho)$, for any $\rho > 0$, [1, 8].

On the other hand, for any $s \in]0, \rho[$, we know that the restriction of f to $d(0,s)$ belongs to $H(d(0,s))$, [1], (Proposition 13.3). We set $D_{\rho,s} = D_\rho \cap d(0,s)$. Let $g = uf$. Then u belongs to $H(D_{\rho,s})$ and in each hole $d(\alpha_n, \rho^-)$ of $D_{\rho,s}$, g is meromorphic in this hole ([1], Chap. 31) but does not admit any pole. Hence $g \in H(d(0,s))$ for any $s < \rho$. Moreover, we see that $|f(x)| = |g(x)|$, for any $x \in d(0,1^-) \setminus \bigcup_{n=0}^{+\infty} d(\alpha_n, r_n^-)$ because $|u(x)| = 1$ in this set. Thus, we have, $\lim_{|x|\to 1} |f(x)| = \lim_{|x|\to 1} |g(x)|$, hence g is bounded in $d(0,1^-)$; i.e. $g \in A$.

Now, by construction, the β_n are the only zeroes of g. So, f and g have no common zero. Let $I = fA + gA$. Next, since $\lim_{n\to+\infty}(\beta_n - \alpha_n) = 0$ by Lemma 2 we see that $\lim_{n\to+\infty} f(\beta_n) = 0$, hence $\lim_{n\to+\infty} \phi(\beta_n) = 0$, $\forall \phi \in I$. Suppose that I is a principal ideal, generated by some $h \in A$. Obviously, $\lim_{n\to+\infty} h(\beta_n) = 0$. But since f and g have no common zero, h does not admit any zero in $d(0,1^-)$ because any zero of h would be a common zero of f and g. Now, by Theorem 23.6 ([1]), any function $\phi \in A$ which does not admit any zero in $d(0,1^-)$ satisfies $|\phi(x)| = |\phi(0)|$, $\forall x \in d(0,1^-)$, hence $|h(\beta_n)| = |h(0)|\ \forall n \in \mathbb{N}$, a contradiction to $\lim_{n\to+\infty} h(\beta_n) = 0$. Hence A is not a Bezout ring.

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