# Regularity in $p$-adic inductive limits

N. De Grande-De Kimpe C. Perez-Garcia *

**Abstract**

This paper consists of a survey of the most important results on $p$-adic inductive limits obtained by the authors in recent years, together with some new results on the subject. It is mainly devoted to regularity properties and their relations with strictness and closedness properties. As a product we get the $p$-adic version of the Dieudonné-Schwartz Theorem.

## Introduction

A first systematic study of $p$-adic inductive limits was carried out in 1997, [9]. Since then the authors of this paper continued the investigation on the subject. A survey of their most important results together with some new ones can be found here.

Although some of the conditions (on strictness, closedness and regularity) are inspired by the real or complex case, the results may be different in the non-archimedean context. This happens especially when the spherical completeness of the underlying field of scalars is involved. These differences are pointed out in this paper, which also provides counterexamples to every implication that appears there.

As usual, the Preliminaries (Section 1) contain the necessary material to read the paper. Then, Section 2 is devoted to conditions of strictness and closedness and the implications between them (for that we follow [13]). In Section 3, which mostly contains the new results, the same is done for regularity conditions. Section 4 covers the essential results of [14], related to the validity of the $p$-adic version of the classical Dieudonné-Schwartz Theorem ("*Every strict LF-space is regular*", [6], Proposition 4) and of its extensions and improvements given later in the archimedean literature (see e.g. [15], Theorem 2.12.2, [16], [17], [18] and [22]). Finally, in Section 5, we present some classes of inductive sequences that turn up to be very useful for the

*Research partially supported by Ministerio de Educación y Ciencia, MTM2006-14786

2000 *Mathematics Subject Classification :* Primary 46S10; Secondary: 46A13.

*Key words and phrases :* $p$-adic inductive limits, regularity.

construction of most of our (counter)examples, which either do not have a classical counterpart or if they have one it has a typically archimedean character.

## 1 Preliminaries

Throughout this paper $K = (K, |\,.\,|)$ is a non-archimedean non-trivially valued field that is complete with respect to the metric induced by the valuation $|\,.\,|$.

Unless explicitly stated otherwise, all the vector spaces and locally convex spaces we will consider in this paper are over $K$.

For fundamentals on normed and locally convex spaces we refer to [24] and [25] respectively.

Let $E$ be a vector space. For a subset $A$ of $E$ we denote by $[A]$ its linear hull. $A$ is **absolutely convex** if $0 \in A$ and $x, y \in A$, $\lambda, \mu \in K$, $\max(|\lambda|, |\mu|) \leq 1$ implies $\lambda\, x + \mu\, y \in A$. For an absolutely convex set $A \subset E$ we define $A^e := A$ if the valuation of $K$ is discrete, $A^e := \bigcap\{\lambda\, A : \lambda \in K,\ |\lambda| > 1\}$ otherwise. $A$ is called **edged** if $A = A^e$. By $E^*$ we mean the **algebraic dual** of $E$ i.e. the vector space of all functionals $E \to K$. For $f \in E^*$, $\operatorname{Ker} f := \{x \in E : f(x) = 0\}$ and, for $A \subset E$, $f|A$ is the restriction of $f$ to $A$.

Let $E = (E, \tau)$ be a locally convex space. For a subset $A$ of $E$ we denote by $\overline{A}^{\tau}$ the closure of $A$ in $E$, and by $\tau|A$ the restriction of $\tau$ to $A$. The set $A$ is called **compactoid** if for every zero neighbourhood $U$ in $E$ there is a finite set $B$ in $E$ such that $A \subset U + \operatorname{aco} B$, where $\operatorname{aco} B$ is the absolutely convex hull of $B$, that is, the smallest absolutely convex set containing $B$. If $F$ is another locally convex space, a linear map $T : E \to F$ is called **compact** if there is a zero neighbourhood $V$ in $E$ for which $T(V)$ is a compactoid in $F$. Also, $E$ is called **nuclear** if for every continuous seminorm $p$ on $E$ there is a continuous seminorm $q$ on $E$, $q \geq p$, such that the natural map $E_q \to E_p$ is compact, where $E_p$ and $E_q$ are the canonical normed spaces associated to $p$ and $q$ respectively.

By $E'$ we denote the **dual** of $E$ i.e. the vector space of all continuous functionals $E \to K$. The following basic result will be frequently used through the paper. We omit its proof, which is a simple adaptation of the classical one given in Theorem III.1.4 of [28].

**Proposition 1.1** *Let $E$ be a locally convex space, let $f \in E^*$. Then $f$ is continuous (i.e. $f \in E'$) if and only if* $\operatorname{Ker} f$ *is closed in $E$.*

The **weak topology** $\sigma(E, E')$ is the locally convex topology on $E$ generated by the family of seminorms $\{|f| : f \in E'\}$. Let $Z$ be a subspace of $E$, endow $Z$ with the induced topology $\tau|Z$. We say that $Z$ has the **Hahn-Banach Extension Property (HBEP)** in $E$ if every $f \in Z'$ has a continuous linear extension to the whole space.

A continuous seminorm $p$ on $E$ is called **polar** if $p = \sup\{|f| : f \in E',\ |f| \leq p\}$. $E$ is called **polar** if its topology is generated by a family of polar seminorms; **strongly polar** if every continuous seminorm $p$ on $E$ with $p(E) \subset \overline{|K|}$ is polar (where $\overline{|K|}$ is the closure in $\mathbb{R}$ of $\{|\lambda| : \lambda \in K\}$). $E$ is called **of countable type** if for every continuous seminorm $p$ on $E$ the associated normed space $E_p$ is of countable type (recall that a normed space is said to be of countable type if it is the closed linear hull of a countable set). If $K$ is spherically complete, every locally convex space is strongly polar. For any $K$, strongly polar spaces are polar and spaces of countable type are strongly polar. Also, every nuclear space, in particular $(E, \sigma(E, E'))$, is of countable type. One verifies that each closed subspace of a strongly polar space (resp. each finite dimensional closed subspace of a polar space) is weakly closed and has the HBEP.

Now suppose that $K$ is not spherically complete. In this case the situation is less satisfactory and at the same time more exciting from the non-archimedean point of view. There exist Banach spaces having a trivial dual, which leads to the construction of polar Banach spaces having proper closed subspaces that are weakly dense (so they are not weakly closed) and that do not have the HBEP. Indeed, $(\ell^\infty/c_0)' = \{0\}$ ([24], Corollary 4.3). It follows that $c_0$ is a weakly dense subspace of the polar Banach space $\ell^\infty$ ([12], Example 3, p. 257) without the HBEP in $\ell^\infty$ ([24], Theorem 4.15). There exist even Banach spaces with orthonormal bases (hence polar) having closed subspaces with the above properties (note that $\ell^\infty$ does not have an orthonormal base, [24], Corollary 5.19). In fact, one can find a set $I$ with cardinality large enough such that $\ell^\infty/c_0$ is isometrically isomorphic to a quotient $c_0(I)/Z$ for some closed subspace $Z$ of $c_0(I)$. Take $x \in c_0(I)$, $x \notin Z$. Then $Z + Kx$ is a closed subspace of $c_0(I)$ that is weakly dense and does not have the HBEP in $c_0(I)$; we leave its verification to the reader. Finally we recall the existence of a proper closed subspace of $\ell^\infty$ that is weakly dense and has the HBEP in $\ell^\infty$ ([26], Remark after Proposition 1.5) and the existence of a set $I$ such that $\ell^\infty(I)$ has a weakly closed subspace without the HBEP in $\ell^\infty(I)$ ([26], Example 3.3).

A very interesting class of locally convex spaces, to which is devoted the present paper, is formed by the locally convex inductive limits. We point out the central role that they play in the definition of a $p$-adic Laplace and Fourier Transform given in [10] and [11] respectively and in the index theory of $p$-adic differential equations (see e.g. [2], [3], [4], [5] and [23]). The last of these references shows also the influence of inductive limits in the study of the $p$-adic Monsky-Washnitzer cohomology.

An **inductive sequence** is an increasing sequence $E_1 \subset E_2 \subset \ldots$ of subspaces of a vector space $E$ such that $E = \bigcup_n E_n$ and where, for each $n$, $E_n$ is provided with a locally convex topology $\tau_n$ in such a way that each inclusion $E_n \to E_{n+1}$ is continuous. The **inductive limit** of the sequence $(E_n)_n$ is the space $E$ endowed with the strongest locally convex topology $\tau_{ind}$ for which all the inclusions $E_n \to E$ are continuous.

If the steps $E_n$ are normed (resp. Banach, metrizable, Fréchet) spaces then $(E_n)_n$ is called an LN (resp. LB, LM, LF)-space. As usual, a **Fréchet** space is a metrizable complete locally convex space.

We finish these Preliminaries with examples of sequence spaces, which will be used through the paper.

Let $B := (b_k^n)_{k,n}$ be an infinite matrix consisting of strictly positive real numbers such that $b_k^n \le b_k^{n+1}$ for all $k, n$. For each $n \in \mathbb{N}$,

$$c_0(\mathbb{N}, 1/b^n) := \{(\lambda_k)_k \in K^{\mathbb{N}} : \lim_k |\lambda_k| / b_k^n = 0\},$$

is a Banach space of countable type under the norm $(\lambda_k)_k \mapsto \max_k |\lambda_k| / b_k^n$. The monotonicity condition we imposed on the matrix $B$ implies that $(c_0(\mathbb{N}, 1/b^n))_n$ is an inductive sequence. Its inductive limit, the so called **Köthe dual space**, is usually denoted by $\Lambda_0(B)$.

Further, for each $j$,

$$c_0(\mathbb{N}, b^j) := \{(\lambda_k)_k \in K^{\mathbb{N}} : \lim_k |\lambda_k| \; b_k^j = 0\},$$

is a Banach space of countable type under the norm $p_j$ defined by

$$p_j((\lambda_k)_k) := \max_k |\lambda_k| \; b_k^j, \qquad (\lambda_k)_k \in c_0(\mathbb{N}, b^j). \tag{1}$$

We consider on the so-called **Köthe space** $\Lambda^0(B) := \bigcap_j c_0(\mathbb{N}, b^j)$ the **normal topology**, $n_{0,\infty}$, which is the one defined by the family of norms $\{p_j : j \in \mathbb{N}\}$. Then $(\Lambda^0(B), n_{0,\infty})$ is a Fréchet space of countable type. Note that if $b_k^n = 1$ for all $n, k$ then $\Lambda^0(B)$ is the space $c_0$ of sequences in $K$ tending to 0, equipped with the usual supremum norm. By $c_{00}$ we denote the subspace of $c_0$ generated by the canonical unit vectors.

When $b_k^n = n^k$, $\Lambda_0(B)$ is the space of germs of analytic functions at zero, and $\Lambda^0(B)$ is the space of entire functions on $K$. For more details on $\Lambda_0(B)$ and $\Lambda^0(B)$, see Section 3.2 of [9].

From now on in this paper $(E_n)_n$ is an inductive sequence of locally convex spaces $(E_n, \tau_n)$ with inductive limit $(E, \tau_{ind})$.

## 2 Strictness and closedness properties

Apart from Examples 2.9, the context of this section is in [13], where there are the details of the proofs.

First we consider some strictness properties.

**Definition 2.1** We say that $(E_n)_n$ is

(i) **strict** if $\tau_{n+1}|E_n = \tau_n$ for each $n$,

(ii) **weakly strict** if $\sigma_{n+1}|E_n = \sigma_n$ for each $n$, or equivalently ([13], Proposition 2.6), if $(E_n, \sigma_{n+1}|E_n)' = E_n'$ for each $n$ (by $\sigma_n$ we denote the weak topology on $E_n$),

(iii) **almost weakly strict** if $(E_n, \tau_{n+1}|E_n)' = E_n'$ for each $n$.

**Proposition 2.2** ([13], Corollary 2.9, Propositions 2.11, 2.12)

(i) *Strictness* $\longrightarrow$ *almost weak strictness.*

*If, in addition, all the* $E_n$ *are metrizable and polar then strictness is equivalent to almost weak strictness.*

(ii) *Weak strictness* $\longrightarrow$ *almost weak strictness.*

*If, in addition, all the* $E_n$ *are strongly polar then weak strictness is equivalent to almost weak strictness.*

In some special cases there are relations between strictness and weak strictness.

**Proposition 2.3** ([13], Proposition 2.3)
(i) *If all the $E_n$ are strongly polar then strictness implies weak strictness.*
(ii) *If all the $E_n$ are metrizable and polar then weak strictness implies strictness.*

But in general strictness and weak strictness are independent properties, as we show in the next examples, which also prove that the converses of (i) and (ii) of Proposition 2.2 are not always true.

**Examples 2.4** ([13], Examples 2.4)

1. *There exist inductive sequences of spaces of countable type that are weakly strict but not strict.*

*Proof.* Take the inductive sequence $(E_n)_n$ or $(F_n)_n$ of 5.1.4 with $F$ of countable type and $Z$ any subspace of $F$.

2. *If $K$ is not spherically complete then there exist inductive sequences of polar Banach spaces (even having orthonormal bases) that are strict but not weakly strict.*

*Proof.* Let $I$ be a set such that $c_0(I)$ has a closed subspace $Z$ without the HBEP in $c_0(I)$. Then take the inductive sequence $(E_n)_n$ of 5.1.3 with $F := c_0(I)$ and $Z$ as above.

**Remark 2.5** It is easily seen that our notion of almost weak strictness coincides with property (H-8) of the classical paper [16]. In Lemma 1 of this paper the authors prove that (H-8) is equivalent to*: For all n, every $f_n \in E'_n$ has an extension* $\overline{f_n} \in E'_{n+1}$. This last property is equivalent to our weak strictness (apply [9], Lemma 1.4.5,(i)). Therefore, the result of [16] is in sharp contrast with the $p$-adic situation for non-spherically complete fields, because we know that in this case almost weak strictness (even strictness) does not imply weak strictness, see Example 2.4.2.

Next we consider some closedness properties. Following [13] and [14]:

**Definition 2.6** We say that $(E_n)_n$ satisfies

**(CI)** if $E_n$ is closed in $E_{n+1}$ for each $n$,

**(CII)** if $E_n$ is closed in $E$ for each $n$,

**(CIII)** if $\overline{E_n}^{\tau_{ind}} \subset E_{n+1}$ for each $n$,

**(C3)** if for each $n$ there exists an $m \geq n$ such that $\overline{E_n}^{\tau_{ind}} \subset E_m$,

**(CIV)** if, for all $n$, every closed absolutely convex subset of $E_n$ is closed in $E_{n+1}$ (or equivalently if, for all $n$, every closed absolutely convex subset of $E_n$ is closed in $E$, [13], Proposition 3.4),

**(CV)** if, for all $n$, every closed absolutely convex and edged subset of $E_n$ is closed in $E_{n+1}$ (or equivalently if, for all $n$, every closed absolutely convex and edged subset of $E_n$ is closed in $E$, [13], Proposition 3.6).

**Remark 2.7** One can easily check that if in the definitions of the strictness properties and of the closedness properties (CI), (CIV) and (CV) we replace the index $n+1$ by an $m \geq n$, then it does not make any difference. However, to do this replacement in the definition of (CIII) leads to (C3), a weaker (and different, see Example 2.9.1) closedness property.

**Proposition 2.8** ([13], Propositions 3.2, 3.9) (CIV) $\Rightarrow$ (CV) $\Rightarrow$ (CII) $\Rightarrow$ (CI),(CIII)*;* (CIII) $\Rightarrow$ (C3)*.*

In the next examples we show that the converses of the above proposition fail. We also prove that (CI) and (CIII) (resp. (CI) and (C3)) are independent properties and that they together do not imply (CII) (resp. (CIII)).

**Examples 2.9**

1. *There exist inductive sequences of normed spaces of countable type that satisfy* (CI) and (C3) *but not* (CIII).

*Proof.* Take the inductive sequence $(E_n)_n$ of 5.1.2, for $m \geq 2$, with $Z := \mathrm{Ker}\, f$ ($f$ as in 5.1.2). Since $Z$ is $\tau'$-closed but not $\tau$-closed (Proposition 1.1), it follows from Theorem 5.1,(v),(vi).(a),(vi).(b) that $(E_n)_n$ satisfies (CI) and (C3) but not (CIII).

2. *There exist inductive sequences of metrizable spaces of countable type that satisfy* (CI) *but not* (C3).

*Proof.* Take the inductive sequence $(F_n)_n$ of 5.1.2 with $Z := \mathrm{Ker}\, f$ ($f$ as in 5.1.2). With similar reasoning as above we obtain that $(F_n)_n$ meets the requirements.

**Examples 2.10** ([13], Examples 3.3, 3.10)

1. *There exist inductive sequences of normed spaces of countable type that satisfy* (CIII) *but not* (CI).

*Proof.* Take the inductive sequence $(E_n)_n$ of 5.1.3, for $m = 1$, with $F$ of countable type and $Z$ a non-norm closed subspace of $F$ (e.g. $Z := \mathrm{Ker}\, g$, where $g$ is a functional on $F$ that is not norm continuous, Proposition 1.1).

2. *There exist inductive sequences of normed spaces of countable type that satisfy* (CI) and (CIII) *but not* (CII).

*Proof.* Take the inductive sequence $(E_n)_n$ of 5.1.2, for $m = 1$, with $Z := \mathrm{Ker}\, f$ ($f$ as in 5.1.2).

3. *There exist inductive sequences of normed spaces of countable type that satisfy* (CII) *but not* (CV).

*Proof.* Take the inductive sequence $(E_n)_n$ or $(F_n)_n$ of 5.1.2 with $Z = \{0\}$.

4. *If $K$ is not spherically complete then there exist inductive sequences of spaces of countable type that satisfy* (CV) *but not* (CIV).

*Proof.* Take the inductive sequence of Example 2.4.1, where we additionally assume that $Z$ is norm closed (i.e. $\tau'$-closed).

It is not possible to give a variant of Example 2.10.4 neither for spherically complete fields nor for metrizable steps. Indeed, we have the following.

**Proposition 2.11** ([13], Proposition 3.11) *Suppose either*

(a) *$K$ is spherically complete,*

*or*

(b) *all the $E_n$ are metrizable and polar.*

*Then* (CIV) *is equivalent to* (CV)*.*

We finish this section by discussing the relations between strictness and closedness properties. For more results in this line, see Section 4.

Firstly we see that each of the properties strict and (almost) weakly strict is in general independent of each of the properties (CI), (CII), (CIII) and (C3).

**Examples 2.12** ([13], Examples 4.1)

1. *There exist inductive sequences of normed spaces of countable type that are strict and weakly strict but do not satisfy neither* (CI) *nor* (C3).

*Proof.* The normed space $c_0/c_{00}$ is infinite dimensional (otherwise, $c_0$ would have countable dimension, which is not the case because of the Baire Category Theorem, see e.g. [7], 3.9.3). So there exists a sequence $(y_n)_n$ in $c_0$ such that $y_1 \notin c_{00}$ and $y_{n+1} \notin c_{00} + [y_1, \ldots, y_n]$ for all $n$.

Then take $E_n := c_{00} + [y_1, \ldots, y_n]$ ($n \in \mathbb{N}$) equipped with the norm induced by $c_0$ (note that in the proof carried out in Example 4.1.1 of [13] in order to see that (CIII) fails for this $(E_n)_n$, it is implicitly proved that (C3) also fails).

2. *There exist inductive sequences of normed spaces of countable type that satisfy* (CII) *but are not almost weakly strict.*

*Proof.* Take the inductive sequences of Example 2.10.3.

Because of the independence we now are going to investigate what happens when we combine some of the strictness properties with some of the closedness properties (CI)−(CIII) or (C3). At the same time we obtain more information about their relations with (CIV) and (CV) (for instance, we see that each of the properties (CIV) and (CV) is not independent of each of strictness properties).

First we start with (CI).

**Theorem 2.13** ([13], Theorem 4.3) *For an inductive sequence $(E_n)_n$ consider the following properties.*

(a) *Strict* + (CI).
(b) (CIV).
(c) (CV).
(d) *Almost weakly strict* + (CI).
(e) *Weakly strict* + (CI).

*Then we have:*

(i) (a) $\longrightarrow$ (b) $\longrightarrow$ (c) $\longrightarrow$ (d) $\Longleftarrow$ (e).

(ii) *If all the $E_n$ are strongly polar, then*

$$\text{(c)} \Longleftrightarrow \text{(d)} \Longleftrightarrow \text{(e)}.$$

(iii) *If all the $E_n$ are metrizable and polar, then*

$$\text{(a)} \Longleftrightarrow \text{(b)} \Longleftrightarrow \text{(c)} \Longleftrightarrow \text{(d)}.$$

*If, in addition, all the $E_n$ are strongly polar, then*

$$\text{(a)} \Longleftrightarrow \text{(b)} \Longleftrightarrow \text{(c)} \Longleftrightarrow \text{(d)} \Longleftrightarrow \text{(e)}.$$

(iv) *If $K$ is spherically complete, then*

$$\text{(b)} \Longleftrightarrow \text{(c)} \Longleftrightarrow \text{(d)} \Longleftrightarrow \text{(e)}.$$

**Remark 2.14** Theorem 2.13 also holds when we replace (CI) by (CII) ([13], Proposition 4.6), but it is not true when we replace (CI) by (CIII) or by (C3) ([13], Remark 4.7.3).

Next we show that the conclusions of (ii), (iii) and (iv) of Theorem 2.13 fail when the assumptions on the $E_n$ or on $K$ are removed.

**Counterexamples 2.15** (Counterexamples to Theorem 2.13, [13], Counterexamples 4.4)

1. *If $K$ is spherically complete then there exist inductive sequences of spaces of countable type for which* (b) *holds but* (a) *fails.*

*Proof.* Take the inductive sequence of Example 2.4.1, where we additionally assume that $Z$ is a norm closed subspace of $F$.

*Now suppose that $K$ is not spherically spherically complete. Then we have the following.*

2. *There exist inductive sequences of spaces of countable type for which* (c) *holds but* (b) *fails.*

*Proof.* This is Example 2.10.4.

3. *There exist inductive sequences of polar spaces for which* (e) *holds* (*even for* (CII)) *but* (c) *fails.*

*Proof.* Take the inductive sequence $(E_n)_n$ or $(F_n)_n$ of 5.1.4 with $F$ being not strongly polar and $Z$ a weakly closed subspace of $F$ with the HBEP in $F$.

4. *There exist inductive sequences of polar Banach spaces* (*even having orthonormal bases*) *for which* (d) *holds* (*even* (a) *holds*) *but* (e) *fails.*

*Proof.* The inductive sequence of Example 2.4.2 satisfies the desired conditions.

**Problem** ([13], Problem 4.5) Suppose $K$ is not spherically complete. Do there exist inductive sequences (of polar spaces or of spaces of countable type) for which (b) holds but (a) fails?

The last result of this section gives information about the behaviour of inductive limits of Fréchet spaces.

**Proposition 2.16** ([13], Proposition 4.8) *Let $(E_n)_n$ be an inductive sequence such that all the $E_n$ are Fréchet spaces. Then each of the properties* (CI), (CII), (CIV) *and* (CV) *characterizes the strictness of the sequence. If, in addition, all the $E_n$ are polar* (*resp. strongly polar*) *then each of these properties characterizes almost weak strictness* (*resp. weak strictness*) *of $(E_n)_n$.*

**Problem** ([13], Problem 4.9) Let $(E_n)_n$ be an inductive sequence whose steps $E_n$ are Fréchet spaces. Does property (CIII) or (C3) characterize strictness of $(E_n)_n$?

## 3 Regularity properties

Following [18] and [20],

**Definition 3.1** We say that $(E_n)_n$ is

(i) **regular** if for every bounded subset $B$ of $E$ there is an $n$ such that $B \subset E_n$ and is bounded in $E_n$,

(ii) **$\alpha$-regular** if for every bounded subset $B$ of $E$ there is an $n$ such that $B \subset E_n$,

(iii) **almost regular** if for all $n$ and every bounded subset $B$ of $E_n$ there is an $m \geq n$ such that $\overline{B}^{\tau_{ind}} \subset E_m$ and is bounded in $E_m$,

(iv) **almost $\alpha$-regular** if for all $n$ and every bounded subset $B$ of $E_n$ there is an $m \geq n$ such that $\overline{B}^{\tau_{ind}} \subset E_m$.

The proofs of the next two propositions are straightforward and left to the reader.

**Proposition 3.2**

(i) *regular* $\longrightarrow$ *$\alpha$-regular* $\longrightarrow$ *almost $\alpha$-regular.*

(ii) *regular* $\longrightarrow$ *almost regular* $\longrightarrow$ *almost $\alpha$-regular.*

**Proposition 3.3**

(i) *$(E_n)_n$ is regular $\Longleftrightarrow$ $(E_n)_n$ is $\alpha$-regular and for every bounded set $B$ in $E$ that is contained in some $E_n$, there exists $m \geq n$ such that $B$ is bounded in $E_m$.*

(ii) *$(E_n)_n$ is almost regular $\Longleftrightarrow$ $(E_n)_n$ is almost $\alpha$-regular and for all $n$ and every bounded set $B$ in $E_n$ such that $\overline{B}^{\tau_{ind}}$ is contained in some $E_m$, $m \geq n$, there exists $r \geq m$ such that $\overline{B}^{\tau_{ind}}$ is bounded in $E_r$.*

(iii) *$(E_n)_n$ is regular $\Longleftrightarrow$ $(E_n)_n$ is almost regular and for every bounded set $B$ in $E$ there exists an $n$ and a bounded set $B_n$ in $E_n$ such that $B \subset \overline{B_n}^{\tau_{ind}}$.*

(iv) *$(E_n)_n$ is almost $\alpha$-regular and for every bounded set $B$ in $E$ there exists an $n$ and a bounded set $B_n$ in $E_n$ such that $B \subset \overline{B_n}^{\tau_{ind}}$ $\longrightarrow$ $(E_n)_n$ is $\alpha$-regular.*

**Remarks 3.4**

1. When $(E_n)_n$ satisfies that *for every bounded set $B$ in $E$ that is contained in some $E_n$, there exists $m \geq n$ such that $B$ is bounded in $E_m$* ((i)), it is said that $(E_n)_n$ is $\beta$-regular, see [20]. Also, when $(E_n)_n$ satisfies that *for every bounded set $B$ in $E$ there exists an $n$ and a bounded set $B_n$ in $E_n$ such that $B \subset \overline{B_n}^{\tau_{ind}}$* ((iii), (iv)), it is said that $(E_n)_n$ has the Qiu property, see [1].

2. The converse of (iv) is not true, see Remark 3.13.

3. With the same philosophy as in Remark 2.7 one could ask what happens if in (iii) and (iv) of Definition 3.1 we replace the index $m \geq n$ by $n$ or by $n+1$. This time the replacements lead to properties that are not adequate for the purposes of this paper. In fact, there exist inductive sequences of normed spaces of countable type that are regular but do not satisfy any of these *too strong* properties. For an example take the inductive sequence $(E_n)_n$ of Example 2.9.1. By Theorem 5.1,(ix) this inductive sequence is regular. Next, we will find a $B \subset Z$ that is $\tau'$-bounded but for which $\overline{B}^{\tau} \not\subseteq Z$. (Then, with a simple adaptation of the proof of the first part of (xii) of Theorem 5.1 we obtain that, for each $n$, $C_n := B^{n+m+1} \times \{0\}^{\mathbb{N}}$ is a bounded set in $E_n$ for which $\overline{C_n}^{\tau_{ind}} \not\subseteq E_{n+1}$, and we are done). Let $B := \{x \in Z : \|x\|_f \leq 1\}$ $(= \{x \in Z : \|x\| \leq 1\}$, recall that $Z = \operatorname{Ker} f)$. Clearly $B$ is $\tau'$-bounded. Also, as

$f$ is not $\| \, . \, \|$-continuous, it follows from Proposition 1.1 that $Z$ is $\| \, . \, \|$-dense in $F$. Hence $\overline{B}^{\tau} = \{x \in F : \|x\| \leq 1\}$. Thus, $\overline{B}^{\tau} \not\subseteq Z$ (otherwise, $Z = F$ i.e. $f = 0$, a contradiction).

**Proposition 3.5** *Suppose either*
(a) $(E_n)_n$ *is strict,*
*or*
(b) $(E_n)_n$ *is weakly strict and all the* $E_n$ *are polar,*
*or*
(c) $(E_n)_n$ *satisfies* (CV) *and all the* $E_n$ *are polar.*
*Then we have the following.*
(i) $(E_n)_n$ *is regular* $\Longleftrightarrow$ $(E_n)_n$ *is* $\alpha$*-regular.*
(ii) $(E_n)_n$ *is almost regular* $\Longleftrightarrow$ $(E_n)_n$ *is almost* $\alpha$*-regular.*

*Proof.* By (i) and (ii) of Proposition 3.3 it suffices to prove that, under the assumptions of (a), (b) or (c), the following holds:

For each $n$, $\tau_{ind}|E_n$ and $\tau_n$ have the same bounded sets.

First assume (a). Then $\tau_n = \tau_{ind}|E_n$ ([9], Theorem 1.4.7,(i)) and clearly the above holds.

Now assume (b). Since $\tau_{ind}|E_n \leq \tau_n$ we have that every $\tau_n$-bounded set is $\tau_{ind}|E_n$-bounded. Now let $B \subset E_n$ be $\tau_{ind}|E_n$-bounded. By weak strictness, $\sigma_n = \sigma|E_n$, where $\sigma_n$ and $\sigma$ denote the weak topology on $E_n$ and $E$ respectively ([9], Theorem 1.4.7,(ii)). Thus, $E_n' = (E_n, \sigma_n)' = (E_n, \sigma|E_n)' \subset (E_n, \tau_{ind}|E_n)'$. Hence $B$ is weakly bounded in $E_n$ and by polarity of $E_n$, $B$ is $\tau_n$-bounded ([25], Theorem 7.5), and we are done.

Finally assume (c). Then $\tau_n$ and $\tau_{ind}|E_n$ have the same closed, absolutely convex and edged sets. By Proposition 1.1, $E_n' = (E_n, \tau_{ind}|E_n)'$. The rest follows as in (b).

**Problem** Are (a) and (b) true when we replace "strict" and "weakly strict" respectively by "almost weakly strict"?

Note that by Proposition 2.2 this question has a partial affirmative answer when either $K$ is spherically complete or all the $E_n$ are of countable type or all the $E_n$ are metrizable and polar.

**Proposition 3.6** *Suppose* $(E_n)_n$ *is an LN-space. Then we have the following.*
(i) $(E_n)_n$ *is regular* $\Longleftrightarrow$ $(E_n)_n$ *is almost regular*
(ii) $(E_n)_n$ *is* $\alpha$*-regular* $\Longleftrightarrow$ $(E_n)_n$ *is almost* $\alpha$*-regular.*

*Proof.* A simple adaptation of the proof done in Theorem 2 of [18] shows that every (LN)-space has the Qiu property (see Remark 3.4.1 for this concept). Then apply (iii) and (iv) of Proposition 3.3.

**Remark 3.7** Proposition 3.6 is not true for LM-spaces. In fact, in Example 3.10 we will prove that there exist LM-spaces that are almost regular but not $\alpha$-regular.

Next we use the inductive sequences $(F_n)_n$ constructed in Theorem 5.1 to give examples showing that $\alpha$-regular and almost regular are independent properties and that the converses of Proposition 3.2 fail if the extra conditions of Propositions 3.5 or 3.6 are removed. First two preliminary lemmas.

**Lemma 3.8** *Let $(F_n)_n$ be the inductive sequence of 5.1.1. Then, for any choice of $Z$, we have that*

$$(F_n)_n \text{ is almost regular} \iff (F_n)_n \text{ is almost } \alpha\text{-regular.}$$

*Proof.* The implication almost regular $\longrightarrow$ almost $\alpha$-regular is always true (Proposition 3.2,(ii)).

Now suppose that the $(F_n)_n$ of 5.1.1 is almost $\alpha$-regular and let us prove that it is almost regular. By Theorem 5.1,(xii),(xiii) it suffices to see that for all $B \subset \Lambda^0(B)$ that is $\tau'$-bounded (i.e. $p_j(B)$ is bounded for all $j$), $\overline{B}^{\tau}$ is $\tau'$-bounded. To this end, choose for each $j$ an $M_j > 0$ such that $p_j(x) \leq M_j$ for all $x \in B$. Then

$$B \subset B_1 := \{(\lambda_k)_k \in \Lambda^0(B) : |\lambda_k| \leq \frac{M_j}{b_k^j} \text{ for all } j,k\} = \{(\lambda_k)_k \in \Lambda^0(B) : |\lambda_k| \leq r_k \text{ for all } k\},$$

where, $r_k := \inf_j \frac{M_j}{b_k^j}$ $(k \in \mathbb{N})$.

Since the projection maps on $\Lambda^0(B)$, $(\lambda_k)_k \mapsto \lambda_k$ $(k \in \mathbb{N})$, are $p_{j_0}$-continuous (=$\tau$-continuous), the set $B_1$ is $\tau$-closed (so it contains $\overline{B}^{\tau}$). Also, $B_1$ is $\tau'$-bounded, hence so is $\overline{B}^{\tau}$.

**Lemma 3.9** *Let $(F_n)_n$ be the inductive sequence of 5.1.2. Then, for any choice of $Z$, we have that*

$$(F_n)_n \text{ is regular} \iff (F_n)_n \text{ is almost regular.}$$

*Proof.* The implication regular $\longrightarrow$ almost regular is always true (Proposition 3.2,(ii)).

Now suppose that the $(F_n)_n$ of 5.1.2 is almost regular and let us see that it is regular. For that we will apply Corollary 5.3.

First we prove that $\tau|Z = \tau'|Z$. Suppose not i.e. $f|Z$ is not $\|\,.\,\|$-continuous; we derive a contradiction. The set

$$B := B_z \cap \operatorname{Ker} f|Z$$

(with $B_Z := \{x \in Z : \|x\| \leq 1\}$) is contained in $Z$ and is $\tau'$-bounded, hence by almost regularity and Theorem 5.1,(xiii), $\overline{B}^{\tau}$ is $\tau'$-bounded. However, as $f|Z$ is not $\|\,.\,\|$-continuous, it follows from Proposition 1.1 that $\operatorname{Ker} f|Z$ is $\|\,.\,\|$-dense in $Z$, so $\overline{B}^{\tau} \supset \overline{B}^{\tau|Z} = B_Z$. Thus, $\tau'$-boundedness of $\overline{B}^{\tau}$, would imply that $B_Z$ is $\tau'$-bounded, from which the normed topologies $\tau|Z$ and $\tau'|Z$ would have the same bounded sets, a contradiction.

Now we prove that $Z$ is $\tau$-closed (and by Corollary 5.3 we are done). Since $(F_n)_n$ is almost regular and $\tau|Z = \tau'|Z$, it follows from Theorem 5.1,(xiii) that for all $B \subset Z$ that is $\tau$-bounded ($=\tau'$-bounded), $\overline{B}^{\tau} \subset Z$. From Proposition 5.2,(iii) we conclude that $Z$ is $\tau$-closed.

**Example 3.10** *There exist inductive sequences of metrizable spaces of countable type that are almost regular but not $\alpha$-regular.*

*Proof.* Let $(F_n)_n$ be the inductive sequence of 5.1.1. Suppose the matrix $B$ satisfies

$$\text{for every } j \text{ there exists } j' > j \text{ such that } \lim_k b_k^j / b_k^{j'} = 0. \tag{2}$$

Let $Z$ be a subspace of $\Lambda^0(B)$ that is $n_{0,\infty}$-closed but not $p_{j_0}$-closed (such as $Z$ exists; in fact, if $n_{0,\infty}$ and $p_{j_0}$ would have the same closed subspaces then, by Proposition 1.1, $(\Lambda^0(B), n_{0,\infty})' = (\Lambda^0(B), p_{j_0})'$, which implies that $p_{j_0}$ and $n_{0,\infty}$ have the same bounded sets ([25], Theorem 7.5). Hence $\tau_{p_{j_0}} = n_{0,\infty}$ ([21], Lemma 4.2): a contradiction because, by (2) and Proposition 3.5 of [8], $(\Lambda^0(B), n_{0,\infty})$ is nuclear and infinite-dimensional, so it cannot be normable).

Since $Z$ is not $\tau$-closed it follows from Proposition 5.2,(iii) that $(F_n)_n$ is not $\alpha$-regular.

Now we pass to get almost regularity of $(F_n)_n$. By Lemma 3.8 this is equivalent to see almost $\alpha$-regularity, and to prove this last one we will use Theorem 5.1,(xii). Let $B \subset Z$ be $n_{0,\infty}$-bounded, we may assume $B$ absolutely convex. Let us prove that $\overline{B}^{\tau} \subset Z$. Since $(\Lambda^0(B), n_{0,\infty})$ is a nuclear Fréchet space we have that $A := \overline{B}^{n_{0,\infty}}$ is absolutely convex, metrizable, compactoid and complete for $n_{0,\infty}$. By [27], Theorem 9.1 we obtain that $\tau_{p_{j_0}}|A = n_{0,\infty}|A$. Hence $A$ is $p_{j_0}$-complete, so $p_{j_0}$-closed. Also, $B \subset A \subset Z$ (the last inclusion because $Z$ is $n_{0,\infty}$-closed). Therefore, $\overline{B}^{p_{j_0}} \subset Z$.

**Example 3.11** *There exist inductive sequences of metrizable spaces of countable type that are $\alpha$-regular but not almost regular.*

*Proof.* Let $(F_n)_n$ be the inductive sequence of 5.1.2. Take a subspace $Z$ of $F$ and an $f \in F^*$ such that $f|Z$ is not $\| . \|$-continuous (it is possible whenever $Z$ is infinite-dimensional). Suppose also that $Z$ is $\| . \|$-closed. Applying Proposition 5.2,(iii) and Corollary 5.3 we obtain that $(F_n)_n$ is $\alpha$-regular but not regular (equivalently, not almost regular, by Lemma 3.9).

**Example 3.12** *There exist inductive sequences of metrizable spaces of countable type that are $\alpha$-regular and almost regular, but not regular.*

*Proof.* Let $(F_n)_n$ be the inductive sequence of 5.1.1. Suppose the matrix $B$ satisfies (2). Let $Z$ be an infinite-dimensional $p_{j_0}$-closed subspace of $\Lambda^0(B)$. By $\tau$-closedness of $Z$ and Proposition 5.2,(iii) we have that $(F_n)_n$ is $\alpha$-regular, which implies that it is almost $\alpha$-regular (Proposition 3.2,(i)), or equivalently almost regular, by Lemma 3.8.

Next we prove that $(F_n)_n$ is not regular. By Corollary 5.3 we have to see that $\tau|Z \neq \tau'|Z$. This inequality is a consequence of the fact that, by (2) and Proposition 3.5 of [8], $(Z, \tau'|Z)$ is nuclear. However, $(Z, \tau|Z)$ is an infinite-dimensional normed space, which cannot be nuclear.

**Remark 3.13** Example 3.12 also shows that the converse of Proposition 3.3,(iv) is not true. In fact, if this converse holds then $(E_n)_n$ would have the Qiu property (see Remark 3.4.1 for this concept). Then by almost regularity and Proposition 3.3,(iii) we would obtain that the $(E_n)_n$ of Example 3.12 is regular, a contradiction.

We finish this section by showing some relations between closedness and regularity properties. More relations will be given in the next section.

**Proposition 3.14**
(i) (C3) $\longrightarrow$ *α-regular.*
(ii) *If* $(E_n)_n$ *is a strict LM-space,* (C3) $\Longleftrightarrow$ *α-regular.*
(iii) (CV) $\longrightarrow$ *almost regular.*
(iv) *If all the* $E_n$ *are polar,* (CV) $\longrightarrow$ *regular.*

*Proof.* The proofs of (i) and (ii) are simple adaptations of the classical ones given in [16], Theorem 1 and [22], Theorem 2 respectively.

(iii). Let $n \in \mathbb{N}$, let $B \subset E_n$ be a bounded set in $E_n$. We may assume that $B$ is absolutely convex. Then $(\overline{B}^{\tau_n})^e$ is a $\tau_n$-closed absolutely convex and edged subset of $E_n$. By (CV) we have that this last set is $\tau_{ind}$-closed. Hence $\overline{B}^{\tau_{ind}} \subset (\overline{B}^{\tau_n})^e$. Since $(\overline{B}^{\tau_n})^e$ is $\tau_n$-bounded then so is $\overline{B}^{\tau_{ind}}$ and we get almost regularity.

(iv). Suppose all the $E_n$ are polar. Since (CV) $\longrightarrow$ (C3) (Proposition 2.8) it follows from (i) that (CV) $\longrightarrow$ α-regular. Now apply Proposition 3.5,(c).

For LF-spaces we have the following.

**Corollary 3.15**
(i) *Every strict LF-space is regular.*
(ii) *Every weakly strict LF-space with polar steps is regular*

*Proof.* Every strict LF-space satisfies (C3), so it is α-regular by Proposition 3.14,(i) and hence regular by Proposition 3.5,(a). Now (ii) follows from (i) and Proposition 2.3,(ii).

**Proposition 3.16** ([14], Proposition 3.4) *Let* $(E_n)_n$ *be an LF-space. Then, any of the closedness properties* (CI)$-$(CV) *and* (C3) *implies regularity.*

Next we show that if in Proposition 3.14 we consider a weaker closedness property or a stronger regularity one then there exist LM-spaces with steps of countable type for which the implications fail (compare with the classical examples of (CII) $\not\Rightarrow$ regular and (CI) $\not\Rightarrow$ α-regular given in [17], in which the steps are not even metrizable).

**Examples 3.17**
1. *There exist inductive sequences of metrizable spaces of countable type that satisfy* (CII) *but are not almost regular.*

*Proof.* The inductive sequence $(F_n)_n$ of Example 3.11 satisfies the conditions. Indeed, in the mentioned example we proved that $(F_n)_n$ is not almost regular. Also, $Z$ is $\tau$-closed i.e. $(F_n)_n$ satisfies (CII) (Theorem 5.1,(vi).(d)).

2. *There exist inductive sequences of metrizable spaces of countable type that satisfy* (CI) *but do not satisfy any of the regularity properties.*

*Proof.* Let $(F_n)_n$ be the inductive sequence of Example 2.9.2. We know that it satisfies (CI). Now we see that almost $\alpha$-regularity fails for this $(F_n)_n$ (and by Proposition 3.2 we are done). Indeed, by Theorem 5.1,(xii) we have to find a $B \subset Z$ that is $\tau'$-bounded but for which $\overline{B}^{\tau} \nsubseteq Z$. Let $B := \{x \in Z : \|x\|_f \leq 1\}$. With the same reasoning as in the example given in Remark 3.4.3 we conclude that this $B$ meets the requirements.

Finally we show that the converses of (i), (iii) and (iv) of Proposition 3.14 are not always true.

**Example 3.18** ([14], Example 3.8.(i)) *There exist inductive sequences of Banach spaces of countable type that are regular and do no satisfy any of the strictness and any of the closedness properties.*

*Proof.* Let $B$ be a matrix satisfying (2). Then, for each $n$, take $E_n := c_0(\mathbb{N}, 1/b^n)$.

**Remark 3.19** Example 3.18 is the $p$-adic substitute of the classical ones given in Example 4 of [16], Example of [18] and Counterexample of [22], all of them with a typically archimedean character.

## 4 The $p$-adic Dieudonné-Schwartz Theorem

The classical Dieudonné-Schwartz Theorem ([6], Proposition 4) states that every strict LF-space is regular. This is still valid in the $p$-adic context (Corollary 3.15). Some extensions were given in Proposition 3.14. Now we mix (weak) strictness and closedness. This leads to new extensions of that theorem. For the proofs of the results and for the details of the examples included in this section, see Section 3 of [14].

**Theorem 4.1** ($p$-adic Dieudonné-Schwartz Theorem, [14], Theorems 3.1, 3.2)

(i) *Strict* + (C3) $\longrightarrow$ *regular.*

(ii) *Strict* + (CI) $\longrightarrow$ (CIV) + *regular.*

(iii) *If all the* $E_n$ *are polar, then*

*Weakly strict* + (C3) $\longrightarrow$ *regular.*

(iv) *If all the* $E_n$ *are strongly polar, then*

*Weakly strict* + (CI) $\longrightarrow$ (CV) + *regular.*

(v) *If either*

(v.a) $K$ *is spherically complete,*

*or*

(v.b) *all the* $E_n$ *are metrizable and polar,*

*then*

*Weakly strict* + (CI) $\longrightarrow$ (CIV) + *regular.*

**Remark 4.2** In (i) and (iii) of Theorem 4.1 we cannot assume other closedness properties (apart from the stated one (C3)). In fact, *there exist inductive sequences of normed spaces of countable type that are strict and weakly strict and satisfy* (C3) (*resp* (CIII)) *but do not satisfy any of the other closedness properties* (*resp. do not satisfy* (CI)). An example is the inductive sequence $(E_n)_n$ of 5.1.3, for $m \geq 2$ (resp. for $m = 1$), with $F$ of countable type and $Z$ a non-norm closed subspace of $F$ (e.g. $Z := \operatorname{Ker} g$, where $g$ is a functional on $F$ that is not norm continuous). To see that this $(E_n)_n$ satisfies the required conditions, apply Theorem 5.1.

The examples presented in this remark are the $p$-adic substitutes of Example 1 of [16].

In the classical case the implication "Weakly strict + (CI) $\longrightarrow$ (CIV) + regular" of Theorem 4.1 is always true (see [16]). In contrast to that, the next examples show that, for $p$-adic inductive limits with polar steps, the conclusions of Theorem 4.1,(iv),(v) may fail when $K$ is not spherically complete.

**Counterexamples 4.3** (Counterexamples to Theorem 4.1, [14], Examples 3.6) *Suppose $K$ is not spherically complete.*

1. *There exist inductive sequences of polar spaces that are weakly strict and satisfy* (CI), *but do not satisfy any of the other closedness properties and any of the regularity properties.*

*Proof.* Let $F := \ell^\infty$. There is a closed subspace $Z$ of $F$ that has the HBEP in $F$, contains $c_0$, and is not weakly closed in $F$. Then take the inductive sequence $(F_n)_n$ of 5.1.4 with $F$ and $Z$ as above.

2. *There exist inductive sequences of polar spaces that are weakly strict and satisfy* (CII) (*hence are regular, Theorem 4.1,*(iii)), *but do not satisfy* (CV).

*Proof.* Take the inductive sequence of Example 2.15.3.

3. *There exist inductive sequences of spaces of countable type that are weakly strict and satisfy* (CV) (*hence are regular, Theorem 4.1,*(iii)), *but do not satisfy* (CIV).

*Proof.* The inductive sequence of Example 2.10.4 satisfies the requirements.

**Remarks 4.4**

1. For real or complex inductive limits we also have that "(CIV) $\longrightarrow$ Weakly strict + (CI)" (see [16]). The same happens in the non-archimedean case when either $K$ is spherically complete or all the $E_n$ are metrizable and strongly polar (Theorem 2.13). However, this is not the case in general (see Counterexample 2.15.4).

This also proves that the converses of (iv) and (v) of Theorem 4.1 may fail (note that by Proposition 3.14,(iv), for polar steps we always have (CIV) $\longrightarrow$ regular).

2. The converse of Theorem 4.1,(ii) holds when all the $E_n$ are metrizable and strongly polar (Theorem 2.13). But it fails even when $K$ is spherically complete (see Counterexample 2.15.1).

3. The failure of the converses in (i) and (iii) of Theorem 4.1 is shown by Example 3.18. Also, *there exist inductive sequences of spaces of countable type that are strict, weakly strict and regular, but do not satisfy any of the closedness properties*, see [14], Example 3.8.(ii). This is the $p$-adic substitute of the classical one given in [16], Example 2, which has a typically archimedean character.

4. By using Proposition 2.2 we obtain the validity of (iv) and (v) of Theorem 4.1 when we replace "weakly strict" by "almost weakly strict". Also, this proposition leads to the validity of (i), (ii) (resp. (iii)) when we replace "strict" (resp. "weakly strict") by "almost weakly strict" and, additionally, either $K$ is spherically complete, or all the steps are metrizable and polar.

On the other hand, the $(F_n)_n$ of Counterexample 4.3.1 shows that (ii) may fail after the above replacement. In fact, we know that this inductive sequence does not satisfy any of the regularity properties and any of the closedness properties (CII)−(CV). However, by Theorem 5.1,(iii), $(F_n)_n$ is almost weakly strict.

But the following is unsolved:

**Problem** Are (i) and (iii) of Theorem 4.1 always true when we replace "strict" and "weakly strict" respectively by "almost weakly strict"?

Note that, by Proposition 3.14,(i), a negative answer to this problem would imply a negative answer to the one posed after Proposition 3.5.

Next, we show that the conclusions of the $p$-adic Dieudonné-Schwartz Theorem 4.1 are not true when the closedness condition is dropped. Examples 3.17 showed that the same occurs when the (weak) strictness condition is the dropped one.

**Examples 4.5** ([14], Examples 3.10)

1. *There exist inductive sequences of normed spaces of countable type that are strict and weakly strict, but do not satisfy any of the regularity and any of the closedness properties.*

*Proof.* Take the inductive sequence $(E_n)_n$ of Example 2.12.1.

2. *There exist inductive sequences of spaces of countable type that are weakly strict, but are not strict and do not satisfy any of the regularity and any of the closedness properties.*

*Proof.* Take the inductive sequence $(F_n)_n$ of 5.1.4 with $F := c_0$, $Z := c_{00}$.

3. *If $K$ is not spherically complete, there exist inductive sequences of polar metrizable spaces that are strict, but are not weakly strict and do not satisfy any of the regularity and any of the closedness properties.*

*Proof.* Take the inductive sequence $(F_n)_n$ of 5.1.3 with $F := \ell^\infty$, $Z := c_{00}$.

**Remarks 4.6**

1. Applying Proposition 2.3 we obtain that the steps of Example 4.5.3 cannot be strongly polar. In particular, they cannot be of countable type, and also it is not possible to give a counterpart of Example 4.5.3 when $K$ is spherically complete.

2. *There exist inductive sequences of Banach spaces of countable type that do not satisfy any of the strictness, closedness and regularity properties*, see [14], Example 3.12, where the inductive sequence is of the form $(c_0(\mathbb{N}, \frac{1}{b^n}))_n$ for a certain matrix $(b_k^n)_{k,n}$.

Observe that when $K$ is not spherically complete the steps of the above sequence are reflexive ([24], Corollary 4.18). This fact is in sharp contrast with the classical case ([18], Theorem 4), where it was proved that any real or complex LB-space with reflexive steps is regular.

However, if $K$ is spherically complete, the non-archimedean version of Theorem 4 of [18] holds. Indeed, in this case if the steps $E_n$ are reflexive Banach spaces then they are finite-dimensional ([24], Theorem 4.1.6). So $(E_n)_n$ is strict, hence regular by Corollary 3.15,(i).

In the classical case there exist also inductive sequences of Fréchet nuclear (hence reflexive) spaces that are not regular (see [19]). This example has a typically archimedean character. In a subsequent paper we will study if it admits a $p$-adic substitute and, if it is the case (which we really hope), we will investigate its applications.

## 5 Very useful examples of inductive sequences

This section is devoted to the classes of inductive sequences constructed in [13] and [14]. They provide most of the examples needed along the paper.

**Theorem 5.1** *Let $\tau, \tau'$ be Hausdorff locally convex topologies on a vector space $F$, $\tau \leq \tau'$, let $X := (F, \tau)$, $Y := (F, \tau')$. Let $Z$ be a subspace of $F$ which we equip with the topology $\tau'|Z$. Let $m \in \mathbb{N}$, and for each $n \in \mathbb{N}$, set*

$$E_n := X^n \times Y \times Z^m \times \{0\}^{\mathbb{N}}.$$

*Also, for each $n \in \mathbb{N}$, set*

$$F_n := X^n \times Y \times \prod_{i>n+1} Z,$$

*where all the product spaces appearing in the definitions of $E_n$ and $F_n$ are endowed with the corresponding product topologies. Then we have the following.*

(i) *$(E_n)_n$ (resp. $(F_n)_n$) is an inductive sequence of Hausdorff locally convex spaces. If $(E, \tau_{ind})$ is its inductive limit then $E \subset X^{\mathbb{N}}$ and $\tau_\pi|E \leq \tau_{ind}$, where $\tau_\pi$ is the product topology on $X^{\mathbb{N}}$. In particular, $(E, \tau_{ind})$ is Hausdorff.*

(ii) *$(E_n)_n$ (resp. $(F_n)_n$) is strict $\Longleftrightarrow \tau = \tau'$.*

(iii) *$(E_n)_n$ (resp. $(F_n)_n$) is almost weakly strict $\Longleftrightarrow X' = Y'$.*

(iv) *$(E_n)_n$ (resp. $(F_n)_n$) is weakly strict $\Longleftrightarrow X' = Y'$ and $Z$ has the HBEP in $Y$.*

(v) *$(E_n)_n$ (resp. $(F_n)_n$) satisfies* (CI) *$\Longleftrightarrow$ $Z$ is $\tau'$-closed.*

(vi) (a) *$(E_n)_n$ always satisfies* (C3).

(b) *If $m \geq 2$, $(E_n)_n$ satisfies* (CII) *$\Longleftrightarrow$ $(E_n)_n$ satisfies* (CIII) *$\Longleftrightarrow$ $Z$ is $\tau$-closed.*

(c) *If $m = 1$, $(E_n)_n$ always satisfies* (CIII)*; $(E_n)_n$ satisfies* (CII) *$\Longleftrightarrow$ $Z$ is $\tau$-closed.*

(d) *$(F_n)_n$ satisfies* (CII) *$\Longleftrightarrow$ $(F_n)_n$ satisfies* (CIII) *$\Longleftrightarrow$ $(F_n)_n$ satisfies* (C3) *$\Longleftrightarrow$ $Z$ is $\tau$-closed.*

(vii) *$(E_n)_n$ (resp. $(F_n)_n$) satisfies* (CIV) *$\longrightarrow$ $Z$ is $\tau'$-closed and every $\tau'$-closed absolutely convex subset of $F$ is $\tau$-closed.*

(viii) *$(E_n)_n$ (resp. $(F_n)_n$) satisfies* (CV) *$\longrightarrow$ $Z$ is $\tau'$-closed and every $\tau'$-closed absolutely convex and edged subset of $F$ is $\tau$-closed.*

(ix) *$(E_n)_n$ is always regular.*

(x) *$Z$ is $\tau$-closed $\longrightarrow$ $(F_n)_n$ is $\alpha$-regular $\longrightarrow$ for all $B \subset Z$ that is $\tau$-bounded, $\overline{B}^{\tau} \subset Z$.*

(xi) *$(F_n)_n$ is regular $\Longleftrightarrow$ $(F_n)_n$ is $\alpha$-regular and every $\tau$-bounded subset of $Z$ is $\tau'$-bounded.*

(xii) *$(F_n)_n$ is almost $\alpha$-regular $\Longleftrightarrow$ for all $B \subset Z$ that is $\tau'$-bounded, $\overline{B}^{\tau} \subset Z$.*

(xiii) *$(F_n)_n$ is almost regular $\Longleftrightarrow$ for all $B \subset Z$ that is $\tau'$-bounded, $\overline{B}^{\tau} \subset Z$ and $\overline{B}^{\tau}$ is $\tau'$-bounded.*

*Proof.* For (i)$-$(v) and (vii)$-$(viii), see [13], Theorems 5.1, 5.1′, Remark 5.2 and [14], Theorem 4.1.

For (vi).(a) (resp. (vi).(b), (vi).(c)), see Remark 3.15 (resp. Theorem 5.1 and Remark 5.2, Theorem 5.1′) of [13].

For (vi).(d), (x), and (xi), see [14], Theorem 4.1.

For (ix), see [14], Remark 4.5.

Before proving (xii) and (xiii) we recall the following (see (b) of the proof of Theorem 4.1 of [14]), which will be used in their proofs: *Let $(E, \tau_{ind})$ be the inductive limit of $(F_n)_n$. Let $r \in \mathbb{N}$ and let $B_1, B_2, \ldots$ be non-empty subsets of $F$ with $B_i \subset Z$ for $i > r+1$. Then*

$$\overline{\prod_i B_i}^{\tau_{ind}} = (\prod_i \overline{B_i}^{\tau}) \bigcap E. \tag{3}$$

(xii). Suppose $(F_n)_n$ is almost $\alpha$-regular. Let $B \subset Z$ be $\tau'$-bounded. Let $n \in \mathbb{N}$. Then $B^{\mathbb{N}}$ is bounded in $F_n$ and by hypothesis there exists an $m \geq n$ such that $\overline{B^{\mathbb{N}}}^{\tau_{ind}} \subset F_m$, which by (3) means that $C^{\mathbb{N}} \cap E \subset F_m$, with $C := \overline{B}^{\tau}$. In particular,

$$C^{\mathbb{N}} \cap F_{m+1} = C^{m+2} \times \prod_{i>m+2} C \cap Z \subset F_m = F^{m+1} \times \prod_{i>m+1} Z,$$

which implies that $\overline{B}^{\tau} = C \subset Z$.

Conversely, suppose that for all $B \subset Z$ that is $\tau'$-bounded, $\overline{B}^{\tau} \subset Z$. Let $n \in \mathbb{N}$ and let $A \subset F_n$ be a bounded subset of $F_n$. Then, for $i > n+1$, $\pi_i(A)$ is a $\tau'$-bounded subset of $Z$ (for each $i$, $\pi_i$ is the $i$-th projection). So by assumption its $\tau$-closure is contained in $Z$. By using (3) we have

$$\begin{aligned} \overline{A}^{\tau_{ind}} &\subset \overline{\prod_i \pi_i(A)}^{\tau_{ind}} = (\prod_i \overline{\pi_i(A)}^{\tau}) \cap E \\ &\subset (F^{n+1} \times \prod_{i>n+1} Z) \cap E = F_n, \end{aligned}$$

which implies that $\overline{A}^{\tau_{ind}} \subset F_n$ and so we get almost $\alpha$-regularity.

(xiii). Suppose $(F_n)_n$ is almost regular. Let $B \subset Z$ be $\tau'$-bounded. By almost $\alpha$-regularity and (xii) we have that $C := \overline{D}^{\tau} \subset Z$. Now let $n \in \mathbb{N}$. Then $B^{\mathbb{N}}$ is bounded in $F_n$. By almost regularity there exists an $m \geq n$ such that $\overline{B^{\mathbb{N}}}^{\tau_{ind}} = C^{\mathbb{N}} \cap E = C^{\mathbb{N}}$ (by (3)) is contained and bounded in $F_m$. Hence $\pi_{m+1}(C^{\mathbb{N}}) = C$ is bounded in $Y$ i.e. $\overline{B}^{\tau}$ is $\tau'$-bounded.

Conversely suppose that the condition after the $\Longleftrightarrow$ of (xiii) is satisfied. By (xii), $(F_n)_n$ is almost $\alpha$-regular. Now let $n \in \mathbb{N}$ and let $B$ be a bounded set in $F_n$ such that $\overline{B}^{\tau_{ind}}$ is contained in some $F_m$, $m \geq n$. Let us find an $r \geq m$ such that $\overline{B}^{\tau_{ind}}$ is bounded in $F_r$ (and by Proposition 3.3,(ii) we get almost regularity of

$(F_n)_n$). For that, since $\overline{B}^{\tau_{ind}} \subset F_m$ we have $\pi_i(\overline{B}^{\tau_{ind}}) \subset Z$ for $i > m+1$. Also, as all the projections $\pi_i : F_n \to X$ are continuous, each $\pi_i(B)$ is $\tau$-bounded (hence, so is its $\tau$-closure). Further, the continuity of the projections $\pi_i : E \to X$ implies that

$$\pi_i(\overline{B}^{\tau_{ind}}) \subset \overline{\pi_i(B)}^{\tau} \quad \text{for all } i. \tag{4}$$

Thus, $\pi_1(\overline{B}^{\tau_{ind}}), \ldots, \pi_{m+1}(\overline{B}^{\tau_{ind}})$ are $\tau$-bounded. Apart from that, it follows from the hypothesis that for the $\tau'$-bounded subsets of $Z$, $\pi_i(B)$ $(i > m+1)$, their $\tau$-closures $\overline{\pi_i(B)}^{\tau}$ are $\tau'$-bounded. Hence by (4) so are $\pi_i(\overline{B}^{\tau_{ind}})$ $(i > m+1)$.

Finally,

$$\overline{B}^{\tau_{ind}} \subset \pi_1(\overline{B}^{\tau_{ind}}) \times \ldots \times \pi_{m+1}(\overline{B}^{\tau_{ind}}) \times \prod_{i>m+1} \pi_i(\overline{B}^{\tau_{ind}}),$$

and as the set after the inclusion is bounded in $F_{m+1}$, we have the same for $\overline{B}^{\tau_{ind}}$. Therefore, $r := m+1$ meets the requirements.

The next proposition gives some partial affirmative answers to the validity of the converses of (vii), (viii) and (x) of Theorem 5.1. The situation in general is unknown.

**Proposition 5.2** *Let $X, Y, Z$ and $(F_n)_n$ be as in Theorem 5.1. Then we have the following.*

(i) *Suppose either*
(a) *$X$ and $Y$ are metrizable and polar,*
*or*
(b) *$K$ is spherically complete.*
*Then the converse of* (vii) *holds.*
(ii) *Suppose either*
(a) *$X$ and $Y$ are metrizable and polar,*
*or*
(b) *$K$ is spherically complete,*
*or*
(c) *$Y$ is of countable type.*
*Then the converse of* (viii) *holds.*
(iii) *Suppose $X$ is metrizable. Then the converses of* (x) *hold i.e.*

*$Z$ is $\tau$-closed $\iff$ $(F_n)_n$ is $\alpha$-regular $\iff$ for all $B \subset Z$ that is $\tau$-bounded, $\overline{B}^{\tau} \subset Z$.*

*Proof.* For (i) and (ii), see [13], Propositions 5.3 and 5.4 for $(E_n)_n$ and [14], Remark 4.2 for $(F_n)_n$.

For (iii), see [14], Proposition 4.3.

**Corollary 5.3** ([14], Corollary 4.4) *Let $X, Y, Z$ and $(F_n)_n$ be as in Theorem 5.1. Suppose $X$ and $Y$ are metrizable. Then*

$$(F_n)_n \textit{ is regular} \iff Z \textit{ is } \tau\textit{-closed and } \tau|Z = \tau'|Z.$$

**Particular cases of Theorem 5.1** The following choices for $F$, $\tau$ and $\tau'$ are frequently used through the paper.

**5.1.1** $F$:= the Köthe space $\Lambda^0(B)$ associated to an infinite matrix $B$, $\tau$:= the topology on $F$ defined by one fixed norm $p_{j_0}$, as defined in (1) for $j = j_0$, $\tau'$:= the normal topology $n_{0,\infty}$.

**5.1.2** $F$:= an infinite dimensional normed space of countable type with norm $\| \, . \, \|$, $\tau$:= the topology on $F$ defined by $\| \, . \, \|$, $\tau'$:= the topology on $F$ defined by the norm $\| \, . \, \|_f : x \mapsto \max(\|x\|, |f(x)|)$, where $f$ is a functional $F \to K$ that is not $\| \, . \, \|$-continuous.

**5.1.3** $F$:= an infinite dimensional polar normed space, $\tau = \tau'$:= the norm topology on $F$.

**5.1.4** $F$:= an infinite dimensional polar normed space, $\tau$:= the weak topology $\sigma(F, F')$ on $F$, $\tau'$:= the norm topology on $F$.

For any of these choices we usually change the subspace $Z$ in the examples given along the paper, according to the purpose of each of these examples.

Clearly the steps of 5.1.1, 5.1.2 and 5.1.3 are metrizable spaces (note that the $E_n$ of 5.1.2 and 5.1.3 are even normed). Using the hereditary properties of spaces of countable type and of polar spaces ([25], Propositions 4.12 and 5.3 respectively) we obtain that, for any $Z$, the steps of 5.1.1 and 5.1.2 are always of countable type, and that the steps of 5.1.3 and 5.1.4 are always polar, being of countable type if and only if $F$ is of countable type.

## A final Remark

Let us reflect on the character of the steps appearing in the examples given along the paper, analyzing how closed they are of being Banach spaces of countable type. For instance, the steps of Example 3.18 have these properties.

Theorems 2.13 and 4.1 tell us that the "quality" of the steps of Counterexamples 2.15 and 4.3 is the best possible one in the sense that if, for some of them, its steps are not (of countable type, normed, Banach, metrizable, Fréchet), it is not possible to make an improvement by requiring to the steps one of this extra (apparently mixed) properties. The same happens with Examples 2.4 and Example 2.10.4, thanks this time to Propositions 2.3 and 2.11 respectively.

Also, it follows from Proposition 2.16 that Examples 2.9.1, 2.10.2, 2.10.3, 2.12 and 4.5.1 (resp. Examples 2.9.2, 4.5.3) in which the steps are normed spaces of countable type (resp. metrizable spaces of countable type, metrizable polar spaces) do not have a counterpart for Banach (resp. Fréchet) steps.

Analogously, applying Proposition 2.3,(ii) (resp. Proposition 3.6, resp. Proposition 3.16) we obtain that Example 4.5.2 (resp. Examples 3.10 and 3.12, resp. Examples 3.17) in which the steps are locally convex spaces (resp. metrizable spaces) of countable type do not have a counterpart for metrizable (resp. normed, resp. Fréchet) steps.

But the following is unknown.

**Problem 1** Do there exist counterparts of Example 2.10.1 (resp. Examples 2.9.2, 3.17, 4.5.3, resp. Examples 3.10 and 3.12) with Banach (resp. normed, resp. Fréchet) steps?

**Problem 2** Does there exist a counterpart of Example 3.11 with normed (Banach, Fréchet) steps?

## References

[1] C. Bosch, J. Kucera, K. McKennon, *Dual characterization of the Dieudonné-Schwartz Theorem on bounded sets*, Internat. J. Math. & Math. Sci. **6** (1983), 189-192.

[2] G. Christol, Z. Mebkhout, *Sur le théorème de l'indice des équations différentielles p-adiques* I, Ann. Inst. Fourier **43** (1993), 1545-1574.

[3] G. Christol, Z. Mebkhout, *Sur le théorème de l'indice des équations différentielles p-adiques* II, Ann. of Math. **146** (1997), 345-410.

[4] G. Christol, Z. Mebkhout, *Sur le théorème de l'indice des équations différentielles p-adiques* III, Ann. of Math. **151** (2000), 385-457.

[5] G. Christol, Z. Mebkhout, *Sur le théorème de l'indice des équations différentielles p-adiques* IV, Invent. Math. **143** (2001), 629-672.

[6] J. Dieudonné, L. Schwartz, *La dualité dans les espaces* $(\mathcal{F})$ *and* $(\mathcal{LF})$, Ann. Inst. Fourier Grenoble **1** (1949), 61-101.

[7] R. Engelking, *General Topology*, Heldermann Verlag, Berlin, 1989.

[8] N. De Grande-De Kimpe, *Non-archimedean Fréchet spaces generalizing spaces of analytic functions*, Indag. Math. **44** (1982), 423-439.

[9] N. De Grande-De Kimpe, J. Kąkol, C. Perez-Garcia, W.H. Schikhof, *p-adic locally convex inductive limits*, Lecture Notes in Pure and Appl. Math., **192**, Marcel Dekker, New York, 1997, 159-222.

[10] N. De Grande-De Kimpe, A. Khrennikov, *The non-archimedean Laplace Transform*, Bull. Belg. Math. Soc. Simon Stevin **3** (1996), 225-237.

[11] N. De Grande-De Kimpe, A. Khrennikov, L. van Hamme, *The Fourier Transform for p-adic tempered distributions*, Lecture Notes in Pure and Appl. Math., **207**, Marcel Dekker, New York, 1999, 97-112.

[12] N. De Grande-De Kimpe, C. Perez-Garcia, *Weakly closed subspaces and the Hahn-Banach extension property in p-adic Analysis*, Indag. Math. **91** (1988), 253-261.

[13] N. De Grande-De Kimpe, C. Perez-Garcia, *Strictness and closedness in p-adic inductive limits*, Contemporary Mathematics, **384**, American Mathematical Society, Providence, 2005, 79-100.

[14] N. De Grande-De Kimpe, C. Perez-Garcia, *The Dieudonné-Schwartz Theorem for p-adic inductive limits*. Bull. Belg. Math. Soc. **14** (2007), 33-50.

[15] J. Horvath, *Topological Vector Spaces and Distributions*, Vol I, Addison Wesley, 1966.

[16] J. Kucera, C. Bosch, *Dieudonné-Schwartz Theorem on bounded sets in inductive limits* II, Proc. Amer. Math. Soc. **86** (1982), 392-394.

[17] J. Kucera, K. McKennon, *Bounded sets in inductive limits*, Proc. Amer. Math. Soc. **69** (1978), 62-64.

[18] J. Kucera, K. McKennon, *Dieudonné-Schwartz Theorem on bounded sets in inductive limits*, Proc. Amer. Math. Soc. **78** (1980), 366-368.

[19] J. Kucera, K. McKennon, *Köthe's Example of an incomplete LB-space*, Proc. Amer. Math. Soc. **93** (1985), 79-81.

[20] B.M. Makarov, *Pathological properties of inductive limits of Banach spaces*, Uspekhi Mat. Nauk. **18** (1963), 171-178.

[21] C. Perez-Garcia, W.H. Schikhof, *Compact operators and the Orlicz-Pettis property in p-adic Analysis*, Report 9101, Department of Mathematics, University of Nijmegen, The Netherlands, 1991, 1-27.

[22] Qui Jing-Hui, *Dieudonné-Schwartz Theorem in inductive limits of metrizable spaces*, Proc. Amer. Math. Soc. **92** (1984), 255-257.

[23] P. Robba, G. Christol, *Équations différentielles p-adiques*, Hermann, Paris, 1994.

[24] A.C.M. van Rooij, *Non-Archimedean Functional Analysis*, Marcel Dekker, New York, 1978.

[25] W.H. Schikhof, *Locally convex spaces over nonspherically complete valued fields* I-II, Bull. Soc. Math. Belg. Sér. B **38** (1986), 187-224.

[26] W.H. Schikhof, *The complementation property of $\ell^\infty$ in p-adic Banach spaces*, Lecture Notes in Math., **1454**, Springer Verlag, Berlin, 1990, 342-350.

[27] W.H. Schikhof, *A perfect duality between p-adic Banach spaces and compactoids*, Indag. Math. N.S. **6** (1995), 325-339.

[28] A.E. Taylor, D.C. Lay, *Introduction to Functional Analysis*, John Wiley & Sons, New York, 1980.

Groene Laan 36 (302)
B 2830 Willebroek
Belgium

Department of Mathematics
Facultad de Ciencias, Universidad de Cantabria
Avda. de los Castros s/n
39071 Santander
Spain
email: perezmc@unican.es