The asymptotical case of certain quasiconformal extension results for holomorphic mappings in $\mathbb{C}^n$

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Abstract

Let $f(z,t)$ be a non-normalized subordination chain and assume that $f(\cdot,t)$ is K-quasiregular on B^n for $t \in [0,\alpha]$. In this paper we obtain a sufficient condition for $f(\cdot,0)$ to be extended to a quasiconformal homeomorphism of $\overline{\mathbb{R}}^{2n}$ onto $\overline{\mathbb{R}}^{2n}$. Finally we obtain certain applications of this result. One of these applications can be considered the asymptotical case of the n-dimensional version of the well known quasiconformal extension result due to Ahlfors and Becker.

1 Introduction and preliminaries

Let $\mathbb{C}^n$ denote the space of n-complex variables $z = (z_1, \dots, z_n)$ with the Euclidean inner product $\langle \cdot, \cdot \rangle$ and the Euclidean norm $\|z\| = \langle z, z \rangle^{1/2}$. Let $B_r^n = \{z \in \mathbb{C}^n : \|z\| < r\}$ and let $B^n = B_1^n$. Also let $\overline{B}^n$ be the closed unit ball in $\mathbb{C}^n$. In the case of one complex variable, B_r^1 is denoted by U_r and B_1^1 by U. Let $\overline{\mathbb{R}}^m = \mathbb{R}^m \cup \{\infty\}$ be the one point compactification of $\mathbb{R}^m$. If $\Omega \subset \mathbb{C}^n$ is a domain, let $H(\Omega)$ be the set of holomorphic mappings from Ω into $\mathbb{C}^n$. If $f \in H(B^n)$, let $J_f(z) = \det Df(z)$ be the complex jacobian determinant of f at z. Also let $L(\mathbb{C}^n, \mathbb{C}^m)$ be the space of continuous linear mappings from $\mathbb{C}^n$ into $\mathbb{C}^m$ with the standard operator norm

$$\|A\| = \sup\{\|A(z)\| : \|z\| = 1\},$$

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and let I_n be the identity in $L(\mathbb{C}^n, \mathbb{C}^n)$. A mapping $f \in H(B^n)$ is said to be normalized if $f(0) = 0$ and $Df(0) = I_n$.

We say that a mapping $f \in H(B^n)$ is K-quasiregular, $K \geq 1$, if

$$\|Df(z)\|^n \leq K|J_f(z)|,\ z \in B^n.$$

In addition, a mapping $f \in H(B^n)$ is called quasiregular if f is K-quasiregular for some $K \geq 1$. It is well known that quasiregular holomorphic mappings are locally biholomorphic.

Definition 1.1. Let Ω and Ω' be domains in $\overline{\mathbb{R}}^m$. A homeomorphism $f : \Omega \to \Omega'$ is said to be K-quasiconformal if it is differentiable a.e., ACL (absolutely continuous on lines) and

$$\|Df(x)\|^m \leq K|\det Df(x)| \text{ a.e. } x \in \Omega,$$

where $Df(x)$ denotes the (real) Jacobian matrix of f, K is a constant and

$$\|Df(x)\| = \sup\{\|Df(x)(a)\| : \ \|a\| = 1\}.$$

We remark that a K-quasiregular biholomorphic mapping is K^2-quasiconformal. For details about quasiregular and quasiconformal mappings, see [17] and [18].

If $f, g \in H(B^n)$, we say that f is subordinate to g (write $f \prec g$) if there is a Schwarz mapping v (i.e. $v \in H(B^n)$ and $\|v(z)\| \leq \|z\|$, $z \in B^n$) such that $f(z) = g(v(z))$, $z \in B^n$.

Definition 1.2. Let $\alpha > 0$. A mapping $f : B^n \times [0, \alpha] \to \mathbb{C}^n$ is called a subordination chain if the following conditions are satisfied:

(i) $f(0, t) = 0$ and $f(\cdot, t) \in H(B^n)$ for $t \in [0, \alpha]$;

(ii) $f(\cdot, s) \prec f(\cdot, t)$ for $0 \leq s \leq t \leq \alpha$.

Moreover, if $f(z, t)$ is a subordination chain such that $f(\cdot, t)$ is biholomorphic on B^n for $t \in [0, \alpha]$, we say that $f(z, t)$ is a Loewner chain (or a univalent subordination chain). In this case, the condition (ii) is equivalent to the fact that there is a unique biholomorphic Schwarz mapping $v = v(z, s, t)$ such that

$$f(z, s) = f(v(z, s, t), t),\ z \in B^n,\ 0 \leq s \leq t \leq \alpha.$$

If $f(z, t)$ is a Loewner chain such that $Df(0, t) = e^t I_n$, we say that $f(z, t)$ is a normalized Loewner chain.

An important role in our discussion is played by the following sets:

$$\mathcal{N} = \Big\{h \in H(B^n) : \ h(0) = 0,\ \mathrm{Re}\ \langle h(z), z\rangle > 0,\ z \in B^n \setminus \{0\}\Big\},$$

$$\mathcal{M} = \Big\{h \in \mathcal{N} : \ Dh(0) = I_n\Big\}.$$

We next use the following results due to Hamada and Kohr [12] (cf. [14, Theorems 2.1 and 2.2]):

Lemma 1.3. *Let $h = h(z,t) : B^n \times [0,\alpha] \to \mathbb{C}^n$ satisfy the following conditions:*

(i) $h(\cdot,t) \in \mathcal{M}$ for each $t \in [0,\alpha]$;

(ii) $h(z,\cdot)$ is measurable on $[0,\alpha]$ for each $z \in B^n$.

Then for each $s \in [0,\alpha)$ and $z \in B^n$, the initial value problem

$$\frac{\partial v}{\partial t} = -h(v,t) \text{ a.e. } t \in [s,\alpha], \ v(z,s,s) = z, \tag{1.1}$$

has a unique solution $v_{s,t}(z) = v(z,s,t) = e^{s-t}z + \cdots$ such that $v(z,s,\cdot)$ is absolutely continuous on $[s,\alpha]$ locally uniformly with respect to $z \in B^n$. Further, for fixed s and t, $v_{s,t}$ is a biholomorphic Schwarz mapping on B^n.

Lemma 1.4. *Let $h(z,t)$ satisfy the assumptions in Lemma 1.3. Also let $f = f(z,t) : B^n \times [0,\alpha] \to \mathbb{C}^n$ be a mapping such that $f(\cdot,t) \in H(B^n)$, $f(0,t) = 0$, $Df(0,t) = e^t I_n$, $t \in [0,\alpha]$, and $f(z,\cdot)$ is absolutely continuous on $[0,\alpha]$ locally uniformly with respect to $z \in B^n$. Assume that*

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t) \text{ a.e. } t \in [0,\alpha], \ \forall\ z \in B^n. \tag{1.2}$$

Then $f(z,s) = f(v(z,s,t),t)$, $z \in B^n$, $0 \le s \le t \le \alpha$, where $v = v(z,s,t)$ is the unique solution of (1.1). Hence $f(z,t)$ is a subordination chain.

Remark 1.5. Let $f(z,t) = e^t z + \cdots$ be a Loewner chain on $[0,\alpha]$. Then there is a mapping $h = h(z,t) : B^n \times [0,\alpha] \to \mathbb{C}^n$ which satisfies the conditions (i) and (ii) in Lemma 1.3 such that the Loewner differential equation (1.2) is satisfied (see [9] and [8]).

Recently, Hamada and Kohr proved the following useful result ([12]; cf. [11]) which extends to the n-dimensional case a well known result due to Becker and Pommerenke [2, Satz 3].

Theorem 1.6. *Let $\alpha > 0$ and $f = f(z,t) : B^n \times [0,\alpha] \to \mathbb{C}^n$ be a mapping such that $f(z,\cdot)$ is absolutely continuous on $[0,\alpha]$ locally uniformly with respect to $z \in B^n$, $f(\cdot,t) \in H(B^n)$, $f(0,t) = 0$, and $Df(0,t) = e^t I_n$ for $t \in [0,\alpha]$. Also let $h = h(z,t) : B^n \times [0,\alpha] \to \mathbb{C}^n$ satisfy the conditions in Lemma 1.3. Assume that $f(z,t)$ satisfies the differential equation (1.2) and $f(\cdot,0)$ is continuous and injective on $\overline{B}^n$. Moreover, assume the following conditions hold:*

(i) There exist constants $M_1 > 0$ and $k \in [0,1)$ such that

$$\|Df(z,t)\| \le \frac{M_1}{(1-\|z\|)^k}, \quad z \in B^n, \ t \in [0,\alpha];$$

(ii) There exists a constant $c_1 > 0$ such that

$$\mathrm{Re}\ \langle h(z,t), z\rangle \ge c_1\|z\|^2, \quad z \in B^n, \ t \in [0,\alpha];$$

(iii) There exists a constant $c_2 > 0$ such that

$$\|h(z,t)\| \le c_2, \quad z \in B^n, \ t \in [0,\alpha];$$

(iv) There exists a constant $K > 0$ such that

$$\|Df(z,t)\|^n \leq K|\det Df(z,t)|, \quad z \in B^n,\ t \in [0,\alpha].$$

Then there exists a constant $\tau \in (0,\alpha)$ such that $f(\cdot,t)$ is continuous and injective on $\overline{B}^n$ for $t \in [0,\tau]$ and there exists a quasiconformal homeomorphism F^* of $\overline{\mathbb{R}}^{2n}$ onto itself such that $F^*|_{B^n} = f(\cdot,0)$.

In this paper we continue the work begun in [12] and obtain a sufficient condition for a normalized quasiconformal biholomorphic mapping f on B^n, which can be imbedded as the first element of a non-normalized subordination chain over $[0,\alpha]$, to be extended to a quasiconformal homeomorphism of $\overline{\mathbb{R}}^{2n}$ onto itself. We also obtain certain applications of this result, including the asymptotical case of the n-dimensional version of the well known quasiconformal extension result due to Ahlfors and Becker.

2 Main results

We begin this section with the following result (cf. [13, Lemma 2.1]).

Lemma 2.1. *Let $\alpha > 0$ and $h = h(z,t) : B^n \times [0,\alpha] \to \mathbb{C}^n$ satisfy the following conditions:*

(i) $h(\cdot,t) \in \mathcal{N}$, $Dh(0,t) = c(t)I_n$ where $c : [0,\alpha] \to \mathbb{C}$ is a continuous function such that Re $c(t) > 0$, $t \in [0,\alpha]$.

(ii) $h(z,\cdot)$ is measurable on $[0,\alpha]$, for all $z \in B^n$.

Then for each $s \in [0,\alpha)$ and $z \in B^n$, the initial value problem

$$\frac{\partial v}{\partial t} = -h(v,t) \ \text{a.e.}\ t \in [s,\alpha],\ v(z,s,s) = z, \tag{2.1}$$

has a unique solution $v = v(z,s,t)$ such that $v(z,s,\cdot)$ is absolutely continuous on $[s,\alpha]$ locally uniformly with respect to $z \in B^n$. Moreover, for fixed s and t, $v(\cdot,s,t)$ is a biholomorphic Schwarz mapping and $Dv(0,s,t) = (a(s)/a(t))I_n$ where $a(t) = \exp\int_0^t c(\tau)d\tau$.

Proof. Let

$$\gamma(t) = \int_0^t \mathrm{Re}\ c(\tau)d\tau \ \text{ and } \ \beta(t) = \int_0^t \mathrm{Im}\ c(\tau)d\tau.$$

Also let

$$z^* = e^{i\beta(t)}z \ \text{ and } \ t^* = \gamma(t),\ z \in B^n,\ t \in [0,\alpha].$$

Then $\|z^*\| = \|z\|$ and since $\dot{\gamma}(t) \geq 0$ for $t \in [0,\alpha]$, it follows that t^* is a function of $[0,\alpha]$ onto $[0,\alpha^*]$ where $\alpha^* = \gamma(\alpha)$.

Further, let $h^* : B^n \times [0,\alpha] \to \mathbb{C}^n$ be given by

$$h^*(z,t^*) = \frac{1}{\mathrm{Re}\ c(t)}\Big[e^{i\beta(t)}h(e^{-i\beta(t)}z,t) - i\mathrm{Im}\ c(t)z\Big].$$

Then $h^*(\cdot,t^*)\in H(B^n)$, $h^*(0,t^*)=0$ and $Dh(0,t^*)=I_n$ for $t^*\in[0,\alpha^*]$. Moreover, since $h(z,\cdot)$ is measurable on $[0,\alpha]$ for $z\in B^n$, it follows that $h^*(z,\cdot)$ is also measurable on $[0,\alpha^*]$. Also

$$\mathrm{Re}\,\langle h^*(z,t^*),z\rangle=\frac{1}{\mathrm{Re}\,c(t)}\mathrm{Re}\,\langle h(e^{-i\beta(t)}z,t),e^{-i\beta(t)}z\rangle>0,\ z\in B^n\setminus\{0\},\ t\in[0,\alpha].$$

Hence $h^*(\cdot,t^*)\in\mathcal{M}$ for $t^*\in[0,\alpha^*]$. Taking into account Lemma 1.3, we deduce that the initial value problem

$$\frac{\partial v^*}{\partial t^*}=-h^*(v^*,t^*)\quad a.e.\quad t^*\in[s^*,\alpha^*],\ v^*(z^*,s^*,s^*)=z^*,\tag{2.2}$$

has a unique solution $v^*=v^*(z^*,s^*,t^*)=e^{s^*-t^*}z^*+\cdots$ such that for fixed s^* and t^*, $v^*(\cdot,s^*,t^*)$ is a biholomorphic Schwarz mapping. Moreover, $v^*(z^*,s^*,\cdot)$ is absolutely continuous on $[s^*,\alpha^*]$ locally uniformly with respect to $z^*\in B^n$. In fact, $v^*(z^*,s^*,\cdot)$ is Lipschitz continuous on $[s^*,\alpha^*]$ locally uniformly with respect to $z^*\in B^n$ (cf. [10, Chapter 8]).

Now, let

$$v(z,s,t)=e^{-i\beta(t)}v^*(e^{i\beta(s)}z,\gamma(s),\gamma(t)),\ z\in B^n,\ 0\le s\le t\le\alpha.$$

It is not difficult to deduce that $v(\cdot,s,t)$ is a biholomorphic Schwarz mapping and

$$Dv(0,s,t)=e^{i(\beta(s)-\beta(t))}Dv^*(0,\gamma(s),\gamma(t))=\frac{a(s)}{a(t)}I_n.$$

Since $v^*(z^*,s^*,\cdot)$ is absolutely continuous on $[s^*,\alpha^*]$ locally uniformly with respect to $z^*\in B^n$, it follows that $v(z,s,\cdot)$ is absolutely continuous on $[s,\alpha]$ locally uniformly with respect to $z\in B^n$. In fact, $v(z,s,\cdot)$ is Lipschitz continuous on $[s,\alpha]$ locally uniformly with respect to $z\in B^n$. Moreover, an elementary computation, based on (2.2), yields that $v=v(z,s,t)$ is a solution of the initial value problem (2.1). Finally, using the uniqueness of solution to the initial value problem (2.2), we deduce that the initial value problem (2.1) has also a unique solution. ∎

We are now able to prove the main result of this paper, which is a generalization of [12, Theorem 3.1] to the case of non-normalized subordination chains.

Theorem 2.2. *Let $c:[0,\alpha]\to\mathbb{C}$ be a continuous function such that $\min\limits_{t\in[0,\alpha]}\mathrm{Re}\,c(t)>0$ and let $h(z,t)$ satisfy the assumptions of Lemma 2.1. Also let $f(z,t)=a(t)z+\cdots$ be a mapping such that $f(\cdot,t)\in H(B^n)$, $f(0,t)=0$, $Df(0,t)=a(t)I_n$, where $a(t)=\exp\int_0^t c(\tau)d\tau$, and $f(z,\cdot)$ is absolutely continuous on $[0,\alpha]$ locally uniformly with respect to $z\in B^n$. Suppose that $f(z,t)$ satisfies the differential equation*

$$\frac{\partial f}{\partial t}(z,t)=Df(z,t)h(z,t)\ a.e.\ t\in[0,\alpha],\ \forall\ z\in B^n.\tag{2.3}$$

Moreover, assume that $f(\cdot,0)$ is continuous and injective on $\overline{B}^n$. Also assume that the following conditions hold:

(i) There exist some constants $M > 0$ and $k \in [0,1)$ such that

$$\|Df(z,t)\| \leq \frac{M|a(t)|}{(1-\|z\|)^k}, \quad z \in B^n, \ t \in [0,\alpha];$$

(ii) There exists a constant $c_1 > 0$ such that

$$\mathrm{Re}\ \langle h(z,t), z\rangle \geq c_1\|z\|^2, \quad z \in B^n, \ t \in [0,\alpha];$$

(iii) There exists a constant $c_2 > 0$ such that

$$\|h(z,t)\| \leq c_2, \quad z \in B^n, \ t \in [0,\alpha];$$

(iv) There exists a constant $K > 0$ such that $f(\cdot,t)$ is K-quasiregular for each $t \in [0,\alpha]$.

Then there exists a constant $\tau \in (0,\alpha)$ such that $f(\cdot,t)$ is continuous and injective on $\overline{B}^n$ for $t \in [0,\tau]$, and there exists a quasiconformal homeomorphism F of $\overline{\mathbb{R}}^{2n}$ onto itself such that $F|_{B^n} = f(\cdot,0)$.

Proof. As in the proof of Lemma 2.1, let

$$\gamma(t) = \int_0^t \mathrm{Re}\ c(\lambda)d\lambda \text{ and } \beta(t) = \int_0^t \mathrm{Im}\ c(\lambda)d\lambda.$$

Also let $t^* = \gamma(t)$, $t \in [0,\alpha]$,

$$f^*(z,t^*) = f(e^{-i\beta(t)}z,t), \ z \in B^n, \ t \in [0,\alpha],$$

and

$$h^*(z,t^*) = \frac{1}{\mathrm{Re}\ c(t)}\Big[e^{i\beta(t)}h(e^{-i\beta(t)}z,t) - i\mathrm{Im}\ c(t)z\Big].$$

Since $f(\cdot,t) \in H(B^n)$, $f(0,t) = 0$, $Df(0,t) = a(t)I_n$, $t \in [0,\alpha]$, it is easy to see that $f^*(\cdot,t^*) \in H(B^n)$, $f^*(0,t^*) = 0$ and

$$Df^*(0,t^*) = e^{-i\beta(t)}a(t)I_n = e^{t^*}I_n, \quad t^* \in [0,\alpha^*],$$

where $\alpha^* = \gamma(\alpha)$. Also, since $f(z,\cdot)$ is absolutely continuous on $[0,\alpha]$ locally uniformly with respect to $z \in B^n$ and β is of class C^1 on $[0,\alpha]$, it is clear that $f^*(z,\cdot)$ is also absolutely continuous on $[0,\alpha^*]$ locally uniformly with respect to $z \in B^n$. Moreover, $f^*(\cdot,0)$ is continuous and injective on $\overline{B}^n$, since $f(\cdot,0)$ is continuous and injective on $\overline{B}^n$. In view of the condition (i) in the hypothesis, we deduce that

$$\|Df^*(z,t^*)\| = \|Df(e^{-i\beta(t)}z,t)\| \leq \frac{M|a(t)|}{(1-\|e^{-i\beta(t)}z\|)^k} = \frac{Me^{t^*}}{(1-\|z\|)^k},$$

for all $z \in B^n$ and $t^* \in [0,\alpha^*]$. Hence $f^*(z,t^*)$ satisfies the assumption (i) in Theorem 1.6.

On the other hand, taking into account the condition (ii), we deduce that

$$\mathrm{Re}\ \langle h^*(z,t^*), z\rangle = \frac{1}{\mathrm{Re}\ c(t)}\mathrm{Re}\ \langle h(e^{-i\beta(t)}z,t), e^{-i\beta(t)}z\rangle \geq \frac{c_1\|z\|^2}{\mathrm{Re}\ c(t)}, \tag{2.4}$$

for all $z \in B^n$ and $t^* \in [0, \alpha^*]$.

Since $\|h(z,t)\| \le c_2$ for $z \in B^n$ and $t \in [0, \alpha]$, it follows in view of Schwarz's lemma that

$$\|Dh(0,t)\| \le c_2,\ t \in [0, \alpha],$$

and thus $|c(t)| \le c_2$, $t \in [0, \alpha]$. Hence, from (2.4) we obtain that

$$\mathrm{Re}\ \langle h^*(z,t^*), z\rangle \ge \frac{c_1}{c_2}\|z\|^2, \quad z \in B^n,\ t^* \in [0, \alpha^*].$$

Further, since

$$\|h^*(z,t^*)\| \le \frac{1}{\mathrm{Re}\ c(t)}[\|h(e^{-i\beta(t)}z,t)\| + |\mathrm{Im}\ c(t)|\|z\|],$$

we deduce from the condition (iii) and the above inequality that

$$\|h^*(z,t^*)\| \le \frac{2c_2}{\mathrm{Re}\ c(t)} \le \frac{2c_2}{\min\limits_{t\in[0,\alpha]} \mathrm{Re}\ c(t)}, \quad z \in B^n,\ t^* \in [0, \alpha^*].$$

Therefore, we have proved that the mapping $h^*(z,t^*)$ satisfies the conditions (ii) and (iii) in Theorem 1.6.

Finally, since $f(\cdot,t)$ is K-quasiregular for $t \in [0, \alpha]$, we deduce that

$$\begin{aligned}\|Df^*(z,t^*)\|^n &= \|Df(e^{-i\beta(t)}z,t)\|^n \le K|\det Df(e^{-i\beta(t)}z,t)| \\ &= K|\det Df^*(z,t^*)|,\ z \in B^n,\ t \in [0,\alpha^*],\end{aligned}$$

and hence $f^*(z,t^*)$ is K-quasiregular too on B^n for $t^* \in [0, \alpha^*]$.

Consequently, taking into account Theorem 1.6, there is a constant $\tau^* \in (0, \alpha^*)$ such that $f^*(z,t^*)$ is continuous and injective on $\overline{B}^n$ for $t^* \in [0, \tau^*]$, and there is a quasiconformal extension F^* of $\overline{\mathbb{R}}^{2n}$ onto itself such that $f^*(z,0) = F^*(z)$ for $z \in B^n$. Since γ is a homeomorphism of $[0, \alpha]$ onto $[0, \alpha^*]$, there is a unique $\tau \in (0, \alpha)$ such that $\gamma(\tau) = \tau^*$, and hence $f(z,t) = f^*(e^{i\beta(t)}z, \gamma(t))$ is continuous and injective on $\overline{B}^n$ for $t \in [0, \tau]$. Finally, since $f(z,0) = f^*(z,0)$, $z \in \overline{B}^n$, it follows that $f(\cdot,0)$ extends to a quasiconformal homeomorphism F of $\overline{\mathbb{R}}^{2n}$ onto itself such that $F|_{B^n} = f(\cdot,0)$, as desired. This completes the proof. ∎

Remark 2.3. If there is a mapping $E(z,t) : B^n \times [0,\alpha] \to L(\mathbb{C}^n, \mathbb{C}^n)$ which is holomorphic in z and such that $E(0,t) = 0$ and $\|E(z,t)\| \le c < 1$ for $z \in B^n$, $t \in [0, \alpha]$, then the mapping $h(z,t)$ given by

$$h(z,t) = [I_n - E(z,t)]^{-1}[I_n + E(z,t)](z), \quad z \in B^n,\ t \in [0, \alpha],$$

satisfies the conditions (ii) and (iii) in Theorem 2.2.

Proof. Indeed, it is clear that

$$\|h(z,t)\| \le \frac{1+c}{1-c},\ z \in B^n,\ t \in [0, \alpha].$$

On the other hand, since

$$h(z,t) - z = E(z,t)(h(z,t)+z),\ z \in B^n,\ t \in [0,\alpha],$$

we deduce that

$$\|h(z,t) - z\| \le c\|h(z,t)+z\|,\ z \in B^n,\ t \in [0,\alpha],$$

and by an elementary computation in the above relation, we obtain that

$$\mathrm{Re}\ \langle h(z,t), z\rangle \ge \|z\|^2 \frac{1-c\|z\|}{1+c\|z\|} \ge \|z\|^2 \frac{1-c}{1+c},\ z \in B^n,\ t \in [0,\alpha].$$

■

3 Applications

In this section we obtain certain applications of Theorem 2.2. The first result may be considered the asymptotical case of the n-dimensional version of Ahlfors' and Becker's quasiconformal extension result [2] (see also [1]) (cf. [6]).

Theorem 3.1. *Let $k \in [0,1)$, $c \in \mathbb{C}$, $|c| \le k$, and let $f : \overline{B}^n \to \mathbb{C}^n$ be a normalized quasiregular holomorphic mapping on B^n and continuous and injective on $\overline{B}^n$. Assume there exists $r \in (0,1)$ such that*

$$\|(1-\|z\|^2)[Df(z)]^{-1}D^2 f(z)(z,\cdot) + c\|z\|^2 I_n\| \le k,\ r \le \|z\| < 1. \tag{3.1}$$

Then f can be extended to a quasiconformal homeomorphism of $\overline{\mathbb{R}}^{2n}$ onto itself.

Proof. We prove that the mapping

$$f(z,t) = f(ze^{-t}) + \frac{1}{1+c}(e^t - e^{-t})Df(ze^{-t})(z)$$

satisfies the conditions of Theorem 2.2 on $B^n \times [0,\alpha]$ where $\alpha = -\ln r$.

Indeed, $f(\cdot,t) \in H(B^n)$, $f(0,t) = 0$, $Df(0,t) = a(t)I_n$, $t \in [0,\alpha]$, where

$$a(t) = \frac{e^t(1+ce^{-2t})}{1+c} = \exp \int_0^t c(\tau)d\tau$$

and $c(t) = (1-ce^{-2t})/(1+ce^{-2t})$. Then Re $c(t) \ge (1-|c|)/(1+|c|)$ for $t \in [0,\alpha]$, and hence $\min\limits_{t\in[0,\alpha]}$ Re $c(t) > 0$. Also $f(z,\cdot) \in C^1([0,\alpha])$ for $z \in B^n$. Next, let

$$E(z,t) = -ce^{-2t}I_n - (1-e^{-2t})[Df(ze^{-t})]^{-1}D^2 f(ze^{-t})(ze^{-t},\cdot),\ z \in B^n,\ t \in [0,\alpha].$$

Then $\|E(z,0)\| = |c| \le k$ for $z \in B^n$. Further, using the maximum modulus theorem for holomorphic mappings into complex Banach spaces and the condition (3.1), we deduce that

$$\|E(z,t)\| \le \max_{\|w\|=1} \|E(w,t)\| \le k,\ z \in B^n,\ t \in (0,\alpha].$$

Therefore,

$$\|E(z,t)\| \leq k,\ z \in B^n,\ t \in [0,\alpha]. \tag{3.2}$$

Straightforward computations yield that

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \quad t \in [0,\alpha],\ z \in B^n,$$

where

$$h(z,t) = [I_n - E(z,t)]^{-1}[I_n + E(z,t)](z).$$

Then $h(\cdot,t) \in H(B^n)$, $h(0,t) = 0$ and $Dh(0,t) = c(t)I_n$ for $t \in [0,\alpha]$. Also $h(z,\cdot)$ is measurable on $[0,\alpha]$ for $z \in B^n$. In view of (3.2), we deduce that the conditions (ii) and (iii) in Theorem 2.2 are satisfied by Remark 2.3.

On the other hand, taking into account the relation (3.1), we have

$$(1 - \|z\|^2)\|[Df(z)]^{-1}D^2f(z)(z,\cdot)\| \leq k + |c| \leq 2k,\ r \leq \|z\| < 1. \tag{3.3}$$

Then, using the above inequality and an argument similar to that in the proof of [12, Theorem 4.1] (cf. [15, Theorem 2.1]), we deduce that there exists some absolute constant $M > 0$ such that

$$\|Df(z,t)\| \leq \frac{M|a(t)|}{(1 - \|z\|)^k}, \quad z \in B^n,\ t \in [0,\alpha]. \tag{3.4}$$

Indeed, fix $w \in \partial B^n$ and let $g(\zeta) = \det A(\zeta)$ where $A(\zeta) = Df(\zeta w)$, $|\zeta| < 1$. Since $g(|\zeta|)$ is uniformly bounded on the disc U_r, there is a constant $C_1 > 0$ such that

$$|g(|\zeta|)| \leq \frac{C_1}{(1 - |\zeta|)^{nk}},\ |\zeta| < r. \tag{3.5}$$

On the other hand, in view of the relation (3.3), we obtain by an argument similar to that in the proof of [15, Theorem 2.1],

$$\left|\zeta \frac{d}{d\zeta} \log g(\zeta)\right| \leq \frac{2nk}{1 - |\zeta|^2},\ r \leq |\zeta| < 1.$$

Then

$$|\log g(|\zeta|)| \leq \int_r^{|\zeta|} \frac{2nk}{(1-\tau^2)\tau} d\tau + C_2 \leq \int_r^{|\zeta|} \frac{nk}{1-\tau} d\tau + C_3 = -nk\log(1 - |\zeta|) + C_4,$$

where C_4 is a constant which does not depend on $w \in \partial B^n$. Consequently, we deduce that

$$|g(|\zeta|)| \leq \frac{e^{C_4}}{(1 - |\zeta|)^{nk}},\ r \leq |\zeta| < 1. \tag{3.6}$$

Combining the relations (3.5) and (3.6), we obtain that

$$|g(|\zeta|)| \leq O\left(\frac{1}{(1 - |\zeta|)^{nk}}\right),\ |\zeta| < 1.$$

Setting $w = z/\|z\|$ and $\zeta = \|z\|$ in the above inequality, we deduce that there is a constant $K > 0$ such that

$$|\det Df(z)| \leq \frac{K}{(1-\|z\|)^{nk}},\ z \in B^n.$$

On the other hand, since f is quasiregular on B^n, there is a constant $L_1 \geq 1$ such that

$$\|Df(z)\|^n \leq L_1 |\det Df(z)|,\ z \in B^n,$$

and hence

$$\|Df(z)\| \leq \frac{L_2}{(1-\|z\|)^k},\ z \in B^n,$$

for some constant $L_2 > 0$. Further, since

$$Df(z,t) = \frac{e^t}{1+c} Df(ze^{-t})[I_n - E(z,t)],\ z \in B^n,\ t \in [0,\alpha],$$

we deduce that

$$\begin{aligned}\|Df(z,t)\| &\leq \frac{e^t L_2(1+k)}{|1+c|(1-\|ze^{-t}\|)^k} \leq \frac{L_2(1+k)e^t}{|1+c|(1-\|z\|)^k} \\ &= \frac{L_2(1+k)|a(t)|}{(1-\|z\|)^k} \cdot \frac{1}{|1+ce^{-2t}|} \leq \frac{M|a(t)|}{(1-\|z\|)^k}\end{aligned}$$

for some constant $M > 0$. Hence the relation (3.4) holds, as claimed.

Finally, we deduce that $f(z,t)$ is $\tilde{L}$-quasiregular for some $\tilde{L} \geq 1$. Indeed,

$$\begin{aligned}\|Df(z,t)\|^n &\leq \frac{e^{nt}}{|1+c|^n} \|Df(ze^{-t})\|^n \|I_n - E(z,t)\|^n \\ &\leq \frac{e^{nt}}{|1+c|^n} L_1 |\det Df(ze^{-t})|(1+k)^n \\ &= \frac{L_1(1+k)^n |\det Df(z,t)|}{|\det[I_n - E(z,t)]|} \leq \left(\frac{1+k}{1-k}\right)^n L_1 |\det Df(z,t)|,\end{aligned}$$

for all $z \in B^n$ and $t \in [0,\alpha]$. Hence $f(\cdot,t)$ is $\widetilde{L}$-quasiregular on B^n for $t \in [0,\alpha]$, where $\widetilde{L} = L_1 \left(\dfrac{1+k}{1-k}\right)^n$.

Therefore $f(z,t)$ satisfies the assumptions of Theorem 2.2, and thus $f = f(\cdot,0)$ extends to a quasiconformal homeomorphism of $\overline{\mathbb{R}}^{2n}$ onto $\overline{\mathbb{R}}^{2n}$, as desired. This completes the proof. ∎

We remark that if $c = 0$ in Theorem 3.1, we obtain [12, Theorem 4.1] that is the asymptotical case of [15, Theorem 2.1]. In the case $n = 1$, this result was obtained by Becker and Pommerenke [2, Satz4]. We have

Corollary 3.2. *Let $f : \overline{B}^n \to \mathbb{C}^n$ be a normalized quasiregular holomorphic mapping on B^n and continuous and injective on $\overline{B}^n$. Assume that*

$$\limsup_{\|z\|\to 1-0} \|(1-\|z\|^2)[Df(z)]^{-1} D^2 f(z)(z,\cdot)\| < 1.$$

Then f can be extended to a quasiconformal homeomorphism of $\overline{\mathbb{R}}^{2n}$ onto itself.

Using arguments similar to those in the proof of Theorem 3.1, we obtain the following result.

Theorem 3.3. *Let $f : \overline{B}^n \to \mathbb{C}^n$ be a normalized quasiregular holomorphic mapping on B^n and continuous and injective on $\overline{B}^n$. Also let $\alpha \geq 2$. Assume there exist some constants r, k with $r \in (0,1)$ and $\frac{\alpha}{2} - 1 \leq k < 2 - \frac{\alpha}{2}$ such that*

$$\left\|(1 - \|z\|^\alpha)[Df(z)]^{-1}D^2f(z)(z,\cdot) + \left(1 - \frac{\alpha}{2}\right) I_n\right\| \leq k,\ r \leq \|z\| < 1.$$

Then f can be extended to a quasiconformal homeomorphism of $\overline{\mathbb{R}}^{2n}$ onto itself.

Proof. To prove this result, it suffices to consider the mapping $f(z,t) : B^n \times [0,\lambda] \to \mathbb{C}^n$ given by

$$f(z,t) = f(ze^{-t}) + (e^{(\alpha-1)t} - e^{-t})Df(ze^{-t})(z),$$

where $\lambda = -\ln r$, and to apply arguments similar to those in the proof of Theorem 3.1. We leave the details for the reader. ∎

More generally, we obtain the following result (cf. [12]; compare with [11], [7] and [16]). Note that if $G(z) = Df(z)$ for $z \in B^n$, Theorem 3.4 reduces to Theorem 3.3. In this case the condition (iii) reduces to the fact that f is quasiregular on B^n. Also if $G(z) = Df(z)$ for $z \in B^n$, and $\alpha = 2$, then Theorem 3.4 reduces to Corollary 3.2.

Theorem 3.4. *Let $f : \overline{B}^n \to \mathbb{C}^n$ be a normalized holomorphic mapping on B^n and let $G(z)$ be a nonsingular $n \times n$ matrix, holomorphic as a function of $z \in B^n$, such that $G(0) = I_n$. Also let $\alpha \geq 2$. Assume f is continuous and injective on $\overline{B}^n$. Also assume there exist some constants r, k with $r \in (0,1)$ and $\frac{\alpha}{2} - 1 \leq k < 2 - \frac{\alpha}{2}$ such that the following conditions hold:*

(i) $\left\|[G(z)]^{-1}Df(z) - \frac{\alpha}{2}I_n\right\| \leq k$ *for* $z \in B^n$*;*

(ii)

$$\left\|\|z\|^\alpha[[G(z)]^{-1}Df(z) - I_n] + (1 - \|z\|^\alpha)[G(z)]^{-1}DG(z)(z,\cdot) + \left(1 - \frac{\alpha}{2}\right) I_n\right\| \leq k$$

for $r \leq \|z\| < 1$*;*

(iii) There exists a constant $K \geq 1$ such that $\|G(z)\|^n \leq K|\det G(z)|$ for $z \in B^n$.

Then f is quasiregular on B^n and can be extended to a quasiconformal homeomorphism of $\overline{\mathbb{R}}^{2n}$ onto itself.

Proof. Let $\lambda = -\ln r$ and

$$f(z,t) = f(ze^{-t}) + (e^{(\alpha-1)t} - e^{-t})G(ze^{-t})(z),\ z \in B^n,\ t \in [0,\lambda].$$

Next we apply arguments similar to those in the proof of Theorem 3.1, to deduce that $f(z,t)$ satisfies the conditions in Theorem 2.2 on $B^n \times [0,\lambda]$.

Indeed, $f(\cdot,t)\in H(B^n)$, $f(0,t)=0$, $Df(0,t)=a(t)I_n$, $t\in[0,\lambda]$, where $a(t)=e^{(\alpha-1)t}=\exp\int_0^t c(\tau)d\tau$ and $c(t)=\alpha-1$. Let

$$E(z,t)=-\frac{2}{\alpha}e^{(\alpha-1)t}\Big[[G(ze^{-t})]^{-1}Df(ze^{-t})-I_n\Big]$$
$$-\frac{2}{\alpha}(1-e^{-\alpha t})[G(ze^{-t})]^{-1}DG(ze^{-t})(ze^{-t},\cdot)+I_n\Big(1-\frac{2}{\alpha}\Big),$$

for all $z\in B^n$ and $t\in[0,\lambda]$. Then

$$\|E(z,0)\|=\frac{2}{\alpha}\Big\|[G(z)]^{-1}Df(z)-\frac{\alpha}{2}I_n\Big\|\leq\frac{2}{\alpha}k<1,\ z\in B^n,$$

by the condition (i). Next, fix $t\in(0,\lambda]$. In view of the maximum modulus theorem for holomorphic mappings into complex Banach spaces, we obtain that

$$\|E(z,t)\|\leq\max_{\|w\|=1}\|E(w,t)\|=$$

$$\frac{2}{\alpha}\max_{\|w\|=1}\Big\|\|we^{-t}\|^{\alpha}\Big[[G(we^{-t})]^{-1}Df(we^{-t})-I_n\Big]+$$
$$+(1-\|we^{-t}\|^{\alpha})[G(we^{-t})]^{-1}DG(we^{-t})(we^{-t},\cdot)+I_n\Big(1-\frac{\alpha}{2}\Big)\Big\|,\ z\in B^n.$$

Hence, we deduce from the condition (ii) that

$$\|E(z,t)\|\leq\frac{2}{\alpha}k<1,\ z\in B^n.$$

Therefore

$$\|E(z,t)\|\leq\frac{2}{\alpha}k,\ z\in B^n,\ t\in[0,\lambda],$$

and hence $I_n-E(z,t)$ is an invertible linear operator and

$$Df(z,t)=\frac{\alpha}{2}e^{(\alpha-1)t}G(ze^{-t})[I_n-E(z,t)],\ z\in B^n,\ t\in[0,\lambda]. \tag{3.7}$$

Straightforward computations yield that

$$\frac{\partial f}{\partial t}(z,t)=Df(z,t)h(z,t),\ \forall t\in[0,\lambda],\ z\in B^n,$$

where

$$h(z,t)=[I_n-E(z,t]^{-1}[I_n+E(z,t)](z).$$

Then $h(z,t)$ satisfies the assumptions of Lemma 2.1 by the same argument as in the proof of Theorem 3.1.

On the other hand, taking into account the conditions (i) and (ii) in the hypothesis, we deduce that

$$(1-\|z\|^{\alpha})\|[G(z)]^{-1}DG(z)(z,\cdot)\|$$

$$\leq k+\Big|1-\frac{\alpha}{2}\Big|+\|z\|^{\alpha}\Big[\Big\|[G(z)]^{-1}Df(z)-\frac{\alpha}{2}I_n\Big\|+\Big|1-\frac{\alpha}{2}\Big|\Big]$$
$$\leq\Big(k+\Big|1-\frac{\alpha}{2}\Big|\Big)(1+\|z\|^{\alpha})<2c,\ r\leq\|z\|<1,$$

where $c = k + (\alpha/2 - 1) < 1$. Since $\alpha \geq 2$, we deduce from the above relation that

$$(1 - \|z\|^2)\|[G(z)]^{-1}DG(z)(z, \cdot)\| \leq 2c,\ r \leq \|z\| < 1.$$

Next, it suffices to use arguments similar to those in the proof of Theorem 3.1 (see also the proof of [11, Theorem 4.1]), to deduce that there exists a constant $M > 0$ such that

$$|\det G(z)| \leq \frac{M}{(1 - \|z\|)^{nc}},\ z \in B^n.$$

Taking into account the condition (iii) and the above relation, we deduce that there exists a constant $L > 0$ such that

$$\|G(z)\| \leq \frac{L}{(1 - \|z\|)^c},\ z \in B^n.$$

Moreover, using the relation (3.7) and the above inequality, we deduce that

$$\|Df(z,t)\| \leq \frac{\alpha}{2}e^{(\alpha-1)t}\Big(1 + \frac{2}{\alpha}k\Big)\frac{L}{(1 - \|z\|)^c} = \frac{L^* a(t)}{(1 - \|z\|)^c},\ z \in B^n,\ t \in [0, \lambda].$$

Therefore, the mapping $f(z,t)$ satisfies the condition (i) in Theorem 2.2.

Further, in view of (3.7) and the condition (iii) in the hypothesis, we obtain that

$$\begin{aligned}
\|Df(z,t)\|^n &\leq \Big(\frac{\alpha}{2}\Big)^n e^{n(\alpha-1)t}\|G(ze^{-t})\|^n(1 + \|E(z,t)\|)^n \\
&\leq \Big(\frac{\alpha}{2}\Big)^n e^{n(\alpha-1)t}K|\det G(ze^{-t})|\Big(1 + \frac{2}{\alpha}k\Big)^n \\
&= \frac{K\Big(1 + \frac{2}{\alpha}k\Big)^n|\det Df(z,t)|}{|\det[I_n - E(z,t)]|} \leq \Big(\frac{1 + k_1}{1 - k_1}\Big)^n K|\det Df(z,t)|,
\end{aligned}$$

for all $z \in B^n$ and $t \in [0, \lambda]$, where $k_1 = 2k/\alpha$. Hence $f(\cdot, t)$ is $\tilde{K}$-quasiregular on B^n for $t \in [0, \lambda]$, where $\tilde{K} = K\Big(\frac{1 + k_1}{1 - k_1}\Big)^n$.

Finally, it suffices to use arguments similar to those in the proof of Theorem 3.1, to deduce that $f(z,t)$ satisfies all assumptions of Theorem 2.2. Consequently, $f = f(\cdot, 0)$ is quasiregular on B^n and can be extended to a quasiconformal homeomorphism of $\overline{\mathbb{R}}^{2n}$ onto itself, as desired. ∎

In particular, from Theorem 3.4 we obtain the following consequence, which generalizes [12, Theorem 4.3] (compare with [3]).

Corollary 3.5. *Let $f : \overline{B}^n \to \mathbb{C}^n$ be a normalized holomorphic mapping on B^n, which is continuous and injective on $\overline{B}^n$. Also let $\alpha \geq 2$ and $a : B^n \to \mathbb{C}$ be a holomorphic function such that $a(z) \neq 0$, $z \in B^n$, and $a(0) = 1$. Assume there exist some constants r, k with $r \in (0, 1)$ and $\frac{\alpha}{2} - 1 \leq k < 2 - \frac{\alpha}{2}$ such that the following conditions hold:*

$$\Big\|[a(z)]^{-1}Df(z) - \frac{\alpha}{2}I_n\Big\| \leq k,\ z \in B^n;$$

and

$$\left\| \|z\|^{\alpha}\{[a(z)]^{-1}Df(z)-I_n\}+(1-\|z\|^{\alpha})[a(z)]^{-1}\frac{da}{dz}(z)z+\left(1-\frac{\alpha}{2}\right)I_n \right\| \leq k,\ r \leq \|z\| < 1.$$

Then f is quasiregular on B^n and can be extended to a quasiconformal homeomorphism of $\overline{\mathbb{R}}^{2n}$ onto itself.

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