The topology of the class of functions representable by Carleman type formulae, duality and applications

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Abstract

We set D to be a simply connected domain and we consider exhaustion function spaces, $X_\infty(D)$ with the projective topology (see §1). We show that the natural topology on the topological dual of $X_\infty(D)$, $(X_\infty(D))'$, is the inductive topology. As a main application we assume that D has a Jordan rectifiable boundary ∂D, and $M \subset \partial D$ to be an open analytic arc whose Lebesgue measure satisfies $0 < m(M) < m(\partial D)$. We prove a result for the dual of $\mathcal{NH}^1_M(D)$, which is the class of holomorphic functions in D which are represented by Carleman formulae on $M \subset \partial D$. Furthermore we show that the Cauchy Integral associated to $f \in \mathcal{NH}^1_M(D)$ is an element of $\mathcal{NH}^1_M(D)$. Lastly, we solve an extremal problem for the dual of $\mathcal{NH}^1_M(D)$.

1 Projective and Inductive topologies

We set D to be a bounded simply connected domain of the complex plane $\mathbb{C}$ and $\{D_i\}_{i=1}$, $i = 1, 2\ldots$ to be an increasing sequence of bounded simply connected domains (i.e. $D_i \subset D_{i+1}$, $i = 1, 2\ldots$) such that $D = \bigcup_i D_i$ with $D_i \subset D$, $i = 1, 2, \ldots$, $\partial D_i \to \partial D$ in the sense that $\{\partial D_i\}_i$ eventually surrounds each compact subdomain of D. Such a sequence of domains $\{D_i\}_i$ is called an exhaustion of D. Furthermore we set $X(D_i)$ to be a function space on D_i with topology $\mathcal{T}_i$.

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For simplicity in the symbolism write X_i for $X(D_i)$, and assume that each X_i carries the topology $\mathcal{T}_i$ for all $i \in \mathbb{N}$. In the following we construct the projective limit associated with X_i and we provide it with the projective topology.

For all $i \leq j$, $f_i \in X_i$, $i, j = 1, 2, \ldots$, define the connecting maps

$$\mu_{ij} : X_j \to X_i,\ i \leq j, \tag{1.1}$$

such that $\mu_{ij}(f_j) = f_i$ is the restriction of f_j on D_i. In addition note that for all $i, j, k = 1, 2 \ldots$, $i \leq j \leq k$, holds $\mu_{ik} = \mu_{ij} \circ \mu_{jk}$. We consider X_∞ to be the subspace of $\prod_i X_i$ whose elements $f = (f_1, f_2, \ldots)$ satisfy the relation $f_i = \mu_{ij}(f_j)$ for all $i \leq j$. X_∞ is called the *projective limit* of the family $\{X_i\}_i$ with respect to the mappings μ_{ij} and is denoted by $X_\infty = \varprojlim \mu_{ij}(X_i, \mathcal{T}_i)$.

We set μ_i to be the restriction to X_∞ of the projection map p_i of $\prod_i X_i$ onto X_i, $i = 1, 2 \ldots$, and we give X_∞ the projective topology $\mathcal{T}_\infty$ with respect to the family $\{(X_i, \mathcal{T}_i), \mu_i\}_i$. That is the coarsest topology on X_∞ for which each of the mappings $\mu_i : X_\infty \to (X_i, \mathcal{T}_i)$, $i \in \mathbb{N}$, is continuous. An element $f_i \in X_i$ is called a *representative* of $f \in X_\infty$, if $\mu_i(f) = f_i$, $i \in \mathbb{N}$. Note that each element of X_∞ has a unique representative in each X_i, but that an element of X_i does not necessarily represent a unique element of X_∞. Furthermore note that there is no restriction of generality in assuming that a projective limit is reduced, in the sense that for each $i \in \mathbb{N}$ the projection $\mu_i(X_\infty)$ is dense in X_i. (An elaborate treatment of the projective limits and of their topologies, can be found in [12].) The following is the definition of the exhaustion space.

Definition 1.1. *We say that a function f belongs to the class $(X_\infty(D), \mathcal{T}_\infty)$ if $f \in (X(D_i), \mathcal{T}_i)$, for each $i = 1, 2, \ldots$, where $\{D_i\}_i$ is an exhaustion of D and $\mathcal{T}_\infty$ is the projective topology with respect to the family $\{(X_i(D), \mathcal{T}_i), \mu_i\}_i$. For simplicity write $(X_\infty, \mathcal{T}_\infty)$ and call it the exhaustion space of $(\{X_i, \mathcal{T}_i)\}, \mu_i)$, $i = 1, 2 \ldots$.*

It is clear that the above definition does not depend on the particular exhaustion of D.

In order to describe the topological dual of $(X_\infty, \mathcal{T}_\infty)$ we need the notion of inductive topologies. For $i \in \mathbb{N}$ set $Y_i \equiv Y(D_i)$ to be the algebraic dual (that is the space of linear functionals) on each X_i. Furthermore suppose that each Y_i has a topology $\mathcal{L}_i$. For all $i, j \in \mathbb{N}$ set ϕ_{ij} to be the dual maps of μ_{ij}. That is $\phi_{ij} = \mu_{ij}^*$ are defined as

$$\phi_{ij} : Y_i \to Y_j,\ i \leq j, \tag{1.2}$$

and if y_i, y_j are elements of Y_i, Y_j respectively, then $y_j = \phi_{ij}(y_i) = y_i \circ \mu_{ij}$. This and the relation (1.1) imply that for $f_i \in X_i$, $f_j \in X_j$, $i \leq j$, the action of y_j on f_j is identified with the action of y_i on f_i.

Now we define the natural injections $g_i : Y_i \hookrightarrow \bigoplus_i Y_i$ and let K denote the (closed) subspace of $\bigoplus_i Y_i$ generated by the closure of the ranges of the linear maps $g_i - g_j \circ \phi_{ij}$ of Y_i into $\bigoplus_i Y_i$ where i, j runs through all pairs such that $i \leq j$. Let also $p : \bigoplus\limits_i Y_i \to (\bigoplus\limits_i Y_i)/K$. The quotient space $Y^\infty \equiv (\bigoplus_i Y_i)/K$ is called the inductive limit of the family $\{Y_i\}_i$ with respect to the mappings ϕ_{ij} and is denoted by

$$Y^\infty = \varinjlim \phi_{ij}(Y_i, \mathcal{L}_i).$$

For all $i \in \mathbb{N}$ set ϕ_i to be the restriction to Y_i of the map $p : \bigoplus_i Y_i \to (\bigoplus_i Y_i)/K$ (that is the imbedding of Y_i into Y^∞). Now provide Y^∞ with the inductive topology $\mathcal{L}^\infty$ with respect to the family $\{(Y_i, \mathcal{L}_i), \phi_i)\}_i$. That is the finest locally convex topology that makes each of the mappings $\phi_i : (Y_i, \mathcal{L}_i) \to Y^\infty$, $i \in \mathbb{N}$, continuous. An element $y_i \in Y_i$ is called a *representative* of $y \in Y^\infty$, if $\phi_i(y_i) = y$. Note that for $i \in \mathbb{N}$, each element of Y_i represents at most one element in Y^∞, but that an element of Y^∞ does not necessarily has a unique representative in Y_i. Furthermore, it is clear that a given $y \in Y^\infty$ need not have a representative in each Y_i, $i \in \mathbb{N}$. Set I to be the subset of $\mathbb{N}$, such that $y \in Y^\infty$ has at least one representative in each element of $\{Y_i\}_{i \in I}$.

In order to prove that there is a certain duality between inductive and projective topologies (as constructed above) we shall need the Mackey-Arens theorem that characterizes the locally convex topologies consistent with a given duality ([12] Chapter IV).

Suppose that F is a vector space over a field K. The *algebraic dual of* F, denoted by F^*, is the vector space of all linear functionals of F. If in addition F is a topological vector space, then the *topological dual* (or briefly *dual*) of F, denoted by F', is the vector space of all continuous linear functionals on F. Recall (see [12]) that if F, G are vector spaces over a field, a locally convex topology $\mathcal{T}$ on F is called **consistent** with the duality $< F, G >$ if the dual of $(F, \mathcal{T})$ is identical with G (G being viewed as a subspace of the algebraic dual F^*). By $\sigma(G, F)$ denote the weak topology on G generated by F.

The following theorem is the *Mackey-Arens* Theorem and is taken from [12].

Theorem 1.1. *There is a finest locally convex topology $\tau(F, G)$ on F consistent with $< F, G >$. This topology is the topology of uniform convergence on all $\sigma(G, F)$-compact convex circled subsets of G.*

This topology on F is called the **Mackey Topology** on F with respect to the dual pair $< F, G >$ and a locally convex space is called **Mackey Space** if its topology is the Mackey topology.

Remark 1.1. *(a). Combining the results [12], Ch.IV,* 3.4 *and* 6.1 *we can easily conclude that if $(F, \mathcal{T})$ is a metrizable space, then its Mackey topology is the topology of uniform convergence on all $\sigma(G, F)$-compact subsets of G and furthermore this topology coincides with $\mathcal{T}$.*

(b). Using the construction of projective limits and [12], Ch.II, 5.3*, it is elementary to show that the projective limit of Fréchet spaces is a Fréchet space.*

In the rest of the paper we suppose that for all $i \in \mathbb{N}$, $(X_i, \mathcal{T}_i)$ and $(X'_i, \mathcal{L}_i)$ are complete metric spaces (Fréchet spaces) themselves.

From Remark 1.1(a) we obtain the following,

Lemma 1.1. *For all $i \in \mathbb{N}$ the Mackey topologies on $(X_i, \mathcal{T}_i)$ and on $(X'_i, \mathcal{L}_i)$ coincide with the metric topologies, $\mathcal{T}_i$ and $\mathcal{L}_i$ respectively.*

The next theorem is of importance. It is a consequence of the result [12], *Ch.IV*, 4.4 once we use Remark 1.1 and Lemma 1.1.

Theorem 1.2. *The topological dual of the reduced projective limit* $X_\infty = \varprojlim \mu_{ij}(X_i, \mathcal{T}_i)$ *under its metric topology, can be identified with the inductive limit of the family* $\{X_i', \mathcal{L}_i\}_{i\in I}$ *with respect to the adjoint mappings* ϕ_{ij} *of* μ_{ij}*. That is* $X_\infty' = \varinjlim \phi_{ij}(X_i', \mathcal{L}_i)$.

2 The Dual of $\mathcal{NH}_M^1(D)$

In this section we define the 'Hardy class $\mathcal{H}^1$ near a subset M of the boundary of the domain D, $M \subset \partial D$' and we denote this class by $\mathcal{NH}_M^1(D)$. It has been shown that if D is a bounded simply connected domain whose boundary ∂D is a Jordan rectifiable curve, and if M is an analytic open arc contained in ∂D whose Lebesgue measure satisfies $0 < m(M) < m(\partial D)$, then this is exactly the class of holomorphic functions in D which are represented by Carleman formulas on $M \subset \partial D$, provided that ∂D is almost regular with respect to M (Theorem 2.14, [7]). Analogous result holds for some other particular domains ([1], [2], [3], [6]).

The main result of this section (Theorem 2.1) is a description of the dual of $\mathcal{NH}_M^1(D)$, that is the dual of the class of functions representable by Carleman formulas.

The following definition is from [11].

Definition 2.1. *A function* $f(z)$ *holomorphic in* D *belongs to the class* $E^p(D)$*,* $p > 0$*, if there exists a sequence of curves* Γ_m *in* D *converging to* ∂D *(in a sense that it eventually surrounds every compact subdomain of* D *) such that*

$$\int_{\Gamma_m} |f(z)|^p |dz| \leq C_1,$$

where C_1 *is independent of* m.

Using the constructions in §1 we provide $\mathcal{NH}_M^1(D)$ with the projective topology. Actually we use a particular exhaustion of the domain D, with a certain subclass of Smirnov domains $D_i \subset D$, $i \in I$. More specifically we consider an Ahlfors regular exhaustion of D attached to M, where M is the stable part of this exhaustion (for details on Ahlfors regular exhaustions see [3]). The following definition is a generalization of a definition in [3].

Definition 2.2. *We say that a holomorphic function* f *on* D *with angular boundary values defined almost everywhere on* M *(denoted also by* f*) belongs to the Hardy class* $\mathcal{H}^p$*,* $p > 0$*, near the set* $M \subset \partial D$ *and denote this class by* $\mathcal{NH}_M^p(D)$ *if* $f \in E^p(D_i)$*, for all* $i \in \mathbb{N}$*,* $p > 0$*, where* $\{D_i\}_i$ *is an Ahlfors-regular exhaustion of* D *attached to* M*.*

From the above definition it is evident that the natural topology associated with $\mathcal{NH}_M^p(D)$ is the projective topology with respect to the mappings μ_{ij}, $i, j \in \mathbb{N}$, as defined in equation (1.1). Hence,

$$(\mathcal{NH}_M^p(D), \mathcal{T}) = \varprojlim \mu_{ij}(E^p(D_i), \mathcal{T}_i),$$

where $\mathcal{T}_i$, $i \in \mathbb{N}$, is the vector space topology on each $E^p(D_i)$ $p > 0$, and $\mathcal{T}$ is the vector space topology of $\mathcal{NH}_M^p(D)$. In case where $p \geq 1$, we consider the associate normed topologies (see also Remark 1.1 (b)). In this paper we are mainly interested in the case of $\mathcal{NH}^1_M(D)$. In light of the Theorem 1.2, in order to describe the dual of $\mathcal{NH}^1_M(D)$ we shall first examine the dual of each $E^1(D_i)$, $i \in \mathbb{N}$. For this we need the following results.

The following lemma is an immediate consequence of [11], Theorem 10.4.

Lemma 2.1. *Let D be a Jordan domain with rectifiable boundary ∂D. Then the set of boundary functions of $f \in E^1(D)$ is precisely the class of functions $f \in L^1(\partial D)$ such that $\int_{\partial D} \zeta^n f(\zeta) d\zeta = 0$, $n = 0, 1, 2, \ldots$.*

The next lemma was originally proved by Smirnov ([13]) in the case of the unit disk (a proof of this can be found in [11]). Here we prove the analogous result for arbitrary Smirnov domains. (The definition of a Smirnov domain can be found in [11]).

Lemma 2.2. *Let D be a Smirnov domain with rectifiable boundary ∂D. The set of boundary functions of $f \in E^p(D)$, $1 < p \leq \infty$, (denoted also by f) is precisely the class of functions $f \in L^p(\partial D)$ such that $\int_{\partial D} \zeta^n f(\zeta) d\zeta = 0$, $n = 0, 1, 2, \ldots$.*

Proof. Suppose that $f \in E^p(D)$, $1 < p \leq \infty$. Since $E^p(D) \subset E^1(D)$, $f \in E^1(D)$ with boundary values in $L^1(\partial D)$. Hence by Lemma 2.1, $\int_{\partial D} \zeta^n f(\zeta) d\zeta = 0$, $n = 0, 1, 2, \ldots$. Conversely, suppose that $f \in L^p(\partial D)$ such that $\int_{\partial D} \zeta^n f(\zeta) d\zeta = 0$, $n = 0, 1, 2, \ldots$. Since $L^p(\partial D) \subset L^1(\partial D)$, by Lemma 2.1, $f \in E^1(D)$. Thus, $f \in E^1(D)$ with boundary function $f \in L^p(\partial D)$, $1 < p \leq \infty$. Now the result follows from Lemma 2.3 in [5], which uses the fact that the conformal mapping appeared in the definition of Smirnov domains, is an outer, and hence a bounded below H^1 function on the unit disk. ∎

The following lemma is well known and shall be used in the sequel. For a proof see [11].

Lemma 2.3. *Suppose X is a Banach space and S a closed subspace of X. If $S^\perp$ is the annihilator of S in X', then the quotient space $X'/S^\perp$ is isometrically isomorphic to S'. Furthermore, for each fixed $T \in X'$,*

$$\sup_{x \in S, \|x\| \leq 1} |T(x)| = \min_{Q \in S^\perp} \|T + Q\|,$$

where '$\min$' indicates that the infimum is attained.

The following theorem was proved for the unit disk. Here we present a proof for arbitrary Smirnov domains.

Theorem 2.1. *Let D be a Smirnov domain with rectifiable boundary ∂D. The dual of $E^1(D)$ is isometrically isomorphic to $L^\infty(\partial D)/H^\infty(D)$. Furthermore, each $T \in (E^1(D))'$ can be represented in the form*

$$T(f) = \int_{\partial D} f(\zeta) \overline{g(\zeta)} d\zeta, \tag{2.1}$$

by some function $g \in L^\infty(\partial D)$.

Proof. Identifying each $f \in E^1(D)$ by its boundary functions, $E^1(D)$ can be regarded as a closed subspace of $L^1(\partial D)$. Using Hahn-Banach theorem we first extend each $T \in (E^1(D))'$ on $L^1(\partial D)$. Thus according to Riesz representation theorem the extended functional has a unique representation such that for all $f \in L^1(\partial D)$,

$$T(f) = \int_{\partial D} f(\zeta)\overline{g(\zeta)}d\zeta,\ g \in L^\infty(\partial D). \tag{2.2}$$

Furthermore, $\|T\|_{(L^1(\partial D))'} = \|g\|_\infty$, and $(L^1(\partial D))'$ is isometrically isomorphic to $L^\infty(\partial D)$. Now Lemma 2.3 can be used to describe the dual $(E^1(D))'$ once the annihilator of $E^1(D)$ in $(L^1(D))'$ can be determined.

If $g \in L^\infty(\partial D)$ annihilates every function in $E^1(\partial D)$ then $\int_{\partial D} \zeta^n g(\zeta)d\zeta = 0$, $n = 0, 1, 2, \ldots$. Therefore by Lemma 2.2, g is a boundary function of an $H^\infty(D)$ function. Conversely, if $g \in H^\infty(D)$, Lemma 2.2 and the fact that for Smirnov domains D, polynomials are dense in $E^1(D)$ ([11],Theorem 10.6), we conclude that $\int_{\partial D} g(\zeta)f(\zeta)d\zeta = 0$, $n = 0, 1, 2, \ldots$ for every $f \in E^1(D)$. Hence $H^\infty(D)$ is the annihilator of $E^1(D)$ in $(L^1(D))'$. Now Lemma 2.3 implies that the dual $(E^1(D))'$ of $E^1(D)$ is isometrically isomorphic to $L^\infty(\partial D)/H^\infty(D)$ and hence the elements of $(E^1(D))'$ may represented in the form (2.1) for some $g \in L^\infty(\partial D)$. Certainly this representation is not unique, since functions belonging to the same coset of $L^\infty(\partial D)/H^\infty(D)$ represent the same functional T on $E^1(D)$. ■

The constructions in §1, Theorem 1.2 and Theorem 2.1 immediately lead to the following main result of this section.

Theorem 2.2. *The dual of $\mathcal{NH}^1_M(D) = \varprojlim \mu_{ij}(E^1(D_i), \mathcal{T}_i)$, under its metric topology, can be identified with the inductive limit of the family $\{(L^\infty(\partial D_i)/H^\infty(D_i), \mathcal{L}_i)\}_i$ with respect to the adjoint mappings ϕ_{ij} of μ_{ij}. That is*

$$(\mathcal{NH}^1_M(D))' = \varinjlim \phi_{ij}(L^\infty(\partial D_i)/H^\infty(D_i), \mathcal{L}_i),$$

where each $\mathcal{L}_i$, $i \in \mathbb{N}$, denotes the quotient normed topology on $L^\infty(\partial D_i)/H^\infty(D_i)$. Furthermore, each element $T_i \in L^\infty(\partial D_i)/H^\infty(D_i)$ can be written in the form

$$T_i(f) = \int_{\partial D} f(\zeta)\overline{g_i(\zeta)}d\zeta,\ f \in E^1(D_i),$$

for some $g_i \in L^\infty(\partial D_i)$, $i \in \mathbb{N}$.

3 Applications

3.1 Cauchy type Integrals

In the following we consider $\mathcal{NH}^1_M(D)$ and we show a result for the Cauchy type integral
$$\frac{1}{2\pi i}\int\limits_M \frac{f(\zeta)}{\zeta - z}d\zeta,\ z \in D,$$
whenever $f \in \mathcal{NH}^1_M(D)$. This result holds for $\mathcal{NH}^1_M(D)$ defined via Ahlfors-regular proper exhaustions attached to M. (For the definition of Ahlfors-regular proper exhaustion of a domain D, see [5]). Such domains are considered in the problems studied in [1], [2], [5], [6]. In the cases [3] and [7] where the exhaustions of D are Ahlfors-regular but not proper, an analogous result holds for almost all the proper subarcs of M.

Theorem 3.1. *Suppose that $f \in \mathcal{NH}^1_M(D)$. If the exhaustion in $\mathcal{NH}^1_M(D)$ is Ahlfors regular and proper then the Cauchy type integral*
$$\frac{1}{2\pi i}\int\limits_M \frac{f(\zeta)}{\zeta - z}d\zeta,\ z \in D,$$
belongs to $\mathcal{NH}^1_M(D)$.

(b) If the Ahlfors regular exhaustion in $\mathcal{NH}^1_M(D)$ is not proper, that is with fixed endpoints (see the case in [3]), we set $L \subset M$ to be any subarc of M satisfying $\overline{L} \subset M$. Then for almost all L the Cauchy type integral $\frac{1}{2\pi i}\int\limits_M \frac{f(\zeta)}{\zeta - z}d\zeta,\ z \in D$, belongs to $\mathcal{NH}^1_L(D)$.

Proof. As proved in [1, 2, 3, 5, 6, 7], if $f \in \mathcal{NH}^1_M(D)$,
$$f(z) = F_+(z) + \mathcal{G}(z) = \frac{1}{2\pi i}\int\limits_M \frac{f(\zeta)d\zeta}{\zeta - z} + \mathcal{G}(z),\ z \in D, \tag{3.1.1}$$
where the holomorphic function $\mathcal{G}(z)$ has analytic continuation across the arc M. In the case (a) where the exhaustion is proper this means that there exists open, bounded, connected set U containing M and satisfying $\partial U \cap \overline{M} = \partial M$, and a function $\hat{\mathcal{G}} \in \mathcal{H}(D \bigcup U)$ such that $\hat{\mathcal{G}}(z) = \mathcal{G}(z)$, $z \in D$. Thus $\mathcal{G}$ is a holomorphic function in $\overline{D_i}$ for every domain D_i, $i \in \mathbb{N}$, of the Ahlfors regular exhaustion of D. In particular, $\mathcal{G} \in E^1(D_i)$, $i \in \mathbb{N}$. Therefore, $\mathcal{G} \in \mathcal{NH}^1_M(D)$. Since by hypothesis $f \in \mathcal{NH}^1_M(D)$, we conclude by equation (3.1.1) that the Cauchy type integral $\frac{1}{2\pi i}\int\limits_M \frac{f(\zeta)}{\zeta - z}d\zeta,\ z \in D$, is an element of $\mathcal{NH}^1_M(D)$. In case (b) where the exhaustion is not proper, (see (2.7) in [3]) the Ahlfors regular exhaustion of D, $D_{L,i}$, $i \in \mathbb{N}$ attached to the arc $L \subset \overline{L} \subset M$ with fixed endpoints ∂L is generated the same way as the proper exhaustion $\{D_i\}$, $i \in \mathbb{N}$ of D. Hence, $D_{L,i} \subset D_j$ for some $j \in \mathbb{N}$. Thus, $\mathcal{G}$ (which has analytic continuation across M) is holomorphic in $\overline{D_{L,i}}$. This as done in case (a) implies that $\mathcal{G} \in \mathcal{NH}^1_L(D)$. ∎

Conclusion: In general whenever $f(\zeta)$, $\zeta \in \partial D$, is an element in $L^1(\partial D)$ it follows from the proof of lemma 3.5 in [6] (see also [8, 9, 10]) that the Cauchy type integral

$$\frac{1}{2\pi i}\int_M \frac{f(\zeta)}{\zeta - z} d\zeta,\ z \in D \tag{3.1.2}$$

belongs to $E^p(D_i)$, $i \in \mathbb{N}$, $0 < p < 1$ for every domain D_i of the Ahlfors regular exhaustion of D, and hence is an element of $\mathcal{NH}^p_M(D)$, for all $0 < p < 1$. Our result states that if in addition the function $f(\zeta)$, $\zeta \in \partial D$, is the boundary function of $f \in \mathcal{NH}^1_M(D)$ and the exhaustion of D is proper, then the above Cauchy integral belongs to $\mathcal{NH}^1_M(D)$. An analogous result holds in the case of nonproper Ahlfors-regular exhaustion of D as described earlier.

3.2 The Extremal problem

Let $T \in (\mathcal{NH}^1_M(D))'$ and recall that$(\mathcal{NH}^1_M(D))' = \varinjlim \phi_{ij}(L^\infty(\partial D_i)/H^\infty(D_i), \mathcal{L}_i)$, where each $\mathcal{L}_i$, $i \in \mathbb{N}$, denotes the quotient normed topology on $L^\infty(\partial D_i)/H^\infty(D_i)$. Furthermore note that T has at least one representative in each element of $(E^1(D_i))' \equiv \{L^\infty(\partial D_i)/H^\infty(D_i)\}$, $i \in I \subset \mathbb{N}$ (see (§1)).

In this section we solve the classical extremal problem on each $(E^1(D_i))'$, $i \in I$. That is to find

$$\|T_i\| = \sup_{f \in E^1(D_i),\ \|f\|_1 \leq 1} |T_i(f)|. \tag{$\star$}$$

We show that under some certain conditions an *extremal function* (that is a function $f \in E^1(D_i)$, $i \in I$, that solves the above problem) exist, and furthermore we prove that the supremum in ($\star$) is attained.

Fix $i \in I$. From Theorem 2.2, the most general bounded linear functional on $E^1(D_i)$ can be expressed in the form

$$T_i(f) = \int_{\partial D_i} f(\zeta)\overline{k(\zeta)} d\zeta,\ f \in E^1(D_i), \tag{3.2.1}$$

where $k \in L^\infty(\partial D_i)$. For a given kernel $k \in L^\infty(\partial D_i)$ the typical extremal problem is given by ($\star$).

A function $h \in L^\infty(\partial D_i)$ is said to be equivalent to the given kernel k ($h \sim k$) if h, k belong to the same coset $L^\infty(\partial D_i)/H^\infty(D_i)$, that is $h - k \in H^\infty(D_i)$. Thus h, k determine the same functional on $E^1(D_i)$ if and only if $h \sim k$. An application of Lemma 2.3 gives

$$\sup_{f \in E^1(D_i),\, \|f\|_1 \leq 1} \left| \int_{\partial D_i} f(\zeta)\overline{k(\zeta)} d\zeta \right| = \min_{g \in H^\infty(D_i)} \|k - g\|_\infty. \tag{3.2.2}$$

This is called the duality relation. It connects the original extremal problem with the dual extremal problem. That is to find the function $g \in H^\infty(D_i)$ which is closest to the kernel $k \in L^\infty(\partial D_i)$. According to Lemma 2.3 the minimum is attained and hence the dual extremal problem has a solution.

In general the original extremal problem ($\star$) need not have a solution in $E^1(D_i)$ for all $k \in L^\infty(\partial D_i)$. We will show the following

Theorem 3.2. *A solution of the extremal problem exists in $E^1(D_i)$ if the given kernel k is continuous, that is if $k \in C(\partial D_i)$.*

The following two lemmata are needed for the proof of the above theorem. The first one was originally proved by F. and M. Riesz in the case of the unit disk. Here we prove the analogous result for arbitrary Jordan domains.

Lemma 3.1. *Suppose that D is a Jordan domain with rectifiable boundary ∂D and μ a complex valued measure of bounded variation on ∂D. If*

$$\int_{\partial D} \zeta^n d\mu(\zeta) = 0, \; n = 0, 1, 2 \ldots, \tag{3.2.3}$$

then $d\mu = f(\zeta)d\zeta$, for some $f \in E^1(D)$.

Proof. First note that the series expansion of the Cauchy kernel shows that the equation (3.2.3) is equivalent to the identical vanishing of $\displaystyle\int_{\partial D} \frac{d\mu(\zeta)}{\zeta - z}$, outside ∂D. Now we modify a part of the technique presented in the proof of [11], Theorem 10.4 (page 171). Since μ is a complex valued measure of bounded variation we can replace the function G appeared there, with $d\mu \circ \phi$, where ϕ is a conformal map from the unit disk onto D. Then we follow the steps of the proof as in Theorem 10.4 of [11], page 171, after equation (6), to conclude that

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{d\mu(\zeta)}{\zeta - z} \in E^1(D).$$

Thus, f has nontangential limits almost everywhere on ∂D, with $f \in L^1(\partial D)$ and hence from [11], Theorem 10.4, (first part), we get $f(z) = \displaystyle\frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)d\zeta}{\zeta - z}$. Using the uniqueness of the Cauchy representation with the integral vanishing identically outside ∂D we conclude that $d\mu = f(\zeta)d\zeta$, $f \in E^1(D)$. ∎

For a proof of the next lemma see [11].

Lemma 3.2. *Suppose X is a Banach space and S is a closed subspace of X. If $S^\perp$ is the annihilator of S in X', then the space $(X/S)'$ is isometrically isomorphic to $S^\perp$. Furthermore, for each fixed $x \in X$,*

$$\max_{\psi \in S^\perp, \|\psi\| \leq 1} |\psi(x)| = \inf_{y \in S} \|x + y\|,$$

*where '*max*' indicates that the supremum is attained.*

Now we proceed with the proof of Theorem 3.2.

Proof. Consider the subspace $P \leq C(\partial D_i)$ which is the uniform closure of the polynomials in $\zeta \in C(\partial D_i)$. Each function $f \in L^1(\partial D_i)$ defines a continuous linear functional on $C(\partial D_i)$,

$$\psi(k) = \int_{\partial D_i} f(\zeta)\overline{k(\zeta)} d\zeta, \; k \in C(\partial D_i), \tag{3.2.4}$$

with $\|\psi\| = \|f\|_1$. We shall describe $P^\perp$ in $(C(\partial D_i))'$. According to Riesz representation theorem ([4]) each element ψ in $(C(\partial D_i))'$ has the form $\psi(k) = \int_{\partial D_i} k(\zeta)d\mu(\zeta)$ for some measure μ of bounded variation. Furthermore, for $\psi \in P^\perp$,

$$\psi(\zeta^n) = \int_{\partial D_i} \zeta^n d\mu(\zeta) = 0, \quad n = 0, 1, 2 \ldots.$$

Thus by Lemma 3.1 $d\mu(\zeta) = f(\zeta)d\zeta$ for some function $f \in E^1(D_i)$. Conversely, every ψ of the form (3.2.4) with $f \in E^1(D)$, annihilates P (Lemma 2.1). Therefore the annihilator of P consists of all continuous linear functionals on $C(\partial D_i)$ of the form (3.2.4) with $f \in E^1(D_i)$. Hence Lemma 3.2 can be invoked to conclude that $\sup_{f\in E^1(D_i),\|f\|_1\le 1} \left|\int_{\partial D_i} f(\zeta)\overline{k(\zeta)}d\zeta\right|$ is attained. To see this observe that

$$\sup_{\psi\in P^\perp,\|\psi\|\le 1} |\psi(k)| = \sup_{f\in E^1(D_i),\|f\|_1\le 1} |\int_{\partial D_i} f(\zeta)\overline{k(\zeta)}d\zeta|.$$

Hence the extremal problem in question has a solution and the supremum is attained.

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