Real analytic zero solutions of linear partial differential operators with constant coefficients

Dietmar Vogt

Dedicated to Jean Schmets on the occasion of his 65th birthday

Abstract

It is shown that for a wide class of linear partial differential operators with constant coefficients the space of real analytic zero solutions does not admit a Schauder basis. This is based on results on the linear topological structure of the space of zero solutions and a careful analysis of the solvability with a real analytic parameter.

Let $\Omega \subset \mathbb{R}^n$ be open and $P \in \mathbb{C}[z_1, \ldots, z_n]$. We consider the linear partial differential operator $P(D)$, where $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$, as acting on the space $A(\Omega)$ of real analytic functions on Ω. We set

$$N_P(\Omega) = \{f \in A(\Omega) \ : \ P(D)f = 0\}.$$

Not very much is known about the structure of the spaces $N_P(\Omega)$. In the present note we show that $N_P(\Omega)$ is never an (LB)-space and it is a Fréchet space if and only if $P(D)$ is elliptic.

As a consequence of the first we get that for a wide class of non-elliptic $P(D)$ the space $N_P(\Omega)$ does not have a (Schauder) basis. In particular it is shown that for any P for which $P(D)$ is surjective on $A(\mathbb{R}^n)$ and the principal part P_m has no elliptic factor $N_P(\mathbb{R}^n)$ has no basis. If, moreover, P_m is irreducible this means that either P is elliptic or $N_P(\mathbb{R}^n)$ has no basis.

The proof uses results of Domański and the author [4], where it was shown that $A(\Omega)$ never has a basis. And it is based on a careful study under which conditions the augmented operator $P^+(D)$ which is $P(D)$ acting on the first n variables of functions

2000 *Mathematics Subject Classification :* Primary: 46E10 Secondary: 26E05, 35E20.

Key words and phrases : real analytic functions, linear partial differential operators, Schauder basis.

in $A(\Omega \times \mathbb{R})$ is surjective. In fact, for $\Omega = \mathbb{R}^n$ these operators are completely characterized. They are those which are surjective on $A(\mathbb{R}^n)$ and its principal part has no elliptic factor.

The author thanks P. Domański for a useful remark.

Preliminaries

We use common notation for the theory of locally convex spaces spaces. For this and general results we refer to [9], for concepts and results from the homological theory of locally convex spaces to [15]. For the theory of linear partial differential operators with constant coefficients we refer to [6].

Throughout the paper we denote by $A(\Omega)$ the linear space of real analytic functions on the open set $\Omega \subset \mathbb{R}^n$ equipped with its natural locally convex topology (see [8]). The same notation applies if Ω is replaced by a real analytic manifold T. If E is a sequentially complete locally convex space then $A(\Omega, E)$ denotes the linear space of E-valued real analytic functions, which are those functions f on Ω for which $\eta \circ f$ is real analytic for any $\eta \in E'$. For these spaces we usually do not specify a topology, however, in case E is Fréchet space, an (LB)-space or, more generally, a (PLB)-space we consider $A(\Omega, E)$ as equipped with a (PLB)-structure in a natural way and apply homological concepts (cf. [14]).

The spaces $N_P(\Omega)$ are understood as closed linear topological subspaces of $A(\Omega)$. As such they are complete (PLS)-spaces. For linear topological invariants on (PLS)-spaces and related results we refer to [1].

1 Linear topological properties of spaces $N_P(\Omega)$

First we show that for dimensions $n > 1$ the space $N_P(\Omega)$ is never an (LB)-space.

Proposition 1.1. *If $N_P(\Omega)$ is an (LB)-space, then $n = 1$.*

For the proof of Proposition 1.1 we use the following Lemma, which is an adaptation of [5, Lemma 4.3]. Let $E \subset A(\Omega)$ be a closed subspace. We put

$$V(E) = \{\zeta \in \mathbb{C}^n \,:\, x \mapsto e^{-i\zeta x} \in E\}.$$

Lemma 1.2. *If $E \subset A(\Omega)$ is an (LB)-subspace, then the following condition is satisfied*

$$\forall\, \varepsilon > 0 \;\exists\, c \;\forall\, \zeta = \xi + i\eta \in V(E) : \; |\eta| \le c + \varepsilon|\zeta|.$$

Proof: If E is an (LB)-space, then from the theory of (PDF)-spaces we know that there exists $K \subset\subset \Omega$, so that the topology induced by $H(K)$ on E coincides with the topology induced by $A(\Omega)$ on E (see [3], pp. 60 and 63).

Therefore we have:

$\exists K \subset\subset \Omega \quad \forall L \subset\subset \Omega, \varepsilon > 0 \quad \exists \delta > 0, C \quad \forall f \in E :$

$$\sum_{\alpha} \sup_{x\in L} \frac{|f^{(\alpha)}(x)|}{\alpha!} \delta^{|\alpha|} \le C \sum_{\alpha} \sup_{x\in K} \frac{|f^{(\alpha)}(x)|}{\alpha!} \varepsilon^{|\alpha|}.$$

In particular we get for $f(x) = e^{-i\zeta x}$ where $\zeta \in V(E)$

$$\sum_\alpha \frac{|\zeta^\alpha|}{\alpha!} \sup_{x\in L} e^{\eta x}\delta^{|\alpha|} \le C \sum_\alpha \frac{|\zeta^\alpha|}{\alpha!} \sup_{x\in K} e^{\eta x}\varepsilon^{|\alpha|}.$$

For $\eta \in \mathbb{R}^n$ we set (also if K and L are not convex)

$$h_K(\eta) = \sup_{x\in K} \eta x, \quad h_L(\eta) = \sup_{x\in L} \eta x$$

and we obtain by taking logarithms, with $c = \log C$,

$$\delta \sum_{j=1}^d |\zeta_j| + h_L(\eta) \le c + \varepsilon \sum_{j=1}^d |\zeta_j| + h_K(\eta).$$

We set $|\zeta|_1 := \sum_{j=1}^d |\zeta_j|$ and obtain
$\exists K \subset\subset \Omega \quad \forall L \subset\subset \Omega, \varepsilon > 0 \quad \exists c \quad \forall \zeta \in V(E)$

$$h_L(\eta) \le c + \varepsilon|\zeta|_1 + h_K(\eta).$$

We choose $L \subset\subset \Omega$ so that $K \subset \mathring{L}$. Then there is $\gamma > 0$ so that $K + B_\gamma \subset L$ where $B_\gamma = \{x \in \mathbb{R}^n \,:\, |x| \le \gamma\}$. Therefore $h_K(\eta) + \gamma|\eta| \le h_L(\eta)$.

Then the last estimate, applied to L, implies that for all $\zeta \in V(E)$

$$\gamma|\eta| \le c + \varepsilon|\zeta|_1 \le c + \varepsilon n|\zeta|.$$

Therefore we finally have the condition

$$\forall \varepsilon > 0 \quad \exists C_\varepsilon \quad \forall \zeta \in V(E) \qquad |\eta| \le C_\varepsilon + \varepsilon|\zeta|$$

which completes the proof. ∎

Now we can prove our Proposition.

Proof of Proposition 1.1: We assume $n > 1$ and $P \in \mathbb{C}[z_1, \dots, z_n]$. We choose a noncharacteristic vector N and may assume that $N = e_1 = (1, 0, \dots, 0)$. Then the polynomial has the form

$$P(z, z') = z^m + Q_{m-1}(z')z^{m-1} + \cdots$$

We choose a vector $e' \in \mathbb{R}^{n-1}$, $|e'| = 1$ and we put $z' = ite'$.

For each $t > 0$ we choose $z_t \in \mathbb{C}$ with $P(z_t, ite') = 0$ and we put $-\zeta_t = (z_t, te')$. Then $\zeta \in V(N_P(\Omega))$, $\operatorname{Im}\zeta_t := \eta_t = (-y_t, -te')$ and therefore

$$|\eta_t| = \sqrt{y_t^2 + t^2} \ge t.$$

On the other hand there is a constant $A \ge 1$ depending only on P, so that for large t

$$0 = |P(z_t, ite')| \ge |z_t|^m - A \max_{j=1,\dots,m} t^j |z_t|^{m-j}.$$

This implies

$$|z_t|^m \le A \max_{j=1,\dots,m} t^j |z_t|^{m-j}$$

and therefore

$$|z_t| \le At \le A|\eta_t|$$

for large t. Therefore, due to Lemma 1.2, $N_P(\Omega)$ is not an (LB)-space. ∎

While we do not need this for further considerations we investigate now the case when $N_P(\Omega)$ is a Fréchet space.

Lemma 1.3. *If $N_P(\Omega)$ is a Fréchet space, then there exists for every $K \subset\subset \Omega$ an $L \subset\subset \Omega$, so that the map $H(L) \cap N_P(\Omega) \to H(K) \cap N_P(\Omega)$ is bounded.*

Proof: For every K there is L, so that the map mentioned in the Lemma factorizes through a local Banach space. ∎

Proposition 1.4. *If $N_P(\Omega)$ is a Fréchet space then $P(D)$ is elliptic.*

Proof: Because of Lemma 1.3 we have: $\forall K \quad \exists L, \varepsilon > 0 \quad \forall \delta > 0 \quad \exists C \quad \forall f \in N_P(\Omega)$

$$\sum_\alpha \sup_{x\in K} \frac{|f^{(\alpha)}(x)|}{\alpha!}\varepsilon^{|\alpha|} \le C \sum_\alpha \sup_{x\in L} \frac{|f^{(\alpha)}(x)|}{\alpha!}\delta^{|\alpha|}.$$

Like in the proof of Proposition 1.1 we obtain: $\forall K \quad \exists L, \varepsilon > 0 \quad \forall \delta > 0 \quad \exists C \quad \forall \zeta \in V_P$

$$\varepsilon|\zeta|_1 + h_K(\eta) \le c + \delta|\zeta|_1 + h_L(\eta)$$

and this is equivalent to $\forall K \quad \exists L, \varepsilon > 0, C > 0 \quad \forall \zeta \in V_P$

$$\varepsilon|\zeta|_1 + h_K(\eta) \le C + h_L(\eta).$$

Since we can find $R > 0$ so that $L \subset K + B_R$, we obtain as a necessary condition

$$\exists C,\ R \quad \forall \zeta \in V_P : \quad |\zeta|_1 \le C + R|\eta|.$$

This implies that P is elliptic (see [6], Theorem 11.4.12). ∎

2 A sufficient condition for the non-existence of bases in $N_P(\Omega)$

We begin with our basic observation. For that we assume T to be a d-dimensional real analytic manifold. For $P \in \mathbb{C}[z_1, \dots, z_n]$ we denote by $P^T(D)$ the differential operator $P(D)$ acting on $A(\Omega \times T)$.

Lemma 2.1. *If $P^T(D) : A(\Omega \times T) \longrightarrow A(\Omega \times T)$ is surjective, then every Fréchet space E which is isomorphic to a complemented subspace of $N_P(\Omega)$ is finite dimensional.*

Proof: Since E is isomorphic to a subspace of $A(\Omega)$ it has property $(\underline{\mathrm{DN}})$ (see [2] or [4]).

Since $P^T(D)$ is surjective on $A(\Omega \times T)$ we have

$$\mathrm{Proj}^1 N_{P^T}(\Omega \times T) = 0.$$

We identify $N_{P^T}(\Omega \times T) = A(T, N_P(\Omega))$. Since $A(T, E)$ is a complemented subspace of $A(T, N_P(\Omega))$

$$\mathrm{Proj}^1 A(T, N_P(\Omega)) = \mathrm{Proj}^1 N_{P^T}(\Omega \times T) = 0$$

implies $\mathrm{Proj}^1 A(T, E) = 0$.

By [14] we conclude that E has property $(\overline{\overline{\Omega}})$. Together with property $(\underline{\mathrm{DN}})$ this implies, by [13], that E is a Banach space hence, being nuclear, finite dimensional. ∎

A crucial result for our further considerations will be the following.

Theorem 2.2. *If $P^T(D)$ is surjective in $A(\Omega \times T)$ and $n > 1$ then $N_P(\Omega)$ has no basis.*

Proof: By Lemma 2.1 every complemented Fréchet subspace of $N_P(\Omega)$ is finite dimensional. Since obviously also $P(D)$ is surjective in $A(\Omega)$ we have $\mathrm{Proj}^1 N_P(\Omega) = 0$. This implies that $N_P(\Omega)$ is ultrabornological (see [15]). If it would have a basis then, by [4, Theorem 2.2], it would be an (LB)-space. This contradicts Proposition 1.1. ∎

3 Solvability with a real analytic parameter

Throughout this section Ω is an open convex subset of $\mathbb{R}^n$. We will use the following notation: For $P \in \mathbb{C}[z_1, \ldots, z_n]$ we set $P^+ = P$, considered as a polynomial in $\mathbb{C}[z_1, \ldots, z_{n+1}]$, and we consider $P^+(D)$ as acting $A(\Omega \times \mathbb{R}) \longrightarrow A(\Omega \times \mathbb{R})$.

We may consider surjectivity of $P^+(D)$ either as solvability of $P(D)$ in $A(\Omega)$ with a real analytic parameter or as solvability in $A(\Omega \times \mathbb{R})$ of an operator with a mute variable. Operators of this type have already played a role. Among the first examples of a non surjective operator on $A(\mathbb{R}^3)$ by di Giorgio, Cattabriga and Piccinini had been the Laplacian in 2 variables and $\partial/\partial\bar{z}$, both acting on $A(\mathbb{R}^3)$.

Motivated by this and by the results of Section 2 we will consider the operator $P^+(D)$ which is $P(D)$ acting on the first n variables and study when it is surjective.

First we state a necessary condition which holds for any open convex Ω.

Proposition 3.1. *If $n > 1$ and $P^+(D) : A(\Omega \times \mathbb{R}) \longrightarrow A(\Omega \times \mathbb{R})$ is surjective then $P_m(D) : A(\Omega) \longrightarrow A(\Omega)$ is surjective and P_m has no elliptic factor.*

Proof: We assume that $P^+(D)$ is surjective. By [7] then also $P_m^+(D)$ is surjective. Therefore also $P_m(D)$ is surjective. If P_m had an elliptic factor Q, then also $Q^+(D)$ (as a factor of $P_m^+(D)$) would be surjective. $N_Q(\Omega)$, being a Fréchet space, would be finite by Lemma 2.1. ∎

Now we derive a sufficient condition which is well evaluated, e.g. in [10].

Proposition 3.2. *If $P_m(D) : C^\infty(\Omega) \longrightarrow C^\infty(\Omega)$ admits a continuous linear right inverse, then $P^+(D)$ is surjective in $A(\Omega \times \mathbb{R})$.*

Proof: If we identify $C^\infty(\Omega \times \mathbb{R}) \cong C^\infty(\Omega)\widehat{\otimes}_\pi C^\infty(\mathbb{R})$ then $P_m^+(D)$ corresponds to $P_m(D) \otimes \mathrm{id}_{C^\infty(\mathbb{R})}$ which has $R \otimes \mathrm{id}_{C^\infty(\mathbb{R})}$ as a right inverse, where R is a continuous linear right inverse for $P_m(D)$. Therefore also $P_m^+(D) : C^\infty(\Omega \times \mathbb{R}) \longrightarrow C^\infty(\Omega \times \mathbb{R})$ has a continuous linear right inverse. By [10], Proposition 4.12 then $P_m^+(D)$ is surjective in $A(\Omega \times \mathbb{R})$. By [7] this implies that $P^+(D)$ is surjective in $A(\Omega \times \mathbb{R})$. ∎

First we concentrate on the case of $\Omega = \mathbb{R}^n$. To express the assumption of Proposition 3.2 without reference to right inverses in $C^\infty(\mathbb{R}^n)$ we recall the following result of [11], Corollary 3.14.

Proposition 3.3. *$P_m(D) : C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$ admits a continuous linear right inverse, if and only if, $P_m(D) : A(\mathbb{R}^n) \longrightarrow A(\mathbb{R}^n)$ is surjective and none of its irreducible factors is elliptic.*

From there now we obtain a complete characterization for the case of $\Omega = \mathbb{R}^n$.

Theorem 3.4. *For $n > 1$ the operator $P^+(D) : A(\mathbb{R}^{n+1}) \longrightarrow A(\mathbb{R}^{n+1})$ is surjective if and only if $P_m(D) : A(\mathbb{R}^n) \longrightarrow A(\mathbb{R}^n)$ is surjective and P_m has no elliptic factor.*

Proof: One implication follows from Propositions 3.2 and 3.3 the other from Proposition 3.1. ∎

We will now study cases where $\Omega \neq \mathbb{R}^n$ and we begin with formulating a consequence of Proposition 3.2.

Lemma 3.5. *If P_m is proportional to a product of real linear forms, then $P^+(D) : A(\Omega \times \mathbb{R}) \longrightarrow A(\Omega \times \mathbb{R})$ is surjective for every open convex Ω.*

Proof: By [10], Theorem 3.8, the assumption of Proposition 3.2 is fulfilled if P_m is proportional to a product of real linear forms. ∎

We restrict now our attention to the case of a bounded convex open set with C^1-boundary and we restrict ourselves to the case of $m = 2$, i.e. to the case of second order equations.

Lemma 3.6. *If $P^+(D) : A(\Omega\times\mathbb{R}) \longrightarrow A(\Omega\times\mathbb{R})$ is surjective, then P_2 is proportional to the product of two real linear forms.*

Proof: Since the assumption implies that $P(D) : A(\Omega) \longrightarrow A(\Omega)$ is surjective,we know by Hörmander [7], Theorem 6.7 that P_m is either elliptic or proportional to a real non-degenerate quadratic form or to the product of two real linear forms. By Proposition 3.1 $P(D)$ cannot be elliptic. So let us assume that that P_2 is a non-degenerate quadratic form, then P_2^+ is a degenerate quadratic form in $n+1$ variables which we may assume of the form

$$P_2(\xi) = \sum a_j \xi_j^2$$

where $a_1 = \cdots = a_k = 1$, $a_{k+1} = \cdots = a_n = -1$, $a_{n+1} = 0$ with $2 \le k < n$.

Now the proof in [7] shows that this is impossible, since the only point where it uses boundedness of the open set is at the end of the proof on page 182. There the surjectivity of the Gauß-map is used to provide a point x_0 in the boundary where the tangent plane contains a plane parallel to the $x_{k+1}, \ldots, x_n$ axis. However it is easily seen that in our case this follows from the boundedness, hence surjectivity of the Gauß-map, of Ω. ∎

This leads to the following theorem, which gives a characterization in this case.

Theorem 3.7. *For a second order differential operator $P(D)$ and bounded convex open set Ω with C^1-boundary the following are equivalent:*

1. *$P^+(D) : A(\Omega \times \mathbb{R}) \longrightarrow A(\Omega \times \mathbb{R})$ is surjective.*
2. *P_2 is, up to a constant factor, the product of two real linear forms.*

Proof: 1. ⇒ 2. is Lemma 3.6, 2. ⇒ 1. follows from Lemma 3.5. ∎

4 Spaces of zero solutions without basis

We will apply the results of Sections 2 and 3 to produce classes of operators $P(D)$ so that $N_P(\Omega)$ has no basis.

For that we rephrase Theorem 2.2 in the following way:

Lemma 4.1. *If $P^+(D)$ is surjective in $A(\Omega \times \mathbb{R})$ and $n > 1$ then $N_P(\Omega)$ has no basis.*

We consider first the case of $\Omega = \mathbb{R}^n$. From Theorem 3.4 we obtain using Lemma 4.1

Theorem 4.2. *If $P_m(D) : A(\mathbb{R}^n) \longrightarrow A(\mathbb{R}^n)$ is surjective and P_m has no elliptic factor then $N_P(\mathbb{R}^n)$ has no basis.*

This yields even a characterization in case P_m is irreducible, an assumption which is quite common in relevant examples. Notice that, by [7], surjectivity of $P(D)$ and $P_m(D)$ in $A(\mathbb{R}^n)$ are the same.

Corollary 4.3. *If P_m is irreducible and $P(D) : A(\mathbb{R}^n) \longrightarrow A(\mathbb{R}^n)$ is surjective, then the following are equivalent:*

1. *$P(D)$ is not elliptic.*
2. *$N_P(\mathbb{R}^n)$ has no basis.*

Proof: 1. $\Rightarrow$ 2. is an immediate consequence of Theorem 4.2.

If, on the other hand, P is elliptic, then $N_P(\mathbb{R}^n) \cong H(\mathbb{C}^{n-1})$ (see Wiechert [16]), hence has a basis. ∎

A more direct access in certain cases gives the following:

Corollary 4.4. *If P_m is irreducible, not elliptic and* $\mathrm{grad} P_m(\xi) \neq 0$ *for $\xi \in \mathbb{R}^n \setminus \{0\}$, then $N_P(\mathbb{R}^n)$ has no basis.*

Proof: This follows from [10], Corollary 4.8., Proposition 3.2 and Lemma 4.1. ∎

From this we get quite a lot of examples:

Corollary 4.5. *If $n > 1$, $m \in \mathbb{N}$, $P_m(z) = \sum_{j=1}^n a_j z_j^m$ with $a_j = \pm 1$ for all j and one of the following is fulfilled*

- *$n = 2$, $m = 2$ and $a_1 a_2 = -1$*
- *$n > 2$ and m odd*
- *$n > 2$, n even and $a_j a_k = -1$ for some j, k*

then $N_P(\mathbb{R}^n)$ has no basis.

Proof: By Proposition 3.3 and [10], Example 4.9, the assumption of Theorem 4.2 is fulfilled in these cases. ∎

In the case of a convex, bounded open set with C^1-boundary, e. g. the n-dimensional unit ball B^n, we can, of course apply Lemma 3.5. The characterization of Theorem 3.7 tells us that we cannot hope fore much more. So for general open convex Ω we state only:

Theorem 4.6. *If P_m is proportional to a product of real linear forms, then $N_P(\Omega)$ has no basis.*

5 Further results

We will make use of the results of Bonet-Domański [1]. For the definition of the linear topological invariant $(P\overline{\overline{\Omega}})$ see [1], Section 5. Ω denotes an arbitrary open subset of $\mathbb{R}^n$.

We will first go back to the argument used in Lemma 2.1.

Lemma 5.1. *If $P^+(D) : A(\Omega \times \mathbb{R}) \longrightarrow A(\Omega \times \mathbb{R})$ is surjective and E a complemented subspace of $A(\mathbb{R})$, then $\mathrm{Ext}^1_{PLS}(E', N_P(\Omega) = 0$.*

Since $H(\mathbb{D})'$ is isomorphic to a complemented subspace of $A(\Omega)$ (see e.g. [14]) and $H(\mathbb{D}) \cong \Lambda_0(\alpha)$ with $\alpha_n = n$ for all n, we obtain from [1], Corollary 7.2:

Proposition 5.2. *If $P^+(D) : A(\Omega \times \mathbb{R}) \longrightarrow A(\Omega \times \mathbb{R})$ is surjective, then $N_P(\Omega)$ has property $(P\overline{\overline{\Omega}})$.*

By [1], Theorem 5.5 we get:

Corollary 5.3. *If $P^+(D) : A(\Omega \times \mathbb{R}) \longrightarrow A(\Omega \times \mathbb{R})$ is surjective then for every nuclear Fréchet space F with property $(\underline{\mathrm{DN}})$ we have $\mathrm{Ext}^1_{PLS}(F, N_P(\Omega)) = 0$.*

We use this to get a result on the structure of the zero space $N_P(\mathbb{R}^n)$ of a homogeneous operator. So let $P = P_m$ be homogeneous and $P = Q_1\, Q_2$, where Q_2 is elliptic and Q_1 contains no elliptic factor.

Proposition 5.4. *If $P(D) : A(\mathbb{R}^n) \longrightarrow A(\mathbb{R}^n)$ is surjective then $N_P(\mathbb{R}^n) = N_{Q_1}(\mathbb{R}^n) \oplus F$ where $F \cong N_{Q_2}(\mathbb{R}^n)$. In particular $N_P(\mathbb{R}^n) \cong N_{Q_1}(\mathbb{R}^n) \oplus N_{Q_2}(\mathbb{R}^n)$.*

Proof: We may assume that $n > 1$ and Q_1 and Q_2 are nontrivial. Since also $Q_1(D)$ is surjective in $A(\mathbb{R}^n)$ we have the exact sequence

$$0 \longrightarrow N_{Q_1}(\mathbb{R}^n) \hookrightarrow N_P(\mathbb{R}^n) \xrightarrow{Q_1(D)} N_{Q_2}(\mathbb{R}^n) \longrightarrow 0.$$

Moreover, by Theorem 3.4, $Q_1^+(D) : A(\mathbb{R}^{n+1}) \longrightarrow A(\mathbb{R}^{n+1})$ is surjective. By [16] $N_{Q_2}(\mathbb{R}^n) \cong H(\mathbb{C}^{n-1})$. Therefore, by Corollary 5.3 the sequence splits. ∎

We put all information together in the following theorem.

Theorem 5.5. *If $n > 1$, $P(D)$ is homogeneous and $P(D) : A(\mathbb{R}^n) \longrightarrow A(\mathbb{R}^n)$ is surjective then:*

1. *$N_P(\mathbb{R}^n) \cong N_{Q_1}(\mathbb{R}^n) \oplus N_{Q_2}(\mathbb{R}^n)$.*
2. *If Q_1 is nontrivial, then $N_{Q_1}(\mathbb{R}^n)$ has no basis.*
3. *If Q_2 is nontrivial, then $N_{Q_2}(\mathbb{R}^n) \cong H(\mathbb{C}^{n-1})$.*

Returning to not necessarily homogeneous operators, let us finally remark that Proposition 5.2 gives us a lot of operators $P(D)$ so that $N_P(\Omega)$ has property $(P\overline{\overline{\Omega}})$. In the case of an irreducible principle part and $\Omega = \mathbb{R}^n$ we obtain even more.

Theorem 5.6. *If $n > 1$, P_m is irreducible and $P(D) : A(\mathbb{R}^n) \longrightarrow A(\mathbb{R}^n)$ is surjective, then either P is elliptic or $N_P(\mathbb{R}^n)$ has property $(P\overline{\overline{\Omega}})$.*

Proof: In view of Theorem 3.4 and Proposition 5.2 it is enough to state that for no elliptic P and no Ω the space $N_P(\Omega)$ has $(P\overline{\overline{\Omega}})$. This is because $N_P(\Omega)$ has property $(\underline{\mathrm{DN}})$ (see e.g. [12]) and both properties together would mean that $N_p(\Omega)$ is finite dimensional (see [1], Proposition 5.3. together with [13]). ■

In fact, we showed a bit more than claimed, namely that for any Ω condition $(P\overline{\overline{\Omega}})$ for $N_p(\Omega)$ excludes ellipticity of $P(D)$.

References

[1] J. Bonet, P. Domański, Parameter dependence of solutions of differential equations on spaces of distributions and the splitting of short exact sequences, *J. Funct. Anal.* **230** (2006), no. 2, 329–381.

[2] P. Domański, M. Langenbruch, Composition operators on spaces of real analytic functions. *Math. Nachr.* **254/255** (2003), 68–86.

[3] P. Domański, D. Vogt, A splitting theory for the space of distributions, *Studia Math.* **140** (2000), 57–77.

[4] P. Domański, D. Vogt, The space of real analytic functions has no basis, *Studia Math.* **142** (2000), 187–200.

[5] P. Domański, D. Vogt, Linear topological properties of the space of analytic functions on the real line, in: *Recent Progress in Functional Analysis*, K. D. Bierstedt, J. Bonet, M. Maestre, J. Schmets (Eds.), Elsevier (2001), 113-132.

[6] L. Hörmander, *The Analysis of Linear Partial Differential Operators II*, Springer, Berlin-Heidelberg-New York-Tokyo, 1983.

[7] L. Hörmander, On the existence of real analytic solutions of partial differential equations with constant coefficients, *Invent. Math.* **21** (1973), 151–183.

[8] A. Martineau, Sur la topologie des espaces de fonctions holomorphes, *Math. Ann.* **163** (1966), 62-88.

[9] R. Meise, D. Vogt, *Introduction to Functional Analysis*, Clarendon Press, Oxford 1997.

[10] R. Meise, B.A. Taylor, D. Vogt: Characterization of the linear partial operators with constant coefficients that admit a continuous linear right inverse, *Ann. Inst. Fourier (Grenoble),* **40** (1990), 619–655.

[11] R. Meise, B.A. Taylor, D. Vogt, Phragmén-Lindelöf-Principles on algebraic varieties, *J. Amer. Math. Soc.* **11** (1998), 1-39.

[12] D. Vogt, Charakterisierung der Unterräume eines nuklearen stabilen Potenzreihenraumes von endlichem Typ, *Studia Math.* **71** (1982), 251-270.

[13] D. Vogt, Frécheträume, zwischen denen jede stetige lineare Abbildung beschränkt ist, *J. reine angew. Math.* **345** (1983), 182-200.

[14] D. Vogt, Fréchet valued real analytic functions, *Bull. Soc. Roy. Sc. Liège* **73** (2004), 155–170.

[15] J. Wengenroth, Derived functors in functional analysis, *Lect. Notes Math.* **1810**, Springer, Berlin 2003.

[16] G. Wiechert, Dualitäts- und Strukturtheorie der Kerne von linearen Differentialoperatoren, Dissertation, Wuppertal 1982.

Bergische Universität Wuppertal,
FB Math.-Nat., Gauss-Str. 20,
D-42097 Wuppertal, Germany
e-mail: dvogt@math.uni-wuppertal.de