On certain (LB)-spaces

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To my friend Jean Schmets on his 65^{th} anniversary

Abstract

Let (X_n) be a sequence of infinite-dimensional Banach spaces. For E being the space $\bigoplus_{n=1}^{\infty} X_n$, the following equivalences are shown: 1. E' $[\mu(E',E)]$ is B-complete. 2. Every separated quotient of E' $[\mu(E',E)]$ is complete. 3. Every separated quotient of E satisfies Mackey's weak condition. 4. X_n is quasi-reflexive, $n \in \mathbb{N}$.

1 Introduction and notation

The linear spaces that we shall use here are assumed to be defined over the field $\mathbb{K}$ of real or complex numbers, and the topologies on them will all be Hausdorff. As usual, $\mathbb{N}$ represents the set of positive integers. If $\langle E, F\rangle$ is a dual pair, then $\sigma(E,F)$, $\mu(E,F)$ and $\beta(E,F)$ denote the weak, Mackey and strong topologies on E, respectively. We shall write $\langle\cdot,\cdot\rangle$ for the bilinear functional associated to $\langle E,F\rangle$. Let E be a locally convex space and let τ be its topology, if A is a subset of E then A $[\tau]$ means the set A endowed with the topology induced by τ, $\overline{A}$ is the closure of A and A° is the polar set of A in the topological dual E' of E. E'' is the topological dual of E' $[\beta(E',E)]$. By $\rho(E,E')$ we denote the topology on E of the uniform convergence over each absolutely convex compact subset of E' $[\beta(E',E)]$. We identify E, in the usual fashion, with a linear subspace of E''. If B is a subset of E, by $\tilde{B}$ we mean the closure of B in E'' $[\sigma(E'',E')]$. A linear functional u on E is said to be bounded if it is bounded on every bounded subset of E.

Let A be a bounded absolutely convex subset of the locally convex space E. Then E_A denotes the linear span of A endowed with the norm defined by the gauge of A.

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The space E is said to be locally complete if E_A is complete for every bounded closed absolutely convex subset A of E; if E is sequentially complete, and, in particular, when it is complete, then it is locally complete. We say the E satisfies Mackey's weak condition if for an arbitrary sequence (x_n) in E which converges to the origin, there is a bounded closed absolutely convex subset A of E such that $x_n \in A$, $n \in \mathbb{N}$, and (x_n) converges to the origin in E_A for the weak topology.

Following Ptak [6], (see also [2, p. 299]), a locally convex space E is B-complete if every subspace F of E' is $\sigma(E', E)$-closed when $F \cap A$ is $\sigma(E', E)$-closed in A for each equicontinuous subset A of E. If E is B-complete, then every separated quotient of E is complete.

We shall say that a Banach space X is quasi-reflexive if it has finite codimension in its bidual X''. In [3], R. C. James gives an example of a quasi-reflexive Banach space that is not reflexive.

A locally convex space E is said to be an (LB)-space if it is the inductive limit of a sequence of Banach spaces, or, equivalently, if it is the separated quotient of the topological direct sum of a sequence of Banach spaces. The first example of an (LB)-space which is not complete is due to Köthe (see [5, pp. 434-435]). In [8] we give the following result: *a) Let (X_n) be a sequence of infinite-dimensional Banach spaces. If $E := \bigoplus_{n=1}^{\infty} X_n$, then the following are equivalent: 1. E is B-complete. 2. Every separated quotient of E is complete. 3. X_n is quasi-reflexive, $n \in \mathbb{N}$.*

In Section 2 of this paper, we obtain a theorem containing an analogous result to that of *a)* replacing E by $E'[\mu(E', E)]$.

Let (x_n) be a sequence in a linear space. We say that (y_n) is a block-convex sequence of (x_n) if there are positive integers

$$1 \; = \; n_1 \; < \; n_2 \; < \; ... \; < \; n_j \; < \; ...$$

and, for each $j \in \mathbb{N}$, there is $a_{jr} \geq 0$, $r = n_j, n_j + 1, ..., n_{j+1} - 1$, such that

$$\sum_{r=n_j}^{n_{j+1}-1} \alpha_{jr} \; = \; 1, \quad y_j \; = \; \sum_{r=n_j}^{n_{j+1}-1} \alpha_{jr} x_r.$$

We shall say that a Schauder basis (x_n) in a Fréchet space E has property P if there is a continuous seminorm p on E such that

$$inf\{p(x_n) \; : \; n \in \mathbb{N}\} \; > \; 0$$

and the set $\{x_1 + x_2 + ... + x_n \; : \; n \in \mathbb{N}\}$ is bounded in E. Property P was introduced by I. Singer in [7] for Banach spaces.

Let E be a locally convex space. A family $\mathcal{A}$ of absolutely convex closed and bounded subsets of E is said to be saturated when the following conditions are satisfied:

1. $\cup\{A \; : \; A \in \mathcal{A}\} = E$.
2. Every finite union of elements of $\mathcal{A}$ is contained in an element of $\mathcal{A}$.
3. Given any $A \in \mathcal{A}$ and $k \in \mathbb{K}$, there is an element B in $\mathcal{A}$ such that $kA \subset B$.

Hence $\{A^\circ \; : \; A \in \mathcal{A}\}$ is a fundamental system of zero neighborhoods in E' for a locally convex topology that we shall denote by $\tau_{\mathcal{A}}$

Proposition 1. *Let E be a locally convex space. Let $\mathcal{A}$ be a saturated family of absolutely convex closed bounded subsets of E. If T is an absolutely convex subset of E such that, for each $A \in \mathcal{A}$, $T \cap A$ is a neighborhood of the origin in A $[\sigma(E, E')]$, then T° is a precompact subset of E' $[\tau_{\mathcal{A}}]$.*

Proof. By G we denote the subspace of E'' given by

$$G \;=\; \cup\{\tilde{A} \;:\; A \in \mathcal{A}\}.$$

By S we represent the closure of T in G $[\sigma(G, E')]$. We fix $A \in \mathcal{A}$. We find an absolutely convex compact subset M of E' $[\sigma(E', E)]$ whose linear hull has finite dimension and, if P is the polar set of M in E, then

$$P \cap A \;\subset\; T \cap A \;\subset\; S \cap \tilde{A}. \tag{1}$$

The convex hull D of $A^\circ \cup M$ is $\sigma(E', E)$-closed and so D is the polar set of $P \cap A$ in E'. Then the polar set of D in G coincides with the closure F of $P \cap A$ in G $[\sigma(G, E')]$ and, after (1), we have that F is contained in $S \cap \tilde{A}$. On the other hand, F coincides with the polar set of $A^\circ \cup M$ in G and so, if Q is the polar set of M in G, having in mind that P is $\sigma(G, E')$-dense in Q, it follows that

$$F \;=\; Q \cap \tilde{A} \;\subset\; S \cap \tilde{A}.$$

Clearly, $Q \cap \tilde{A}$ is a neighborhood of the origin in $\tilde{A}[\sigma(G, E')]$ and thus $S \cap \tilde{A}$ is a neighborhood of the origin in $\tilde{A}[\sigma(G, E')]$.

If B is an arbitrary subset of E', we write B° to denote the polar set of B in G. We consider now the locally convex space E' $[\tau_{\mathcal{A}}]$. It is clear that we may identify the topological dual of this space with G, with $\{\tilde{A} : \; A \in \mathcal{A}\}$ being a fundamental system of equicontinuous subsets. Let $\mathcal{B}$ stand for the family of all absolutely convex closed and bounded subsets of E' $[\tau_{\mathcal{A}}]$ such that $B^\circ \cap \tilde{A}$ is a zero neighborhood in $\tilde{A}[\sigma(G, E')]$ for every $B \in \mathcal{B}$ and every $A \in \mathcal{A}$. It is no hard job to see that $\mathcal{B}$ is saturated. Let $\tau_{\mathcal{B}}$ be the topology on G given by the uniform convergence over every element of $\mathcal{B}$. We fix now $A \in \mathcal{A}$ and take in $\tilde{A}$ a net

$$\{x_j \;:\; j \in J, \; \geq\} \tag{2}$$

such that it $\sigma(G, E')$-converges to x. We then find an element A_1 in $\mathcal{A}$ such that $2A \subset A_1$. Then the net

$$\{x_j - x \;:\; j \in J, \; \geq\} \tag{3}$$

is in $\tilde{A}_1$ and $\sigma(G, E')$-converges to the origin. Consequently, the net (3) $\tau_{\mathcal{B}}$-converges to the origin and so the net in (2) $\tau_{\mathcal{B}}$-converges to x. It then follows that $\tau_{\mathcal{B}}$ and $\sigma(G, E')$ coincide in $\tilde{A}$. Therefore, the elements of $\mathcal{B}$ are $\tau_{\mathcal{A}}$-precompact (see [4, 8. Proposition, p. 180]). We then deduce that the polar set of S in E', which coincides with T°, is $\tau_{\mathcal{A}}$-precompact. ∎

Let E be a locally convex space. We say that a subset A of E is a Banach disk if it is absolutely convex, bounded and E_A is a Banach space.

Let us consider now a dual pair $\langle F, G\rangle$. Let $\mathcal{A}$ be the family of all absolutely convex closed and bounded subsets of F $[\sigma(F, G)]$ such that $A \in \mathcal{A}$ if and only if there is a Banach disk B in F $[\sigma(F, G)]$ such that $A \subset B$ and A is weakly compact in E_B. Clearly, $\mathcal{A}$ is a saturated family. We represent by $\delta(G, F)$ the topology on G of the uniform convergence over every element of $\mathcal{A}$.

2 Mackey's weak condition in (LB)-spaces

Theorem 1. *Let (X_n) be a sequence of Banach spaces of infinite dimension. If X_1 is not quasi-reflexive, then there is a separated quotient H of $\bigoplus_{n=1}^{\infty} X_n$ such that it does not satisfy Mackey's weak condition.*

Before giving the proof of this theorem, we shall construct H as it is done in [8] and we shall establish some previous propositions. Hence, proceeding as in [8], we find in X_1 an increasing sequence of separable closed subspaces (F_n) such that

$$X_1 \neq X_1 + \tilde{F}_1, \quad X_1 + \tilde{F}_n \neq X_1 + \tilde{F}_{n+1}, \quad n \in \mathbb{N}.$$

Let E be the closed linear hull of $\cup_{n=1}^{\infty} F_n$ in X_1. As usual, we identify E'' with $\tilde{E}$. Let $E_n := X_{n+1}, \quad n \in \mathbb{N}$. We take

$$x_1 \in \tilde{F}_1, \quad x_1 \notin E, \quad x_{n+1} \in \tilde{F}_{n+1}, \quad x_{n+1} \notin E + \tilde{F}_n, \quad n \in \mathbb{N}.$$

We write Z for the linear hull of $E \cup \{x_n : n \in \mathbb{N}\}$. Let B be the closed unit ball of E. We put B_n for the closed unit ball of E_n, $n \in \mathbb{N}$. It follows that $B^\circ\ [\sigma(E', Z)]$ is metrizable and separable. By T_m we denote the subspace of E' orthogonal to F_m. In $(B^\circ \cap T_m)[\sigma(E', Z)]$ we choose a dense subset $\{u_{mn} : n \in \mathbb{N}\}$. We then define a mapping h from E into $\ell^\infty(\mathbb{N} \times \mathbb{N})$ by setting

$$h(z) \;=\; (\langle z, u_{mn}\rangle)_{m,n\in\mathbb{N}}, \quad z \in E.$$

For each $j \in \mathbb{N}$, we find ([9, Lemma 1]) a one-to-one continuous linear mapping φ_j from $\ell^\infty[\mu(\ell^\infty, \ell^1)]$ into E_j. Let

$$\Phi_j : \ \ell^\infty(\mathbb{N} \times \mathbb{N}) \longrightarrow E_j$$

be such that

$$\Phi_j((a_{mn})) = \ \varphi_j((a_{jn})), \quad (a_{mn}) \in \ell^\infty(\mathbb{N} \times \mathbb{N}).$$

We now define

$$\zeta : \ E \times \bigoplus_{n=1}^{\infty} E_n \longrightarrow \prod_{n=1}^{\infty} E_n$$

as

$$\zeta((z, (z_1, z_2, ..., z_n, ...))) \;=\; ((\Phi_1 \circ h)(z) + z_1, (\Phi_2 \circ h)(z) + z_2, ..., (\Phi_n \circ h)(z) + z_n, ...),$$

$$z \in E, \quad (z_1, z_2, ..., z_n ...) \in \bigoplus_{n=1}^{\infty} E_n.$$

Obviously, ζ is continuous. We write

$$H \;:=\; (E \times \bigoplus_{n=1}^{\infty} E_n)/\zeta^{-1}(0).$$

Proposition 2. [8]. H is an (LB)-space which is not locally complete.

In the sequel, we shall consider, in the usual manner, E, $\bigoplus_{n=1}^{r} E_n$, $r \in \mathbb{N}$, and $\bigoplus_{n=1}^{\infty} E_n$ as subspaces of $E \times \bigoplus_{n=1}^{\infty} E_n$. We take a closed absolutely convex neighborhood of the origin U in $\prod_{n=1}^{\infty} E_n$. We then put

$$T \;:=\; E \,\cap\, \zeta^{-1}(U).$$

It is plain that, for each $m \in \mathbb{N}$,

$$\zeta_{|mB} \;:\; (mB)\;[\sigma(E,E')] \longrightarrow \prod_{n=1}^{\infty} E_n$$

is continuous and thus the barrel T of E meets (mB) $[\sigma(E,E')]$ in a neighborhood of the origin. We apply now Proposition 1 for

$$\mathcal{A} \;:=\; \{mB \;:\; m \in \mathbb{N}\}$$

to obtain that the polar set of T in E' $[\beta(E',E)]$ is compact. Consequently,

$$\zeta :\; E[\rho(E,E')] \times \bigoplus_{n=1}^{\infty} E_n \longrightarrow \prod_{n=1}^{\infty} E_n$$

is continuous. Let τ be the locally convex topology on H such that

$$H\;[\tau] \;=\; (E[\rho(E,E')] \times \bigoplus_{n=1}^{\infty} E_n)/\zeta^{-1}(0).$$

Clearly, τ is compatible with the duality $\langle H, H' \rangle$. If we set $F \;:=\; \cup_{n=1}^{\infty} F_n$, it follows that $\zeta^{-1}(0)$ is contained in $F \times \bigoplus_{n=1}^{\infty} E_n$. If λ denotes the restriction of ζ to $F \times \bigoplus_{n=1}^{\infty} E_n$ and $\mathcal{S}$ is the topology induced in $\bigoplus_{n=1}^{\infty} E_n$ by the topology of $\prod_{n=1}^{\infty} E_n$, we then have

$$\lambda \;:\; F\;[\rho(E,E')] \times \bigoplus_{n=1}^{\infty} E_n \longrightarrow (\bigoplus_{n=1}^{\infty} E_n)[\mathcal{S}]$$

is continuous and onto. If η is the canonical mapping from $E \times \bigoplus_{n=1}^{\infty} E_n$ onto $(E \times \bigoplus_{n=1}^{\infty} E_n)/\zeta^{-1}(0)$, we denote by G the subspace of H $[\tau]$ given by the image under η of $F \times \bigoplus_{n=1}^{\infty} E_n$.

Given an arbitrary x in G, we find y in $F \times \bigoplus_{n=1}^{\infty} E_n$ such that $\eta(y) = x$, and we put $\varphi(x) = \lambda(y)$. Then

$$\varphi \;:\; G \longrightarrow (\bigoplus_{n=1}^{\infty} E_n)[\mathcal{S}]$$

is linear continuous one-to-one and onto. For an arbitrary element z of $\bigoplus_{n=1}^{\infty} E_n$, we fix the vector $0 \in E$ and write $\alpha(z) := \eta((0,z))$. Then

$$\alpha \;:\; \bigoplus_{n=1}^{\infty} E_n \longrightarrow G$$

is linear continuous one-to-one and onto. It is immediate to see that $\varphi \circ \alpha$ is the canonical injection from $\bigoplus_{n=1}^{\infty} E_n$ into $(\bigoplus_{n=1}^{\infty} E_n)[\mathcal{S}]$. We set

$$L := \eta(E),\;\; L_r := \eta(\bigoplus_{n=1}^{r} E_n),\;\; D := \eta(B),\;\; D_r := \eta(\bigoplus_{n=1}^{r} B_n),\;\; A_r := D + D_r,\;\; r \in \mathbb{N}.$$

Let M stand for the closure of D in H $[\tau]$.

Proposition 3. *In $H\ [\tau]$, L_r is a subspace isomorphic to $\bigoplus_{n=1}^r E_n$.*

Proof. It is an immediate consequence of the fact that $\varphi \circ \alpha_{|\bigoplus_{n=1}^r E_n}$ is a topological isomorphism from $\bigoplus_{n=1}^r E_n$ onto $(\bigoplus_{n=1}^r E_n)[\mathcal{S}]$. ■

Proposition 4. *$D\ [\tau]$ is a precompact topological space.*

Proof. Let $\{x_i : i \in I, \geq\}$ be a net in D. We take y_i in B such that $\eta(y_i) = x_i$, $i \in I$. Since $B\ [\rho(E, E')]$ is precompact, we find a Cauchy subnet $\{z_j : j \in J, \succeq\}$ of $\{y_i : i \in I, \geq\}$. Then, $\{\eta(z_j) : j \in J, \succeq\}$ is a Cauchy subnet of $\{x_i : i \in I, \geq\}$ in $D\ [\tau]$. ■

Proposition 5. *In $H\ [\tau]$, $M + L_r$ and $M + D_r$ are closed subsets.*

Proof. Let x be a point in the closure of $M + L_r$ in $H\ [\tau]$. We take a net $\{x_i : i \in I, \geq\}$ in $M + L_r$ converging to x. We then write

$$x_i = y_i + z_i, \quad y_i \in M, \quad z_i \in L_r, \quad i \in I.$$

Since M is precompact, there is a subnet of $\{z_i = x_i - y_i : i \in I, \geq\}$ which is Cauchy and, since L_r is complete, it follows that $\{z_i : i \in I, \geq\}$ has an adherent point $z \in L_r$. Then, $\{y_i : i \in I, \geq\}$ has $x - z$ as adherent point and, consequently, $x = (x - z) + z$ belongs to $M + L_r$. The same proof works for $M + D_r$, just replacing L_r by D_r. ■

Proposition 6. *If A is bounded in H, then there is $r \in \mathbb{N}$ such that A is contained in $r(M + D_r)$.*

Proof. It is immediate that H is the inductive limit of the sequence of Banach spaces (H_{A_n}). Therefore, if U_n is the polar set of nA_n in H', it follows that

$$\{U_n \ : \ n \in \mathbb{N}\}$$

is a fundamental system of zero neighborhoods in H' for a metrizable locally convex topology $\mathcal{V}$. Let K_n be the closure of nA_n in H. We then have that

$$\mathcal{K} := \ \{K_n \ : \ n \in \mathbb{N}\}.$$

is a saturated family of absolutely convex closed and bounded subsets of H such that $\mathcal{V}$ coincides in H' with the topology of the uniform convergence over the elements of $\mathcal{K}$. Let u be an element in the completion of $H'\ [\mathcal{V}]$. After Grothendieck's completion theorem ([5, p. 270]), $u^{-1}(0) \cap K_n$ is closed in H, $n \in \mathbb{N}$, and thus the restriction of u to H_{A_n} is continuous, hence we have that u belongs to H'. Consequently, $H'[\mathcal{V}]$ is a Fréchet space. If A° is the polar set of A in H', it follows that A° is a barrel in $H'[\mathcal{V}]$ and so it is a neighborhood of the origin, from where we deduce that there is a positive integer r such that A is contained in K_r. Now, since $r(M + D_r)$ is closed and contains rA_r, we have that A is contained in $r(M + D_r)$. ■

Proposition 7. *For each $r \in \mathbb{N}$, $M + D_r$ is not a Banach disk.*

Proof. After Proposition 1, there is a subset A of H which is absolutely convex closed and bounded and is not a Banach disk. Applying the former proposition we obtain $s \in \mathbb{N}$ such that A is contained in $s(M + D_s)$, hence we have that $M + D_s$ is not a Banach disk and so, having in mind that D_s is a Banach disk, it follows that M is not a Banach disk. Finally, given $r \in \mathbb{N}$, if $M + D_r$ was a Banach disk, since M is closed in H_{M+D_r}, we would have that M would then be a Banach disk, which is a contradiction. ∎

Proof of Theorem 1. Let us assume that H satisfies Mackey's weak condition. Since

$$\{r(M + D_r) \; : \; r \in \mathbb{N}\}$$

is a fundamental system of bounded sets in H, we apply ([10, (10), p.161]) to obtain $s \in \mathbb{N}$ such that the weak topology of H and the weak topology of H_{M+D_s} coincide in M. Let ψ be the canonical injection of the Banach space $H_{A_s} := H_{D+D_s}$ into H_{M+D_s}. It follows that D is dense in M for the weak topology of H_{M+D_s}. Then $D + D_s$ is dense in $M + D_s$ in the normed space H_{M+D_s} and so ψ is almost open. We then apply ([2, p.296]) to obtain that ψ is a topological isomorphism from H_{D+D_s} onto H_{M+D_s}. Hence H_{M+D_s} is a Banach space, which is a contradiction. ∎

We shall need later the following result that we proved in [12]: *b) Let E be a separable Fréchet space. Let (u_n) be a sequence in E' $[\sigma(E', E)]$ converging to the origin. If (u_n) does not converge to the origin in Mackey's weak sense, then there is a block-convex sequence (w_n) of (u_n) such that it satisfies the following properties:*

1. *(w_n) is $\sigma(E', E)$-basic.*
2. *If F is the $\sigma(E', E)$-closed linear hull of $\{w_n : n \in \mathbb{N}\}$ and $F^\perp$ is the subspace of E orthogonal to F, then the sequence (x_n) of $E/F^\perp$ such that*

$$\langle x_n, w_n \rangle = 1, \quad \langle x_n, w_m \rangle = 0, \quad m \neq n, \; m, n \in \mathbb{N},$$

is a Schauder basis with property P in $E/F^\perp$.

Lemma 1. *Let E be a Fréchet space. Let (u_n) be a sequence in E' $[\sigma(E', E)]$ which converges to the origin. If (u_n) does not converge in Mackey's weak sense, then there is a block-convex sequence (w_n) of (u_n) such that, if F is the subspace of E' $[\sigma(E', E)]$ given by the closed linear hull of $\{w_n : n \in \mathbb{N}\}$, then there is a bounded linear functional x on F such that $\langle x, w_n \rangle = 1$, $n \in \mathbb{N}$.*

Proof. We take in E' $[\sigma(E', E)]$ a fundamental system of absolutely convex compact subsets

$$A_1 \subset A_2 \subset ... \subset A_n \subset ...$$

so that $u_n \in A_1$, $n \in \mathbb{N}$. By $\| \cdot \|_n$ we denote the norm in E'_{A_n}. Let A_n° be the polar set of A_n in E. We write H for the linear hull of $\{u_n : n \in \mathbb{N}\}$. In E'_{A_m} we take a dense subset $\{u_{mn} : n \in \mathbb{N}\}$ of H. For every $m, n, r \in \mathbb{N}$, we choose in A_m° an element x_{mnr} such that

$$| \langle x_{mnr}, u_{mn} \rangle | > \| u_{mn} \|_m - \frac{1}{r}.$$

We denote by G the closed linear span of

$$\{x_{mnr} \ : \ m, n, r \in \mathbb{N}\}$$

in E. Let $G^\perp$ be the subspace of E' orthogonal to G and let φ be the canonical mapping from E' onto $E'/G^\perp$. We identify, in the usual manner, $E'/G^\perp$ with the topological dual of G. It follows that $\varphi(A_n)$, $n \in \mathbb{N}$, is a fundamental system of compact absolutely convex subsets of $(E'/G^\perp)$ $[\sigma(E'/G^\perp, G)]$. It is immediate that $(\varphi(u_n))$ converges to the origin in $(E'/G^\perp)$ $[\sigma(E'/G^\perp, G)]$.For an arbitrary $n \in \mathbb{N}$, we show that φ is an isometry from the normed subspace H of E'_{A_m} onto the normed subspace $\varphi(H)$ of $(E'/G^\perp)_{\varphi(A_m)}$. We put $| \cdot |_m$ to denote the norm of this Banach space. If $u \in H$, we clearly have that $| \varphi(u) |_m \leq \| u \|_m$. Given $r \in \mathbb{N}$, we find an element u_{mn} in H such that

$$\| u - u_{mn} \|_m < \frac{1}{r}.$$

Then,

$$\begin{aligned} | \varphi(u) |_m &= sup\{| \langle z, \varphi(u) \rangle |: \ z \in A_m^\circ \cap G\} \\ &= sup\{| \langle z, u \rangle |: \ z \in A_m^\circ \cap G\} \geq | \langle x_{mnr}, u \rangle | \\ &\geq | \langle x_{mnr}, u_{mn} \rangle | - | \langle x_{mnr}, u - u_{mn} \rangle | \\ &\geq \| u_{mn} \|_m - \frac{1}{r} - \| u - u_{mn} \|_m \geq \| u_{mn} \|_m - \frac{2}{r} \\ &\geq \| u \|_m - \| u_{mn} - u \|_m - \frac{2}{r} \geq \| u \|_m - \frac{3}{r}. \end{aligned}$$

Consequently, $\| u \|_m = | \varphi(u) |_m$. We deduce from here that $(\varphi(u_n))$ does not converge to the origin in Mackey's weak sense in $(E'/G^\perp)$ $[\sigma(E'/G^\perp, G)]$. We then apply result *b)* to obtain a block-convex sequence (w_n) of (u_n) such that $(\varphi(w_n))$ is basic in $(E'/G^\perp)$ $[\sigma(E'/G^\perp, G)]$ and, if L represents the closed linear hull in this space of $\{\varphi(w_n) : \ n \in \mathbb{N}\}$ and $L^\perp$ is the subspace of G orthogonal to L, then the sequence (z_n) of $G/L^\perp$ such that

$$\langle z_n, \varphi(w_n) \rangle = 1, \quad \langle z_m, \varphi(w_n) \rangle = 0, \quad m \neq n, \quad m, n \in \mathbb{N},$$

is a Schauder basis with property P of $G/L^\perp$. Thus, the sequence $(z_1 + z_2 + ... + z_n)_{n=1}^\infty$ is bounded in this space. Let y be an adherent point of this sequence in $(G/L^\perp)''$ $[\sigma((G/L^\perp)'', L)]$. It follows that y is a bounded linear functional in L $[\sigma(L, G/L^\perp)]$. Let F be the subspace of E' $[\sigma(E', E)]$ given by the closed linear hull of $\{w_n : n \in \mathbb{N}\}$. Let x be the linear functional on F such that

$$\langle x, u \rangle = \langle y, \varphi(u) \rangle, \quad u \in F.$$

Clearly, x is bounded in F. On the other hand,

$$\langle x, w_n \rangle = \langle y, \varphi(w_n) \rangle = \lim_m \langle z_1 + z_2 + ... + z_m, \varphi(w_n) \rangle = 1. \qquad \blacksquare$$

Lemma 2. *Let E be an (LB)-space. If E does not satisfy Mackey's weak condition, then there is a separated quotient of E' $[\delta(E', E)]$ which is not complete.*

Proof. We take a sequence (x_n) in E converging to the origin and not doing so in Mackey's weak sense. Let

$$A_1 \subset A_2 \subset ... \subset A_n \subset ...$$

a fundamental system of Banach disks in E. Then, E' $[\beta(E', E)]$ is a Fréchet space and A_n°, $n \in \mathbb{N}$, is a fundamental system of zero-neighborhoods in this space. Clearly, E is a subspace of E'' $[\beta(E'', E')]$ and so (x_n) is a sequence in E'' $[\sigma(E'', E')]$ that converges to the origin and does not converge in Mackey's weak sense. Applying the former lemma we obtain a block-convex sequence (y_n) of (x_n) such that, if F is the subspace of E'' $[\sigma(E'', E')]$ given by the closed linear hull of $\{y_n : n \in \mathbb{N}\}$, then there is a bounded linear functional u on F such that

$$\langle y_n, u \rangle = 1, \quad n \in \mathbb{N}. \tag{4}$$

We write V for $F \cap E$ with the topology induced by that of E. It follows that $A_n \cap V$, $n \in \mathbb{N}$, is a fundamental system of Banach disks in V. We see next that V' $[\delta(V', V)]$ is not complete. Let v be the restriction of u to V. Given $n \in \mathbb{N}$, we take in $A_n \cap V$ an absolutely convex subset D weakly compact in $V_{A_n \cap V}$. Since v is bounded in V, we have that $v^{-1}(0) \cap D$ is $\sigma(V, V')$-closed and, applying Grothendieck's completeness theorem, we have that v belongs to the completion of V' $[\delta(V', V)]$. Clearly, (y_n) converges to the origin in V and, in light of (4), v does not belong to V'. Finally, if $V^\perp$ is the subspace of E' orthogonal to V, it means no difficulty to show that E' $[\delta(E', E)]/V^\perp$ is isomorphic to V' $[\delta(V', V)]$ and the result now follows. ■

Theorem 2. *Let E be the direct topological sum of a sequence (X_n) of infinite-dimensional Banach spaces. The following conditions are then equivalent:*

1. *E' $[\mu(E', E)]$ is B-complete.*
2. *Every separated quotient of E' $[\mu(E', E)]$ is complete.*
3. *Every separated quotient of E satisfies Mackey's weak condition.*
4. *X_n is quasi-reflexive, $n \in \mathbb{N}$.*

Proof. It is plain that 1 ⇒ 2. We show now that 2 ⇒ 3. Let us assume that condition 3 does not hold. We find a closed subspace L of E such that E/L does not satisfy Mackey's weak condition. Let $L^\perp$ be the subspace of E' orthogonal to L. We apply Lemma 2 to obtain a closed subspace M of $L^\perp$ $[\sigma(L^\perp, E/L)]$ so that $L^\perp$ $[\delta(L^\perp, E/L)]/M$ is not complete. Let τ be the restriction of $\mu(E', E)$ to $L^\perp$. It is immediate that τ is coarser than $\delta(L^\perp, E/L)$ and, since both topologies are compatible with the duality $\langle E/L, L^\perp \rangle$, it follows that $L^\perp[\tau]/M$ is not complete. Hence, E $[\mu(E', E)]/M$ is not complete either. 3 ⇒ 4. It is an immediate consequence of Theorem 1. 4 ⇒ 1. Let F be a subspace of E such that every absolutely convex weakly compact subset of E meets F in a closed set. We consider $E_n := X_1 + X_2 + ... + X_n$ as a subspace of E, $n \in \mathbb{N}$. Then, $F_n := F \cap E_n$ is closed in E. We set $n_1 := 1$. Proceeding inductively, let us assume that, for a positive integer j, we have found the positive integer n_j. Since E_{n_j} has finite codimension in $\widetilde{E}_{n_j}$, there is an integer $n_{j+1} > n_j$ such that

$$\widetilde{F}_n \cap \widetilde{E}_{n_j} = \widetilde{F}_{n_{j+1}} \cap \widetilde{E}_{n_j}, \quad n \in \mathbb{N}, \quad n \geq n_{j+1}.$$

We see now that $H := \cup_{n=1}^{\infty} \widetilde{F}_n$ is $\sigma(E'', E')$-closed, making use for this purpose of Krein-Smulian's theorem (see [2, p.246]) applied to the Fréchet space E' $[\beta(E', E)]$. Let A be an absolutely convex compact subset of E'' $[\sigma(E'', E')]$. We find a positive integer r such that A is contained in $\widetilde{E}_r$. Then, $A \cap H = A \cap \widetilde{F}_{n_{r+1}}$ is $\sigma(E'', E')$-closed. Consequently, H is $\sigma(E'', E')$-closed and so $F = H \cap E$ is closed in E. ■

Note. It is said in [1] that a Fréchet space E is totally reflexive when every separated quotient of E is reflexive and then the following problem is posed ([1, probl. 9]): *Let E_1 and E_2 be totally reflexive Fréchet spaces. Is the product $E_1 \times E_2$ also totally reflexive? We proved in [11] that a Fréchet space is totally reflexive if and only if it is isomorphic to a closed subspace of a countable product of reflexive Banach spaces.* This property is thus adequate to give a positive answer to Grothendieck's question. Lemma 1 can be used to obtain our characterization of the totally reflexive Fréchet spaces in the following way: Let E be a totally reflexive Fréchet space and let F be a closed subspace of E' $[\beta(E', E)]$. If $F^{\perp}$ is the subspace of E orthogonal to F, then $E/F^{\perp}$ is reflexive and thus every bounded linear functional u on F extends to to a continuous linear functional on E' $[\beta(E', E)]$. Hence, after Lemma 1, every sequence that converges to the origin in E' $[\beta(E', E)]$ converges also in the weak sense of Mackey. Then, given an absolutely convex compact subset A of E' $[\sigma(E', E)]$, there is a subset B in E' $[\sigma(E', E)]$, absolutely convex and compact, such that A is contained in B and it is weakly compact in E'_B. (see [10, (10),p.161]). Proceeding now as in [11] we obtain that E' $[\beta(E', E)]$ is the inductive limit of a sequence of reflexive Banach spaces and so E is isomorphic to a closed subspace of a countable product of reflexive Banach spaces.

References

[1] GROTHENDIECK, A.: Sur les espaces (F) et (DF), *Summa Brasil. Math.* **3**, 57-123 (1954).

[2] HORVÁTH, J.: Topological Vector Spaces and Distributions I, *Reading, Massachussets, Palo Alto-London: Addison-Wesley Publishing Co.* 1966.

[3] JAMES, R. C.: Bases and reflexivity of Banach spaces, *Ann. Math.* **52**, 518-527 (1950).

[4] JARCHOW, H.: Locally Convex Spaces. *Stuttgart: Teubner* 1981.

[5] KÖTHE, G.: Topological Vector Spaces I. *Berlin-Heidelberg-New York: Springer* 1969.

[6] PTÁK, V.: Completeness and the open mapping theorem, *Bull. Soc. Math. Fr.***86**, 41-74 (1958).

[7] SINGER, I.: Basic sequences and reflexivity of Banach spaces, *Studia Math.* **21**, 351-369 (1962).

[8] VALDIVIA, M.: Countable direct sums of Banach spaces, *Math. Nachr.* **141**, 73-79 (1989).

[9] VALDIVIA, M.: On B_r-completeness, *Ann. Inst. Fourier* **25**, 235-248 (1975).

[10] VALDIVIA, M.: Topics in Locally Convex Spaces, *North Holland, Amsterdam*, 1982.

[11] VALDIVIA, M.: A characterization of totally reflexive Fréchet spaces, *Math. Z.* **200**, 327-346 (1989).

[12] VALDIVIA, M.: Basic sequences in the dual of a Fréchet space, *Math. Nachr.* **231**, 169-185 (2001).

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