A Banach-Stone Theorem for completely regular spaces

Hamid Reza Shatery Jafar Zafarani

This paper is dedicated to Professor Jean Schmets for his 65th birthday

Abstract

In this paper, we study some Banach algebras with this property that each linear isometry between them induces a Banach algebra isometry. We obtain a Banach-Stone type theorem between Baire functions defined on completely regular spaces. As a consequence, a similar result for the space of continuous functions is deduced.

1 Introduction

Let X be a completely regular Hausdorff space and E a Banach space. We designate by $C(X, E)$ (resp. $C(X)$) the space of all E-valued (resp. real-valued) continuous functions on X. The space of all members of $C(X, E)$ (resp. $C(X)$) with relatively compact ranges is denoted by $C^\circ(X, E)$ (resp. $C^\circ(X)$).
The celebrated Banach-Stone Theorem says that if K and Q are compact Hausdorff spaces and T is a linear isometric isomorphism of $C(Q)$ onto $C(K)$, then there is a homeomorphism φ from K onto Q and a continuous unimodular function h on K such that for each $f \in C(Q)$,

$$Tf(t) = h(t)f(\varphi(t)) \text{ for each } t \in K.$$

This Theorem has been generalized in three directions:

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1. Holsztynski [4] proved a similar result for nonsurjective isometries.

2. Behrends and some other authors investigated and extended this result for vector-valued cases ([2, 3, 5, 14, 17, 19]).

3. Kadison [11] gave a similar theorem for C^*-algebras. The Kadison's Theorem is the following: Let $\mathcal{A}$ and $\mathcal{B}$ be two C^*-algebras. Then T is a linear isometry from $\mathcal{A}$ onto $\mathcal{B}$, if and only if there is a unitary element v of $\mathcal{B}$ and a C^*-isomorphism τ such that $T = v\tau$.

Jayne [16] proved that each linear isometry isomorphism between $\beta_\alpha(X)$, space of Baire functions of class α on X and $\beta_\alpha(Y)$, space of Baire functions of class α on Y induces a Banach algebra isometry isomorphism between them, when X and Y are compact. In [29], the authors proved that any isometry ring isomorphism between $\beta_\alpha(X)$ and $\beta_\alpha(Y)$ rises to a Banach-Stone type theorem when X and Y are perfectly normal spaces . In this paper, we establish a similar result for the spaces of bounded Baire functions defined on completely regular Hausdorff spaces.

Let X be a topological space and $\beta_0(X) = C(X)$. For each ordinal number α, we define the real-valued Baire functions of class α as the following: (see [28])

$$\beta_\alpha(X) = \{f : X \to \mathbb{R}; \exists (f_n)_{n=1}^\infty \subseteq \cup_{n=1}^\infty \beta_{\alpha_n}(X), \text{ and}, \\ \forall n;\ \alpha_n < \alpha\ ,\ f_n \in \beta_{\alpha_n}(X)\ ,\ \lim_{n\to\infty} f_n(x) = f(x) \text{ for each } x \in X\}.$$

The vector valued Baire functions of class α is defined as the following: (see [29])

$$\beta_0(X, E) = C(X, E),$$

$$\beta_\alpha(X, E) = \{f : X \to E; \exists (f_n)_{n=1}^\infty \subseteq \cup_{n=1}^\infty \beta_{\alpha_n}(X, E), \text{ and}, \\ \forall n;\ \alpha_n < \alpha\ ,\ f_n \in \beta_{\alpha_n}(X, E)\ ,\ \lim_{n\to\infty} f_n(x) = f(x) \text{ for each } x \in X\}.$$

The set of all elements of $\beta_\alpha(X, E)$ with relatively compact ranges is denoted by $\beta_\alpha^\circ(X, E)$. For each $f \in \beta_\alpha^\circ(X, E)$, $\|f\|$ is defined as

$$\|f\| = \sup\{\|f(x)\| :\ x \in X\}.$$

For an ordinal number α, we denote the Borel sets of multiplicative (additive) class α by $\mathcal{P}_\alpha$ ($\mathcal{S}_\alpha$), beginning with $\mathcal{P}_0 = \mathcal{F}$ ($\mathcal{S}_0 = \mathcal{G}$), as the following (see [18]):

$$\mathcal{P}_\alpha : \mathcal{F}, \mathcal{G}_\delta, \mathcal{F}_{\sigma\delta}, \ldots$$

$$\mathcal{S}_\alpha : \mathcal{G}, \mathcal{F}_\sigma, \mathcal{G}_{\delta\sigma}, \ldots$$

The ambiguous sets of class α is denoted by $\mathcal{H}_\alpha$ ([18]) and defined as

$$\mathcal{H}_\alpha = \mathcal{S}_\alpha \cap \mathcal{P}_\alpha.$$

We also define Borel functions of class α as:

$$B_0(X) = C(X) \quad , \qquad B_0(X, E) = C(X, E),$$

$$B_\alpha(X) = \{f : X \to \mathbb{R}\ \ : f^{-1}(F) \in \mathcal{P}_\alpha \text{ for each closed set } F \subseteq \mathbb{R}\},$$

$$B_\alpha(X,E)=\{f:X\to E: f^{-1}(F)\in\mathcal{P}_\alpha \text{ for each closed set } F\subseteq E\}.$$

The set of all elements of $B_\alpha(X,E)$ (resp. $B_\alpha(X)$) with relatively compact ranges is denoted by $B_\alpha^\circ(X,E)$ (resp. $B_\alpha^\circ(X)$).

Let X be a completely regular space. For every $f:X\to\mathbb{R}$, we define $Z(f)=f^{-1}(\{0\})$ [1, 10, 12, 25, 26], and

$$\mathcal{Z}_\alpha(X)=\{Z(f):f\in\beta_\alpha(X)\}\ ,\ \mathcal{CZ}_\alpha(X)=\{X\setminus Z(f):f\in\beta_\alpha(X)\},$$

$$\text{and } \mathcal{A}_\alpha(X)=\mathcal{Z}_\alpha(X)\cap\mathcal{CZ}_\alpha(X).$$

$$(\mathcal{Z}(X)=\{Z(f):f\in C(X)\}\ ,\ \mathcal{CZ}(X)=\{X\setminus Z(f):f\in C(X)\},$$

$$\text{and } \mathcal{A}(X)=\mathcal{Z}(X)\cap\mathcal{CZ}(X).)$$

The class $\mathcal{Z}_\alpha(X)$(resp. class $\mathcal{CZ}_\alpha(X)$) is a multiplicative (resp. additive) class α of the Baire sets of X. The elements of $\mathcal{A}_\alpha(X)$ are called the two-sided Baire sets of class α. One can show that the results about separation of Borel sets in metric spaces are also valid for $\mathcal{A}_\alpha(X)$([6, 15, 16]). Some algebraic properties of Baire-1 functions are investigated in [7, 27].

Let E be a Banach space. The space E is canonically embedded in the second dual E^{**} of E. For every set $H\subseteq E^{**}$, we denote by $w_1^*(H)$ the set of all limits in E^{**} of w^*-convergent sequences in H. Denote $w_0^*E=E\subseteq E^{**}$ and $w_\alpha^*(E)=w_1^*(\cup\{w_\beta^*E:\ \beta<\alpha\})$ for every $\alpha\le\Omega$. By construction $w_\Omega^*(E)=w_1^*(\cup\{w_\alpha^*E:\ \alpha<\Omega\})$. The space $w_\alpha^*(E)$ is called the α-Baire space for E ([6]). McWilliams proved that $w_\alpha^*(E)$ is a closed subspace of E^{**} for every ordinal α ([21]). Also, we have $w_\alpha^*(C(X))=\beta_\alpha(X)$ for a pseudocompact space X and $w_\alpha^*(C(Y))=\beta_\alpha(\beta Y)$ for a completely regular space Y.

Let $f:X\to Y$, we say that f is a Baire(resp. Borel) α-continuous map if one of the following equivalent statements holds:
(1)- Inverse image by f of every $\mathcal{CZ}_\alpha$(resp. $\mathcal{S}_\alpha$) set in Y is a $\mathcal{CZ}_\alpha$(resp. $\mathcal{S}_\alpha$) set in X.
(2)- Inverse image by f of every $\mathcal{Z}_\alpha$(resp. $\mathcal{P}_\alpha$) set in Y is a $\mathcal{Z}_\alpha$(resp. $\mathcal{P}_\alpha$) set in X.
(3)- Inverse image by f of every $\mathcal{A}_\alpha$(resp. $\mathcal{H}_\alpha$) set in Y is an $\mathcal{A}_\alpha$(resp. $\mathcal{H}_\alpha$) set in X.

When f is bijective, if f and f^{-1} are both Baire(resp. Borel) α-continuous, then we say that f is a Baire(resp. Borel) α-homeomorphism between X and Y, and we say that X and Y are Baire(resp. Borel) α-homeomorphic. It is trivial that every continuous function is Baire(resp. Borel) α-continuous for each ordinal α .

2 Tietze extension Theorem for Baire functions

In general, β_α is not equal to B_α, see [28]. The Lebesgue-Hausdorff theorem ([18], page 391) says that if X is metric and Y is either an n-dimensional cube $[0,1]^n$, or the Hilbert cube $[0,1]^{\mathbb{N}}$, then the first Baire and Borel classes of functions from X to Y coincide. Rolewicz showed in [24] that if Y is a separable convex subset of a normal linear space, the first Baire and Borel classes of functions from X to Y coincide. Here, as a consequence of a Tietze extension type theorem, we obtain an extension of this result for the spaces of bounded vector-valued Baire and bounded vector-valued Borel functions on completely regular Hausdorff spaces.

We first mention some of the properties of the completely regular spaces.

Remark 2.1. By the same argument as in metric spaces [18], one can easily show that in a completely regular hausdorff space X, we have:

(a) Every set in $\mathcal{CZ}_\alpha$ $(\alpha \geq 1)$ is the union of some countable disjoint $\mathcal{A}_\alpha$ sets. Moreover, for each $A \in \mathcal{Z}_\alpha$, there exists a sequence $(G_n)_{n=1}^\infty \subseteq \cup_{\alpha_n<\alpha}\mathcal{CZ}_{\alpha_n}$ such that

$$A = \cap_{n=1}^\infty G_n.$$

Similarly, for additive sets, "$\mathcal{CZ}$", "$\mathcal{Z}$", and "$\cap$" are replaced by "$\mathcal{Z}$", "$\mathcal{CZ}$", and "$\cup$", respectively.

(b) For each sequence $(G_n)_{n=1}^\infty \subseteq \mathcal{CZ}_\alpha$ $(\alpha \geq 1)$, there exists a mutually disjoint sequence $(H_n)_{n=1}^\infty$ in $\mathcal{CZ}_\alpha$ such that $\cup_{i=1}^\infty H_i = \cup_{i=1}^\infty G_i$ and $H_i \subseteq G_i$ for each i. In addition, if $X = \cup_{i=1}^\infty H_i$, then H_i's belong to $\mathcal{A}_\alpha$.

(c) For every sequence $(F_n)_{n=1}^\infty$ in $\mathcal{Z}_\alpha$ $(\alpha \geq 1)$ such that $\cap_{n=1}^\infty F_n = \emptyset$, there exists a sequence $(E_n)_{n=1}^\infty \subseteq \mathcal{A}_\alpha$ such that $\cap_{n=1}^\infty E_n = \emptyset$ and $F_n \subseteq E_n$ for each n. In particular, if A and B are two disjoint $\mathcal{Z}_\alpha$ sets, then there exists E in $\mathcal{A}_\alpha$ such that $A \subseteq E$ and $B \cap E = \emptyset$. That is, if $A \in \mathcal{Z}_\alpha$, $C \in \mathcal{CZ}_\alpha$ and $A \subseteq C$, then there exists $E \in \mathcal{A}_\alpha$ such that $A \subseteq E \subseteq C$ ([18]).

Remark 2.2. By a similar proof as that of Lemmas 1.2 and 1.3 in [28], one can show the following facts hold for $\mathcal{CZ}_\alpha$ sets. Let $B \subseteq A \subseteq X$. If $B \in \mathcal{CZ}_\alpha(A)$ (resp. $\mathcal{Z}_\alpha(A)$), then there is an element $G \in \mathcal{CZ}_\alpha(X)$ (resp. $\mathcal{Z}_\alpha(X)$) such that $A \cap G = B$. Consequently, if $A \subseteq X$ and $B \in \mathcal{Z}_\alpha(A)$ (resp. $\mathcal{CZ}_\alpha(A)$ or $\mathcal{A}_\alpha(\mathrm{A})$), then B belongs to $\mathcal{Z}_\alpha(X)$ (resp. $\mathcal{CZ}_\alpha(X)$ or $\mathcal{A}_\alpha(\mathrm{X})$). Moreover, if $A \in \mathcal{Z}_\alpha(X)$ and K is $\mathcal{A}_\alpha(A)$, then there exists $H \in \mathcal{A}_\alpha(X)$ such that $K = A \cap H$.

In the sequel, we give a refinement of Remark 2.2 for a countable partition of A.

Lemma 2.3. *Let $A \in \mathcal{Z}_\alpha(X)$ and $(A_n)_{n=1}^\infty$ be a partition of A consisting of $\mathcal{A}_\alpha(A)$ sets. Then there is a partition $(O_n)_{n=1}^\infty$ of X consisting of $\mathcal{CZ}_\alpha(X)$ sets such that $A_n = O_n \cap A$ for each n in $\mathbb{N}$.*

Proof. By Remark 2.2, there exists a sequence $(C_n)_{n=1}^\infty$ consisting of $\mathcal{A}_\alpha(X)$ sets such that $A_n = C_n \cap A$. Let $O = \cup_{n=1}^\infty C_n$. Then $O \in \mathcal{CZ}_\alpha(X)$ and $A \subseteq O$. From

Remark 2.1(c), there is an element $C \in \mathcal{A}_\alpha(X)$ such that $A \subseteq C \subseteq O$. Therefore, $\{C_n : n \in \mathbb{N}\} \cup \{C^c\}$ is a cover of X by $\mathcal{A}_\alpha(X)$ sets. By Remark 2.1(b), there is partition $\{D_j : j \geq 1\}$ of X by $\mathcal{A}_\alpha(X)$ sets such that each D_i is a subset of some sets in the above cover. It's obvious that each D_j that intersects A is contained in some C_i and therefore intersects A_n if and only if $n = i$. For each $i \in \mathbb{N}$, we define:

$$O_i^* = \cup\{D_j : D_j \cap A_i \neq \emptyset\}.$$

Thus, O_i^*'s are $\mathcal{CZ}_\alpha(X)$ sets and we have $O_i^* \cap O_j^* = \emptyset$ for $i \neq j$. Also, for each positive integer i, we have $O_i^* \cap A = A_i$. Let O^* be $C \cup (\cup_{i=1}^\infty O_i^*)$. Then O^* is in $\mathcal{CZ}_\alpha(X)$. We choose an $i_0 \in \mathbb{N}$ and set $O_{i_0} = O_{i_0}^* \cup O^*$ and for $i \neq i_0$, we set $O_i = O_i^*$. Thus $(O_i)_{i=1}^\infty$ is the desired partition.

We can give a Baire-α characterization of $\mathcal{A}_\alpha(X)$ sets similar to the case of perfectly normal spaces [29].

Remark 2.4. Let X be a completely regular topological space and E be a Banach space. We have $H \in \mathcal{A}_\alpha(X)$ if and only if $\chi_H \in \beta_\alpha(X)$. Furthermore, if $e \in E$ and $H \in \mathcal{A}_\alpha(X)$, then $e\,\chi_H \in \beta_\alpha(X, E)$.

The Tietze extension theorem has been generalized for many cases of continuous functions [9, 22]. Hausdorff proved a Tietze extension type theorem for real valued Borel functions [13]. Here, we prove a Tietze extension type theorem for vector valued Baire(Borel) functions. This theorem improves a result of Leung and Tang in [20] for vector-valued Baire functions.

Theorem 2.5. *Let $A \in \mathcal{Z}_\alpha(X)$(resp. $\mathcal{P}_\alpha(X)$) and E be a Banach space. Each f in $\beta_\alpha^\circ(A, E)$(resp. $B_\alpha^\circ(A, E)$) has an extension g in $\beta_\alpha^\circ(X, E)$(resp. $B_\alpha^\circ(A, E)$).*

Proof. We prove the theorem for vector valued Baire functions. The proof for the vector valued Borel functions is similar and therefore is omitted. As the $range(f)$ is relatively compact, therefore there exists a countable set $D \subseteq E$ such that the $range(f)$ is in the norm closure of D. For each positive integer n, let $\mathcal{C}_n$ be the collection of open balls of radius $\frac{1}{n}$ in E with centers in D. Hence the $range(f)$ is covered by finite members of $\mathcal{C}_n$, denoted by $\mathcal{B}_n$. Set $\mathcal{B}_n' = f^{-1}(\mathcal{B}_n)$. Thus $\mathcal{B}_n'$ is a cover of A and its elements are $\mathcal{CZ}_\alpha(A)$ sets. Part (b) of Remark 2.1 implies that $\mathcal{B}_n'$ has a countable refinement consisting of mutually disjoint elements of $\mathcal{A}_\alpha(A)$ sets. Denote this refinement by $\mathcal{W}_n = \{A_{i,n} : i \in I_n\}$. We can suppose that for any $n \geq 2$, $\mathcal{W}_n$ refines $\mathcal{W}_{n-1}$. For $\mathcal{W}_1$, Lemma 2.3 implies that there is a partition $\mathcal{K}_1 = \{O_{i,1} : i \in I_1\}$ of X consisting of $\mathcal{CZ}_\alpha(X)$ sets such that:

$$A_{i,1} = O_{i,1} \cap A.$$

By induction, we will obtain a partition $\mathcal{K}_n = \{O_{i,n} : i \in I_n\}$ of X consisting of $\mathcal{CZ}_\alpha(X)$ sets such that $\mathcal{K}_n$ is a refinement of $\mathcal{K}_{n-1}$, and for each $i \in I_n$

$$A_{i,n} = O_{i,n} \cap A.$$

By part (b) of Remark 2.1, elements of $\mathcal{K}_n$ are $\mathcal{A}_\alpha(X)$ sets. Let $n > 1$ and assume that $\mathcal{K}_{n-1}$ has been defined. For each $i \in I_{n-1}$, let

$$J_{i,n} = \{j \in I_n : A_{j,n} \subseteq A_{i,n-1}\}.$$

Thus $\{A_{j,n} : j \in J_{i,n}\}$ is a finite $\mathcal{CZ}_\alpha(X)$ partition of $A_{i,n}$. From Lemma 2.3, there is a finite $\mathcal{CZ}_\alpha(X)$ partition $\{O_{j,n} : j \in J_{i,n}\}$ of $O_{i,n-1}$ such that for each $j \in J_{i,n}$:

$$O_{j,n} \cap A_{i,n-1} = A_{j,n}.$$

Then $I_n = \cup\{J_{i,n} : i \in I_{n-1}\}$, and $\mathcal{K}_n = \{O_{j,n} : j \in I_n\}$ is the desired $\mathcal{A}_\alpha(X)$ partition of X.
Now, for each $n \geq 1$ and each $i \in I_n$, choose $y_{i,n} \in f(A_{i,n})$. Let $x \in X$ and for each $n \geq 1$ let $i(x,n) \in I_n$ be such that $x \in O_{i(x,n),n}$. If $m \geq n$, then $O_{i(x,m),m} \subseteq O_{i(x,n),n}$ and so $A_{i(x,m),m} \subseteq A_{i(x,n),n}$. Consequently, since $f(A_{i(x,n),n})$ has diameter at most $2/n$, $\{y_{i(x,n),n} : n \geq 1\}$ is a Cauchy sequence.
We define:

$$g(x) = \lim_{n\to\infty} y_{i(x,n),n}.$$

Notice that

$$\|g(x) - y_{i(x,n),n}\| \leq 2/n.$$

If $x' \in O_{i(x,n),n}$, then $i(x,n) = i(x',n)$ and so

$$\|g(x) - g(x')\| \leq \|g(x') - y_{i(x,n),n}\| + \|g(x) - y_{i(x,n),n}\| \leq 4/n.$$

If $x \in A$, then for each $n \geq 1$, $x \in A_{i(x,n),n}$ and therefore

$$\|g(x) - f(x)\| \leq 4/n.$$

Hence g is an extension of f.
Now, for each $n \in \mathbb{N}$, we define

$$f_n = \sum_{i\in I_n} y_{i,n}\chi_{A_{i,n}}.$$

Remark 2.4 implies that f_n's are in $\beta^\circ_\alpha(X,E)$. Also, notice that g is the uniform limit of f_n's and therefore, $g \in \beta_\alpha(X,E)$. Now, we prove that the $range(g)$ is relatively compact and therefore, $g \in \beta^\circ_\alpha(X,E)$. Let $\varepsilon > 0$ be given. There is an n_0 such that $\|g - f_{n_0}\| < \varepsilon$. Set $L = \{y_{i,n_0} : i \in I_{n_0}\}$ and suppose that B_E is the open unit ball of E. Therefore, L is finite and $range(g) \subseteq L + \varepsilon B_E$. Consequently, the range of g is totally bounded and therefore, it's relatively compact.

As an application of Theorem 2.5, we obtain an approximation theorem for $\beta^\circ_\alpha(X,E)$ and $B^\circ_\alpha(X,E)$. We define:

$$\Sigma\beta_{\alpha,E} = \{\sum_{i=1}^{n} e_i\chi_{H_i} : n \in \mathbb{N}, e_i \in E \text{ and } H_i \in \mathcal{A}_\alpha \text{ for each } i \leq n\}.$$

$$\Sigma B_{\alpha,E} = \{\sum_{i=1}^{n} e_i\chi_{H_i} : n \in \mathbb{N}, e_i \in E \text{ and } H_i \in \mathcal{H}_\alpha \text{ for each } i \leq n\}.$$

Corollary 2.6. *For a Banach space E, the uniform closure of $\Sigma\beta_{\alpha,E}$ is $\beta^\circ_\alpha(X,E)$ and the uniform closure of $\Sigma B_{\alpha,E}$ is $B^\circ_\alpha(X,E)$.*

Jayne [15] obtained a necessary and sufficient condition for the equality between Baire and Borel functions in the real case. As a consequence of Corollary 2.6, we deduce a vector-valued version of this result.

Theorem 2.7. *Let X be a completely regular space and E be a Banach space. Then $\beta^\circ_\alpha(X,E) = B^\circ_\alpha(X,E)$ for finite ordinal numbers(resp. $\beta^\circ_\alpha(X,E) = B^\circ_{\alpha+1}(X,E)$ for infinite ordinal numbers) if and only if $\beta^\circ_\alpha(X) = B^\circ_\alpha(X)$ for finite ordinal numbers(resp. $\beta^\circ_\alpha(X) = B^\circ_{\alpha+1}(X)$ for infinite ordinal numbers) if and only if $\mathcal{A}_\alpha(X) = \mathcal{H}_\alpha(X)$ for finite ordinal numbers (resp. $\mathcal{A}_\alpha(X) = \mathcal{H}_{\alpha+1}(X)$ for infinite ordinal numbers).*

In the following result we obtain a Banach-Stone type theorem between spaces of bounded Baire functions defined on completely regular Hausdorff spaces.

Theorem 2.8. *Let X and Y be two completely regular spaces with the property that each singleton is in $\mathcal{Z}_\alpha(X)$ and $\mathcal{Z}_\alpha(Y)$, respectively. If $\varphi : \beta^\circ_\alpha(Y) \to \beta^\circ_\alpha(X)$ is a surjective isometric linear isomorphism, then there exists a Baire α-homeomorphism, $\tau : X \to Y$, and a unimodular Baire-α function $h \in \beta^\circ_\alpha(X)$ such that*

$$\varphi(f)(x) = h(x)\ (f \circ \tau)(x) \quad \textit{for each} \quad f \in \beta^\circ_\alpha(Y).$$

Proof. Kadison's theorem implies that there is a unimodular Baire-α function $h \in \beta^\circ_\alpha$ and a Banach algebra isometry ψ such that $\varphi(f) = h\ \psi(f)$. Now by the same argument as that of Theorem 2.1 in [29], there exists a Baire-α homeomorphism τ such that $\psi(f) = f \circ \tau$. Therefore, the proof is complete.

Remark 2.9 Let X and Y be two completely regular spaces with the property that each singleton is in $\mathcal{P}_\alpha(X)$ and $\mathcal{P}_\alpha(Y)$, respectively. By the same argument as that of Theorem 2.1 in [29], we can show that if $\varphi : B^\circ_\alpha(Y) \to B^\circ_\alpha(X)$ is a surjective isometric linear isomorphism, then there exists a Borel α-homeomorphism $\tau : X \to Y$ and a unimodular Borel-α function $h \in B^\circ_\alpha(X)$ such that, we have

$$\varphi(f)(x) = h(x)\ (f \circ \tau)(x), \text{ for each } f \text{ in } B^\circ_\alpha(Y).$$

Choban [6] investigated some properties of the following compactification for a completely regular space X denoted by $b_\alpha X$. Let PX be the set X with the topology generated by the $\mathcal{G}_\delta(X)$ sets for a completely regular Hausdorff space X. The topology of the space PX is called the Baire topology of the space X. If $\beta_1(X) \subseteq K \subseteq \beta_\Omega(X)$, where Ω is the first uncountable ordinal number, then $PX = (X, \tau_K)$, where τ_K is the weakest topology on X generated by K. We define

$$b_\alpha X = \text{The compactification of } \ (X, \tau_{\beta_\alpha(X)}).$$

The compact space $b_\alpha X$ is called the maximal ideal space of the α-th Baire class $\beta_\alpha(X)$. We say that the Banach space E has the Banach-Stone property if for each (locally) compact Hausdorff spaces X_1 and X_2, $C(X_1, E)$ and $C(X_2, E)$ are linearly

isometric, then X_1 and X_2 are topologically homeomorphic. Jerison [17] proved that if E is a real strictly convex Banach space, then E has the Banach-Stone property. Behrends proved that every Banach space E such that $Z(E)$ (the centralizer of E) is one dimensional has the Banach-Stone property ([2], Theorem 8.11).
Now, we obtain the following vector version of a result of Choban [6] for completely regular spaces.

Theorem 2.10. *Let X and Y be two completely regular spaces such that each singleton is in $\mathcal{Z}_\alpha(X)$ and $\mathcal{Z}_\alpha(Y)$ respectively (resp. $\mathcal{P}_\alpha(X)$ and $\mathcal{P}_\alpha(Y)$) and E be a Banach space with the Banach-Stone property. If $\beta^\circ_\alpha(X, E)$ (resp. $B^\circ_\alpha(X, E)$) and $\beta^\circ_\alpha(Y, E)$ (resp. $B^\circ_\alpha(X, E)$) are isometric, then the compact spaces $b_\alpha X$ and $b_\alpha Y$ are homeomorphic.*

Proof. The proof is similar to that of the Theorem 3.2 in [29].

We establish now a Banach-Stone type theorem for the spaces of bounded continuous functions defined on completely regular Hausdorff spaces. Eilenberg [11] proved that for completely regular first countable Hausdorff topological spaces X and Y such that each singleton is $\mathcal{G}_\delta(X)$ and $\mathcal{G}_\delta(Y)$ respectively, if $C^\circ(X)$ and $C^\circ(Y)$ are linear isometric isomorphic, then X and Y are homeomorphic. We obtained the following similar result. Here βX denotes the Stone-Čech compactification of X.

Theorem 2.11. *Let X and Y be two completely regular topological spaces such that every singleton is in $\mathcal{Z}_1(X)$ and $\mathcal{Z}_1(Y)$ respectively. If T is a linear isometry from $C^\circ(X)$ onto $C^\circ(Y)$, then there is a homeomorphism imbedding φ from Y into βX and a continuous unimodular function h on Y such that*

$$Tf(y) = h(y)f(\varphi(y)), \quad y \in Y, \ f \in C^\circ(X).$$

Proof. By a result of McWilliams, $w_1^*(C^\circ(X)) = \beta_1^\circ(\beta X)$ (see [6]). Hence, the restriction of the linear isometry T^{**} to $w_1^*(C^\circ(X)) = \beta_1^\circ(\beta X)$ induces the linear isometry T^β from $\beta_1^\circ(\beta X)$ onto $\beta_1^\circ(\beta Y)$. Therefore, by Theorem 2.8, $T^\beta(f) = h\ (f \circ \tau)$, where $\tau : \beta Y \to \beta X$ is a Baire-1 homeomorphism and h is a unimodular element of $\beta_1^\circ(\beta Y)$. Note that the map T^{**} is $w^* - w^*$ continuous thus, its restriction T^β to $C(\beta X)$ is the bijection linear isometry Δ:

$$\Delta : C(\beta X) \to C(\beta Y) \qquad \text{defined as} \ \ \Delta(f) = h\ (f \circ \tau).$$

Obviously, $h = \Delta(1_{\beta X})$ is continuous on βY and therefore it is continuous on Y. Note that the restriction of Δ to $C^\circ(X)$ is T. Moreover, for each $f \in C(\beta X)$, $f \circ \tau = h\Delta(f)$ is continuous on βY. We claim that τ is continuous. If τ is not continuous, there exists a zero set $Z(f) \in \mathcal{Z}(\beta X)$ such that $\tau^{-1}(Z(f))$ is not closed in βY. But the set $(f \circ \tau)^{-1}(0) = \tau^{-1}(Z(f))$ must be closed in βY because $f \circ \tau$ is continuous and this is a contradiction. As $\tau : \beta Y \to \beta X$ is continuous, therefore $\phi = \tau|_Y$ is an embedding of Y in βX.

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Faculty of Khansar,
University of Isfahan,
81745-163, Isfahan, Iran
shatery@sci.ui.ac.ir

Department of Mathematics,
University of Isfahan,
81745-163, Isfahan, Iran
jzaf@math.ui.ac.ir