Non-isomorphism of some algebras of holomorphic functions

M. Nawrocki*

Abstract

Suppose that $\mathcal{X}$ is a family of spaces of holomorphic functions such that each $X = X(D) \in \mathcal{X}$ can be defined on a domain D belonging to some class $\mathcal{D}$ of domains. Then for any two concrete domains D_1 and $D_2 \in \mathcal{D}$ and $X \in \mathcal{X}$ one can ask the following natural question if corresponding spaces $X(D_1)$ and $X(D_2)$ are isomorphic as topological vector spaces. Similarly, for a fixed $D \in \mathcal{D}$ and two different spaces $X_1, X_2 \in \mathcal{X}$ one can consider the existence of an isomorphism between $X_1(D)$ and $X_2(D)$. We answer these questions when $\mathcal{X}$ consists of Hardy $N_*^p(D)$, maximal Hardy $MN_*^p(D)$, Bergman $\mathbb{N}^p(D)$, and Lumer's Hardy $LN_*^p(D)$ algebras, $p \geq 1$, and $\mathcal{D} = \{\mathbb{B}_n, \mathbb{U}^n, n \in \mathbb{N}\}$ is the family of the unit balls and the unit polydiscs in C^n.

1 Introduction

In the paper we use standard notation like in [R]. Moreover, we assume that $D = \mathbb{B}_{n_1} \times \ldots \times \mathbb{B}_{n_k}$ is the product of k open unit balls $\mathbb{B}_{n_j}$ in $\mathbb{C}^{n_j}$, $j = 1, \ldots, k$. In particular, if $k = n$ and $n_1 = \ldots n_k = 1$ than D is the unit polydisk in $\mathbb{C}^n$.

For each j, σ_j is the rotation–invariant probability Borel measure on the unit sphere $\mathbb{S}_{n_j}$ in $\mathbb{C}^{n_j}$. Moreover, let $S = \mathbb{S}_{n_1} \times \ldots \times \mathbb{S}_{n_k}$ and let $\sigma = \sigma_1 \otimes \ldots \otimes \sigma_k$ be the corresponding product measure on S.

Let us first recall definitions of spaces which are the subject of our consideration.

*This research was supported in part by Komitet Badań Naukowych (State Committee for Scientific Research), Poland, grant no. P03A 02225

1991 *Mathematics Subject Classification :* Primary 32A22, 32A35, 46A06; Secondary 32A05, 46A12.

Key words and phrases : Nevanlinna class, Smirnov class, Hardy algebra, Bergman algebra, Fréchet envelope, nuclear power series spaces.

The Hardy algebra $N_*^p(D)$, $p \geq 1$, is defined as the space of all holomorphic functions f on $D = \mathbb{B}_{n_1} \times \ldots \times \mathbb{B}_{n_k}$ such that

$$\|f\| = \sup_{(r_i)\in(0,1)^k} \left(\int_S \log^p(1 + |f(r_1\zeta_1, \ldots, r_k\zeta_k)|)\, d\sigma(\zeta_1, \ldots, \zeta_k) \right)^{1/p} < \infty .$$

For $p \geq 1$ the *maximal Hardy algebra* $MN_*^p(D)$ consists of all holomorphic functions on D such that

$$\|f\|^p = \int_S \log^p(1 + Mf(\zeta))\, d\sigma(\zeta) < \infty$$

where

$$Mf(\zeta) = \sup_{0<r<1} |f(r\zeta)| \quad \text{for} \quad \zeta \in S$$

is the maximal radial function of f. Contrary to the case of the classical Hardy space $H^p(D)$, which coincides with the corresponding space defined by the maximal function, $MN_*^p(D)$ is a proper subspace of $N_*^p(D)$.

Let A_i be the normalized Lebesgue measure on $\mathbb{B}_{n_i}$ and $A = A_1 \otimes \ldots \otimes A_k$ be the product measure on D. For $p \geq 1$, we define the *Bergman algebra* $\mathcal{N}^p(D)$ as the space of all holomorphic functions in D such that

$$\|f\|^p = \int_D \log^p(1 + |f(z)|)\, dA(z) < \infty .$$

We recall that a holomorphic function f on D belongs to the *Lumer's Hardy* algebra $LN^p(D)$ if $\log^p(1 + |f|) \leq u$ for some pluriharmonic function u on D. It is known that $LN^p(D)$ endowed with the metric $d(f, g) = \|f - g\|$, where

$$\|f\|^p = \inf\{u(0) : u \text{ pluriharmonic, } \log^p(1 + |f|) \leq u\}$$

is a topological group. For multi–dimensional domains D the space of polynomials $P(D)$ is not dense in $LN^p(D)$. We denote $LN_0^p(D)$ the closure of $P(D)$ in $LN^p(D)$. The group topology on $LN_0^p(D)$ is linear.

Let us recall, that if D is the unit disc in $\mathbb{C}$, then $LN_0^p(D)$ coincides with the standard Hardy algebra $N_*^p(D)$. For more information on Lumer-Hardy spaces see [R, N2, N3].

All the spaces $N_*^p(D)$, $MN_*^p(D)$, $\mathcal{N}^p(D)$, $LN_0^p(D)$, $p \geq 1$, equipped with the topology induced by the corresponding metric $d(f, g) = \|f - g\|$ are complete topological vector spaces (F-spaces). In the present note we are showing that these spaces are not isomorphic to each other.

Theorem. *(a) Let $D \in \{\mathbb{B}_n, \mathbb{U}^n, n \in \mathbb{N}\}$. Moreover, let $X^p(D)$ be one of the spaces $N_*^p(D)$, $MN_*^p(D)$, $\mathcal{N}^p(D)$, and $LN_0^p(D)$, $p \geq 1$, and $X^q(D)$ the corresponding space defined by the parameter $q \geq 1, q \neq p$. Then $X^p(D)$ is not isomorphic to $X^q(D)$.*

(b) Let $D \in \{\mathbb{B}_n, \mathbb{U}^n, n \in \mathbb{N}\}$ and $p \geq 1$. Then the spaces $N_*^p(D)$, $\mathcal{N}^p(D)$, and $LN_0^p(D)$ are pair-wisely non-isomorphic.

(c) Let $\mathcal{D}_n = \{\mathbb{B}_n, \mathbb{U}^n\}, n \in \mathbb{N}$, $D_n \in \mathcal{D}_n$, $D_m \in \mathcal{D}_m$. Moreover, let $X(D_n)$ be one of the spaces $N_*^p(D_n)$, $MN_*^p(D_n)$, $\mathcal{N}^p(D_m)$, $LN_0^p(D_m)$, and $X(D_m)$ the corresponding space defined on the domain D_m, $m \in \mathbb{N}, m \neq n$. Then $X(D_m)$ is not isomorphic to $X(D_n)$.

(d) Let $X(\mathbb{B}_n)$ be one of the spaces N_*^p, MN_*^p, $\mathcal{N}^p$, $n > 1$, defined on the unit ball $\mathbb{B}_n$, and $X(\mathbb{U}^n)$ the corresponding space defined on the unit polydisc $\mathbb{U}^n$. Then $X(\mathbb{B}_m)$ is not isomorphic to $X(\mathbb{U}^n)$.

2 Proof of Theorem

All the spaces $N_*^p(D)$, $MN_*^p(D)$, $\mathcal{N}^p(D)$, $LN_0^p(D)$, $p \geq 1$, are not locally convex. However, the corresponding locally convex structures of these spaces play a crucial role in the study of their isomorphisms.

Let us recall that if $X = (X, \tau)$ is an F–space whose topological dual X' separates the points of X, then its *Fréchet envelope* $\widehat{X}$ is defined to be the completion of the space (X, τ^c), where τ^c is the strongest locally convex topology on X which is weaker than τ. If $\mathcal{U}$ is a base of neighborhoods of zero for τ, then the family $\{coU : U \in \mathcal{U}\}$ of convex hulls is a base of neighborhoods of zero for τ^c. This immediately implies the following lemma:

Lemma 1. *If two F-spaces $X_j, j = 1, 2$, are isomorphic, then their Fréchet envelopes $\widehat{X}_j, j = 1, 2$, are also isomorphic.*

It turns out that the Fréchet envelopes of the spaces $N_*^p(D)$, $MN_*^p(D)$, $\mathcal{N}^p(D)$, $LN_0^p(D)$, $p \geq 1$, can be identify with appropriate weighted space of holomorphic functions.

Let $(s) = (s_1, \ldots, s_k)$ be a fixed sequence of positive numbers. For each holomorphic function f on $D = \mathbb{B}_{n_1} \times \ldots \times \mathbb{B}_{n_k}$ and $m \in \mathbb{N}$ we define

$$\| f \|_m = \sup_{(z_i) \in D} |f(z_1, \ldots, z_k)| \exp\left(- \prod_{i=1}^{k} (1 - |z_i|)^{-s_j}/m \right).$$

The *weighted space* $F_{(s)}(D)$ consists of all holomorphic functions f on D such that $\| f \|_m < \infty$ for each $m \in \mathbb{N}$. If for a fixed number $s > 0$ and $m \in \mathbb{N}$ we define

$$\| f \|_m = \sup_{(z_i) \in D} |f(z_1, \ldots, z_k)| \exp\left(- \left(1 - \max_{i=1,\ldots,k} |z_i| \right)^{-s} /m \right),$$

then we get another *weighted space* $LF_s(D)$ of holomorphic functions on D.

Lemma 2. *Let $D = \mathbb{B}_{n_1} \times \ldots \times \mathbb{B}_{n_k}$.*

(a) The Fréchet envelopes of $N_*^p(D)$ and $MN_*^p(D)$ are isomorphic to $F_{(s)}(D)$ where $s = (s_j)$, $s_j = n_j/p, j = 1, \ldots, k$ (cf. [N5, Theorem 5.1, Theorem 6.2]).

(b) The Fréchet envelope of $\mathcal{N}^p(D)$ is isomorphic to $F_{(s)}(D)$ where $s = (s_j)$, $s_j = (n_j + 1)/p, j = 1, \ldots, k$ (cf. [N5, Theorem 7.1]).

(c) The Fréchet envelope of $LN_0^p(D)$ is isomorphic to $LF_s(D)$ where $s = 1/p$ (cf. [N5, Theorem 9.1]).

The above lemma suggests that one can try to distinguish F-spaces by looking for a topological vector invariant in the class of Fréchet spaces. In our case the Λ-nuclearity type is the suitable one.

Let E be a Fréchet space and let $\mathcal{U}$ be a base of neighbourhoods of zero in E. For every $U, V \in \mathcal{U}$, $U \supseteq V$, and $j \in \mathbb{N}$, the j-th Kolmogorov diameter of V with respect to U is defined by

$$\delta_j(V,U) = \inf\{\delta(V,U,F) : F \text{ is a linear subspace of } E, \dim F \le j\},$$

where $\delta(V,U,F) = \inf\{\delta > 0 : V \subseteq \delta U + F\}$.

Let us suppose that $\rho = \{\rho_j\}$ is a given non-decreasing sequence of positive numbers. Then a Fréchet space E is said to be $\Lambda_1(\rho)$-nuclear if for every $U \in \mathcal{U}$ there are $V \in \mathcal{U}$ and $R > 1$ such that $\lim_j R^{\rho_j}\delta_j(V,U) = 0$.

The power series space $\Lambda_1(\rho)$ consisting of all complex sequences $x = \{x_j\}$ such that

$$\| x \|_m = \sup_j |x_j| \exp(-\rho_j/m) < \infty \quad \text{for all } m \in \mathbb{N}$$

is a standard Fréchet space which is $\Lambda_1(\rho)$-nuclear. It is well known that a power series space $\Lambda_1(\rho')$ is $\Lambda_1(\rho)$-nuclear if and only if $\sup_j \rho'_j/\rho_j < \infty$ (see [RO, Proposition 3.4]).

Lemma 3. *Let $p > 0$, $n \in \mathbb{N}$.*

(a) $F_{n/p}(\mathbb{B}_n)$ is isomorphic to the power series space $\Lambda_1(j^{1/(p+n)})$.

(b) $F_{(s)}(\mathbb{U}^n)$, where $s = (s_j), s_j = 1/p, j = 1,\dots,n$, is $\Lambda_1(j^c)$-nuclear for each $c < 1/(p+n)$ but is not $\Lambda_1(j^{1/(p+n)})$-nuclear if $n > 1$.

(c) The spaces $LF_{1/p}(\mathbb{B}_n)$, $LF_{1/p}(\mathbb{U}^n)$ are isomorphic to the power series space $\Lambda_1(j^{1/(np+n)})$.

Proof. (a). The weighted space $F_{n/p}(\mathbb{B}_n)$ is isomorphic to the nuclear power series space (Köthe space) consisting of all sequences $x = \{x(\alpha)) : \alpha \in \mathbb{Z}_+^n\}$ such that

$$\| x \|_m = \sup_\alpha |x(\alpha)| \exp(-|\alpha|^{n/(p+n)}/m) < \infty$$

for each $m \in \mathbb{N}$ (see [N4, Corollary 1]). If we rearrange $\mathbb{Z}_+^n$ in a sequence $(\rho_j) = (\rho_{j(\alpha)})$ such that $\rho_{j(\alpha)} \le \rho_{j(\alpha')}$ if $|\alpha| \le |\alpha'|$, then $\rho_j \sim j^{1/n}$ (cf. [RO, p. 362]). Thus $F_{n/p}(\mathbb{B}_n)$ is isomorphic to $\Lambda_1(j^{1/(p+n)})$.

(b). This assertion follows from [N4, Corrolary 3, Theorem 3] and from (a).

(c) By [N5, Theorem 8.1] the spaces $LF_{1/p}(\mathbb{B}_n)$, $LF_{1/p}(\mathbb{U}^n)$ are isomorphic to the Köthe space consisting of all sequences $x = \{x(\alpha)) : \alpha \in \mathbb{Z}_+^n\}$ such that

$$\| x \|_m = \sup_\alpha |x(\alpha)| \exp(-|\alpha|^{1/(p+1)}/m) < \infty$$

for each $m \in \mathbb{N}$. However, this space is isomorphic to $\Lambda_1(j^{1/(np+n)})$ (cf. (a)). ∎

Lemma 4. *(a)* $\widehat{N_*^p(\mathbb{B}_n)}$ *is isomorphic to* $\Lambda_1(j^c)$ *where* $c = 1/(p+n)$.

(b) $\widehat{N_*^p(\mathbb{U}^n)}$ *is not* $\Lambda_1(j^{1/(p+n)})$*-nuclear if* $n > 1$*, but it is* $\Lambda_1(j^c)$*-nuclear for any* $c < 1/(p+n)$.

(c) $\widehat{\mathcal{N}_*^p(\mathbb{B}_n)}$ *is isomorphic to* $\Lambda_1(j^c)$*, where* $c = 1/(\frac{np}{n+1} + n)$.

(d) $\widehat{\mathcal{N}_*^p(\mathbb{U}^n)}$ *is not* $\Lambda_1(j^{1/(\frac{p}{2}+n)})$*-nuclear if* $n > 1$*, but it is* $\Lambda_1(j^c)$*-nuclear for any* $c < 1/(\frac{p}{2} + n)$.

(e) $\widehat{LN_0^p(\mathbb{B}_n)}$ *and* $\widehat{LN_0^p(\mathbb{U}^n)}$ *are isomorphic to* $\Lambda_1(j^{1/(np+n)})$.

Proof. (a), (b) and (e) immediately follow from Lemma 2 and Lemma 3. For the proof of (c) it is enough to apply Lemma 2 (b) and observe that $\widehat{\mathcal{N}_*^p(\mathbb{B}_n)} = F_{((n+1)/p)}(\mathbb{B}_n) = F_{(n/q)}(\mathbb{B}_n)$, where $q = np/(n+1)$. Now, (c) is a consequence of Lemma 3 (a). The assertion (d) follows from Lemma 2 (b) and Lemma 3 (b), since $\widehat{\mathcal{N}_*^p(\mathbb{U}^n)} = F_{(2/p,\ldots,2/p)}(\mathbb{U}^n) = F_{(1/q,\ldots,1/q)}(\mathbb{B}_n)$, where $q = p/2$. ∎

Now the proof of the Theorem follows from Lemma 1 and Lemma 4.

Remark: Since the Fréchet envelopes of $N_*^p(D)$ and $MN_*^p(D)$ coincide, in Theorem (b) one can replace $N_*^p(D)$ by $MN_*^p(D)$.

Open problems: 1. Is $N_*^p(D)$ isomorphic to $MN_*^p(D)$?
The answer is not known even in the one dimensional case, i.e. if D is the unit disk in the plane.
2. Is $LN_0^p(\mathbb{B}_n)$ isomorphic to $LN_0^p(\mathbb{U}^n)$ if $n > 1$?

Bibliography

[HJ] A. Haldimann, H. Jarchow, *Nevanlinna Algebras*, Studia Math. **147** (2001), 211-235;

[KPR] N.J. Kalton, N.T. Peck, and J.W.Roberts, *F-spaces sampler*, London Math. Soc. Lecture Notes. **89** (1984);

[JMWX] H. Jarchow, V. Montesinos, K. J. Wirths, J. Xiao, *Duality for some laree spaces of analytic functions*, Proc. Edinb. Math. Soc. **44** (2001), 571–583;

[N1] M. Nawrocki, *The non-isomorphism of the Smirnov Classes of different balls and polydiscs*, Bull. Soc. Math. Belg. **41 (3, ser. B)** (1989), 307-315;

[N2] M. Nawrocki, *Gleason-type decompositions for* $H^\infty(\mathbb{B}_n)$ *and Lumer's ardy algebra of the ball*, Annales Acadeniae Scientiarum Fennicae **16 (2)** (1991), 311–322;

[N3] M. Nawrocki, *Linear topological properties of the Lumer-Smirnov class of the polydisc*, Studia Math. **102 (1)** (1992), 87–102;

[N4] M. Nawrocki, *Köthe sequence space representations of some weighted algebras of holomorphic functions, in Functional Analysis, Proceedings of the First International Workshop held at Trier University, Germany*, W. De Gruyter Berlin - New York, 1996;

[N5] M. Nawrocki, *Locally convex structure of some algebras of holomorphic functions of several variables*, Math. Nachr **246-247** (2002), 170–187;

[RO] Rolewicz, S., *Metric linear spaces.*, PWN-Polish Scientific Publishers, Warszawa; D. Reidel Publishing Company, Dordrecht-Boston-Lancaster, 1984.;

[R] W. Rudin, *Function Theory in the Unit Ball of* $\mathbb{C}^n$, Springer-Verlag, New York, Heidelberg, Berlin, 1980;

[S] J. H. Shapiro, *Mackey topologies, reproducing kernels, and diagonal maps on Hardy and Bergman spaces*, Duke Math. J. **43** (1976), 187–202;

Faculty of Mathematics and Informatics,
A. Mickiewicz University,
ul. Umultowska 87, Poznań, Poland
email: nawrocki@math.amu.edu.pl