# One-sided interpolation of injective tensor products of Banach spaces

Carsten Michels

*To Professor Jean Schmets on the occasion of his 65th birthday*

**Abstract**

The Kouba formula for the complex interpolation of injective tensor products requires all spaces involved to be at least of cotype 2. We show that this can be weakened when one side of the tensor products is fixed.

## 1 Introduction

Let $(E_0, E_1)$ and $(F_0, F_1)$ be regular compatible couples of Banach spaces. Kouba in [13] proved the equality

$$[E_0 \tilde{\otimes}_\varepsilon F_0, E_1 \tilde{\otimes}_\varepsilon F_1]_\theta = [E_0, F_1]_\theta \tilde{\otimes}_\varepsilon [E_0, F_1]_\theta$$

for all $0 < \theta < 1$ whenever one of the following holds:

(i) $E_0', E_1', F_0', F_1'$ are type 2 spaces;

(ii) $E_0', E_1'$ are type 2 spaces and $F_0, F_1$ are 2-concave Banach function spaces;

(iii) $E_0, E_1, F_0, F_1$ are 2-concave Banach function spaces.

Since the dual of a type 2 space has cotype 2 (see, e.g., [12, 11.10]) and a 2-concave Banach function also has cotype 2 (see, e.g., [15, 1.f.16]), all spaces involved in the above have cotype 2. This is also the case in more recent extensions of Kouba's result (see, e.g., [7, 8, 9, 10, 11]).

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However, cotype 2 is not necessary: Using the well-known complex interpolation formula for vector-valued $\ell_\infty^n$'s, one can deduce that the one-sided interpolation formula

$$[c_0\tilde{\otimes}_\varepsilon E_0, c_0\tilde{\otimes}_\varepsilon E_1]_\theta = c_0\tilde{\otimes}_\varepsilon [E_0, E_1]_\theta$$

holds for all regular compatible Banach couples $(E_0, E_1)$, where $c_0$ denotes the space of all null sequences. In this short note we show that in the above, $c_0$ can be substituted by a Banach space of cotype $q$, $2 < q < \infty$, whenever $(E_0, E_1)$ is a compatible couple of $p$-concave Banach function spaces, $1 \le p < q'$.

For $1 \le p \le \infty$, let $p'$ be defined by $1/p + 1/p' = 1$. We refer to [5, 12, 15] for all notions and notations needed within Banach space theory, and to [1] for the basic concepts of interpolation theory of Banach spaces. For a Banach space $E$, we write $E'$ for its topological dual. $E \otimes_\varepsilon F$ denotes the injective tensor product of two Banach spaces $E, F$, and $E\tilde{\otimes}_\varepsilon F$ its completion. $\mathbf{C_q}(E)$ stands for the cotype $q$ constant of a Banach space $E$, $2 \le q < \infty$, and $\mathbf{M_{(p)}}(X)$ for the $p$-concavity constant of a Banach function space $X$, $1 \le p < \infty$. A compatible couple $(E_0, E_1)$ of Banach spaces is called regular if $E_0 \cap E_1$ is dense in both $E_0$ and $E_1$. With $[E_0, E_1]_\theta$ for $0 < \theta < 1$ we denote the complex interpolation space associated to such a couple. By an $n$-dimensional lattice $E^n$ we mean the vector space $\mathbb{K}^n$ equipped with some lattice norm. If $E^n$ and $F^n$ are $n$-dimensional lattices, we denote by $M(E^n, F^n)$ the space of all multiplication operators from $E^n$ to $F^n$, endowed with the norm induced by $\mathcal{L}(E^n, F^n)$. Here, for two Banach spaces $E$ and $F$ we mean $\mathcal{L}(E, F)$ to be the space of all bounded and linear operators from $E$ to $F$ endowed with the usual operator norm.

## 2 The result

The following variant of a factorization theorem due to Maurey and Rosenthal was shown in [4, 4.2]. Recall that for $1 \le p < \infty$ the identity map $\mathrm{id}_U$ on a Banach space $U$ is called $(p, 1)$-mixing if every $p$-summing operator on $U$ is 1-summing. We then denote by $\mu_{p,1}(U)$ the $(p, 1)$-mixing norm of $\mathrm{id}_U$. For $p = 2$, basic examples are Banach spaces of cotype 2 (see, e.g., [5, 32.2]), and for $1 < p < 2$, the identity map on a Banach space of cotype $p'$ is $(r, 1)$-mixing for all $r < p$. In particular, $\mathrm{id}_{\ell_p}$ is $(2, 1)$-mixing whenever $1 \le p \le 2$, and if $2 < p < \infty$, it is $(r, 1)$-mixing for all $r < p'$. Note that by [17, 20.1.17] there exists no infinite-dimensional Banach space $U$ such that $\mathrm{id}_U$ is $(p, 1)$-mixing whenever $p > 2$.

**Proposition 1.** *For $1 \le p < \infty$ let $U$ be a Banach space such that the identity map* $\mathrm{id} : U' \to U'$ *is $(p, 1)$-mixing. Then there exists a universal constant $C > 0$ such that for any $p$-concave Banach function space $X(\mu)$ every bounded operator $T : U \to X$ factorizes as follows:*

$$\begin{tikzcd} U \arrow[rr, "T"] \arrow[dr, "R"'] & & X(\mu)\ , \\ & L_p(\mu) \arrow[ur, "M_g"'] & \end{tikzcd}$$

*where $\|R\|\,\|M_g\| \le C\,\mu_{p,1}(U')\,\mathbf{M_{(p)}}(X)\,\|T\|$ and $M_g$ a positive multiplication operator.*

**Lemma 1.** *For $1 \le p < \infty$ let $U$ be a Banach space such that $\mathrm{id}_{U'}$ is $(p,1)$-mixing and $E^n$ be an $n$-dimensional lattice. Then for any couple $(E_0^n, E_1^n)$ of $n$-dimensional lattices and any exact interpolation functor $\mathcal{F}$, it holds*

$$\begin{aligned}&\|\mathrm{id} : \mathcal{L}(U, E^n) \hookrightarrow \mathcal{F}(\mathcal{L}(U, E_0^n), \mathcal{L}(U, E_1^n))\| \\ &\qquad \le C\, \mu_{p,1}(U')\, \mathbf{M}_{(\mathbf{p})}(E^n)\, \|M(\ell_p^n, E^n) \hookrightarrow \mathcal{F}(M(\ell_p^n, E_0^n), M(\ell_p^n, E_1^n))\|,\end{aligned}$$

*where $C > 0$ is a universal constant.*

*Proof.* According to Proposition 1 there exists $C > 0$ such that for any $T \in \mathcal{L}(U, E^n)$ there are $R : U \to \ell_p^n$ and $\lambda \in \mathbb{K}^n$ such that $T = M_\lambda R$ and $\|R\|\, \|\lambda\|_{M(\ell_p^n, E^n)} \le C\, \mu_{p,1}(U')\mathbf{M}_{(\mathbf{p})}(E^n)\, \|T\|$. Obviously the map $\Psi$ defined by

$$\Psi(\mu) := M_\mu R, \quad \mu \in \mathbb{K}^n,$$

maps the couple $(M(\ell_p^n, E_0^n), M(\ell_p^n, E_0^n))$ into the couple $(\mathcal{L}(U, E_0^n), \mathcal{L}(U, E_1^n))$ such that both restrictions have norm less or equal to $\|R\|$. Hence, by the interpolation property the map

$$\Psi : M(\ell_p^n, E^n) \to \mathcal{F}(\mathcal{L}(U, E_0^n), \mathcal{L}(U, E_1^n))$$

has norm less than or equal to $\|R\|\, \|M(\ell_p^n, E^n) \hookrightarrow \mathcal{F}(M(\ell_p^n, E_0^n), M(\ell_p^n, E_1^n))\|$. Thus we obtain

$$\begin{aligned}\|T\|_{\mathcal{F}(\mathcal{L}(U,E_0^n),\mathcal{L}(U,E_1^n))} &= \|M_\lambda R\|_{\mathcal{F}(\mathcal{L}(U,E_0^n),\mathcal{L}(U,E_1^n))} \\ &\le \|R\|\, \|\lambda\|_{M(\ell_p^n,E^n)}\, \|M(\ell_p^n, E^n) \hookrightarrow \mathcal{F}(M(\ell_p^n, E_0^n), M(\ell_p^n, E_1^n))\| \\ &\le C(p, U, E^n)\, \|T\|_{\mathcal{L}(U,E^n)} \|M(\ell_p^n, E^n) \hookrightarrow \mathcal{F}(M(\ell_p^n, E_0^n), M(\ell_p^n, E_1^n))\|,\end{aligned}$$

where $C(p, U, E^n) := C\, \mu_{p,1}(U')\mathbf{M}_{(\mathbf{p})}(E^n)$. ∎

**Theorem 1.** *Let $2 < q < \infty$ and $1 \le p < q'$. Then for any compatible couple $(X_0(\mu), X_1(\mu))$ of $p$-concave Banach function spaces and any Banach space $F$ of cotype $q$, it holds*

$$[F \tilde{\otimes}_\varepsilon X_0, F \tilde{\otimes}_\varepsilon X_1]_\theta = F \tilde{\otimes}_\varepsilon [X_0, X_1]_\theta$$

*for all $0 < \theta < 1$.*

*Proof.* Let $U$ be a finite-dimensional subspace of $F$. Then by [5, 32.2] there exists a universal constant $C_0 > 0$ such that

$$\mu_{p,1}((U')') = \mu_{p,1}(U) \le C_0\, \mathbf{C_q}(U) \le C_0\, \mathbf{C_q}(F).$$

Now let $E_0^n$ and $E_1^n$ be $n$-dimensional lattices with $\mathbf{M}_{(\mathbf{p})}(E_0^n) = \mathbf{M}_{(\mathbf{p})}(E_1^n) = 1$. Then by [18, p. 218/219] it is $\mathbf{M}_{(\mathbf{p})}([E_0^n, E_1^n]_\theta) = 1$. Furthermore, by [6, 3.5] and [11, Lemma 4] it holds

$$M(\ell_p^n, [E_0^n, E_1^n]_\theta) = [M(\ell_p^n, E_0^n), M(\ell_p^n, E_1^n)]_\theta$$

isometrically, which together with the above lemma gives

$$\|\mathrm{id} : \mathcal{L}(U', [E_0^n, E_1^n]_\theta) \hookrightarrow [\mathcal{L}(U', E_0^n), \mathcal{L}(U', E_1^n)]_\theta\| \le C\, \mu_{p,1}(F),$$

where $C > 0$ is a universal constant. Equivalently, this means that

$$\|\mathrm{id} : U \otimes_\varepsilon [E_0^n, E_1^n]_\theta \hookrightarrow [U \otimes_\varepsilon E_0^n, U \otimes_\varepsilon E_1^n]_\theta\| \le C\, \mu_{p,1}(F).$$

Now proceed as in [11] to obtain the infinite-dimensional case. ∎

An application of the following example can be found in [16].

**Example 1.** Let $2 < q < \infty$ and $1 \le p_0, p_1 < q'$. Then for $0 < \theta < 1$

$$[\ell_q \tilde{\otimes}_\varepsilon \ell_{p_0}, \ell_q \tilde{\otimes}_\varepsilon \ell_{p_1}]_\theta = \ell_q \tilde{\otimes}_\varepsilon \ell_p,$$

where $1/p = (1-\theta)/p_0 + \theta/p_1$.

*Proof.* It is well-known (see, e.g., [15]) that $\ell_q$ has cotype $q$, and that $\ell_{p_0}$ and $\ell_{p_1}$ are $\max(p_0, p_1)$-concave. Furthermore, $[\ell_{p_0}, \ell_{p_1}]_\theta = \ell_p$ (see, e.g., [1]). Thus, the above theorem applies. ■

## 3 Some counterexamples

We conclude with some counterexamples that show that the conditions in the above cannot be essentially weakened. For this purpose, we need a refinement of [10, Proposition 7.1] – its proof goes along the same lines as in the mentioned article, and we leave the details to the reader. For $1 \le p \le s \le \infty$ and an $(s,p)$-mixing (respectively, $(s,p)$-summing) operator $T$, its $(s,p)$-mixing (respectively, $(s,p)$-summing) norm is denoted by $\mu_{s,p}(T)$ (respectively, by $\pi_{s,p}(T)$). A couple $(E_0, E_1)$ is called a finite-dimensional interpolation couple if $E_i = (\mathbb{K}^n, \|\cdot\|_i)$, $i = 0, 1$, for some (finite) dimension $n$.

**Lemma 2.** *Let* $1 \le p_i \le s_i \le \infty$, $i = 0, 1$, *and* $0 < \theta < 1$. *Then for* $s_\theta$ *and* $p_\theta$ *defined by* $1/s_\theta = (1-\theta)/s_0 + \theta/s_1$ *and* $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$, *respectively, two finite-dimensional interpolation couples* $(E_0, E_1)$, $(F_0, F_1)$ *and each* $T : (E_0, F_0) \to (F_0, F_1)$, *it holds*

$$\begin{aligned}&\mu_{s_\theta,p_\theta}(T : [E_0, E_1]_\theta \to [F_0, F_1]_\theta) \\ &\qquad \le d_\theta[p_0, p_1, E_0, E_1]\, \mu_{s_0,p_0}(T : E_0 \to F_0)^{1-\theta} \mu_{s_1,p_1}(T : E_1 \to F_1)^\theta\end{aligned}$$

*and*

$$\begin{aligned}&\pi_{s_\theta,p_\theta}(T : [E_0, E_1]_\theta \to [F_0, F_1]_\theta) \\ &\qquad \le d_\theta[p_0, p_1, E_0, E_1]\, \pi_{s_0,p_0}(T : E_0 \to F_0)^{1-\theta} \pi_{s_1,p_1}(T : E_1 \to F_1)^\theta,\end{aligned}$$

*where*

$$d_\theta[p_0, p_1, E_0, E_1] := \|\mathrm{id} : \ell_{p_\theta} \otimes_\varepsilon [E_0, E_1]_\theta \to [\ell_{p_0} \otimes_\varepsilon E_0, \ell_{p_1} \otimes_\varepsilon E_1]_\theta\|.$$

**Example 2.** Let $2 < q < \infty$ and $1 \le p_0 < q' \le p_1 \le 2$. Then for all $0 < \theta < 1$

$$[\ell_q \tilde{\otimes}_\varepsilon \ell_{p_0}, \ell_q \tilde{\otimes}_\varepsilon \ell_{p_1}]_\theta \ne \ell_q \tilde{\otimes}_\varepsilon [\ell_{p_0}, \ell_{p_1}]_\theta.$$

*Proof.* The idea is to show that $d_\theta[p_0, p_1, \ell_q^n, \ell_q^n]$ is unbounded when viewed as a function in $n$. We have to treat two cases.

(i) Let $p_1 = q'$. Consider the identity map $\mathrm{id}_q^n : \ell_q^n \hookrightarrow \ell_q^n$. By [2] it is known that

$$\mu_{q',p_0}(\mathrm{id}_q^n) \asymp (1 + \log n)^{1/q'}.$$

Trivially, since any bounded linear operator is $(q', q')$-mixing, we have $\mu_{q',q'}(\mathrm{id}_q^n) \asymp 1$. Thus, the above lemma applied to $s_0 = s_1 = q'$, $p_0 = p_0$, $p_1 = q'$ and $E_0 = E_1 = F_0 = F_1 = \ell_q^n$ gives

$$(1+\log n)^{1/q'} \asymp \mu_{q',p_\theta}(\mathrm{id}_q^n) \prec d_\theta[p, q', \ell_q^n, \ell_q^n]\, \mu_{q',p_0}(\mathrm{id}_q^n)^{1-\theta} \asymp d_\theta[p_0, q', \ell_q^n, \ell_q^n]\,(1+\log n)^{(1-\theta)/q'},$$

where $1/p_\theta = (1-\theta)/p_0 + \theta/q'$. Hence,

$$d_\theta[p_0, q', \ell_q^n, \ell_q^n] \succ (1+\log n)^{\theta/q'}.$$

(ii) Let $p_0 < q' < p_1$. Then by [17, p. 312] we know that

$$\pi_{p_0,p_0}(\mathrm{id} : \ell_q^n \hookrightarrow \ell_\infty^n) \asymp n^{1/q'} \quad \text{and} \quad \pi_{p_1,p_1}(\mathrm{id} : \ell_q^n \hookrightarrow \ell_\infty^n) \asymp n^{1/p_1}.$$

Thus, the above lemma gives

$$\pi_{p_\theta,p_\theta}(\mathrm{id} : \ell_q^n \hookrightarrow \ell_\infty^n) \prec d_\theta[p_0, p_1, \ell_q^n, \ell_q^n]\, n^{(1-\theta)/q' + \theta/p_1},$$

where $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$. Now if $p_\theta \le q'$, then again by [17, p. 312] it is

$$\pi_{p_\theta,p_\theta}(\mathrm{id} : \ell_q^n \hookrightarrow \ell_\infty^n) \asymp n^{1/q'}.$$

This gives

$$d_\theta[p_0, p_1, \ell_q^n, \ell_q^n] \succ n^{\theta(1/q' - 1/p_1)}.$$

If $p_\theta > q'$, then

$$\pi_{p_\theta,p_\theta}(\mathrm{id} : \ell_q^n \hookrightarrow \ell_\infty^n) \asymp n^{1/p_\theta},$$

which gives

$$d_\theta[p_0, p_1, \ell_q^n, \ell_q^n] \succ n^{(1-\theta)(1/p_0 - 1/q')}.$$

This gives the claim, since $p_0 < q' < p_1$. ∎

A similar strategy can be used to give the following list of counterexamples in the spirit of the interpolation formulas contained in the article of Kouba – recall that equality holds whenever all indices involved are less than or equal to 2. Kouba himself only provided a very particular counterexample: the case $q_0 = q_1 = 2$ and $p_0 = 2$, $p_1 = \infty$. Le Merdy in [14] actually showed that $[\ell_1 \tilde{\otimes}_\varepsilon \ell_2, \ell_\infty \tilde{\otimes}_\varepsilon \ell_2]_{\frac{1}{2}} = \mathcal{S}_4$ (the latter denoting the 4th Schatten class).

**Example 3.** Let $1 \le q_0, q_1 < \infty$ and $1 \le p_0 < p_1 \le \infty$ be such that one of the following holds:

(i) $1 \le p_0 < 2 < p_1 \le \infty$;

(ii) $1 \le q_0, q_1 \le 2$ and $2 \le p_0 < p_1 \le \infty$;

(iii) $2 \le q_1 \le q_0 < \infty$ and $2 \le p_0 < p_1 \le \infty$.

Then for all $0 < \theta < 1$

$$[\ell_{q_0} \tilde{\otimes}_\varepsilon \ell_{p_0}, \ell_{q_1} \tilde{\otimes}_\varepsilon \ell_{p_1}]_\theta \ne [\ell_{q_0}, \ell_{q_1}]_\theta \tilde{\otimes}_\varepsilon [\ell_{p_0}, \ell_{p_1}]_\theta.$$

*Proof.* We consider again finite-dimensional identity maps, this time with range space $\ell_1^n$, to show that $d_\theta[q_0, q_1, \ell_{p_0}^n, \ell_{p_1}^n]$ is not bounded. In the following, $p_\theta$ and $q_\theta$ are defined by $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$ and $1/q_\theta = (1-\theta)/q_0 + \theta/q_1$.

(i) Under the given assumption we know again by [17, p. 312] that

$$\pi_{q_0,q_0}(\mathrm{id} : \ell_{p_0}^n \hookrightarrow \ell_1^n) \asymp n^{1-1/p_0+1/2} \quad \text{and} \quad \pi_{q_1,q_1}(\mathrm{id} : \ell_{p_1}^n \hookrightarrow \ell_1^n) \asymp n^1.$$

Thus, the above lemma gives

$$\pi_{q_\theta,q_\theta}(\mathrm{id} : \ell_{p_\theta}^n \hookrightarrow \ell_1^n) \prec d_\theta[q_0, q_1, \ell_{p_0}^n, \ell_{p_1}^n]\, n^{1-(1-\theta)/p_0+(1-\theta)/2}.$$

However, if $p_\theta \le 2$, then

$$\pi_{q_\theta,q_\theta}(\mathrm{id} : \ell_{p_\theta}^n \hookrightarrow \ell_1^n) \asymp n^{1-1/p_\theta+1/2},$$

which gives

$$d_\theta[q_0, q_1, \ell_{p_0}^n, \ell_{p_1}^n] \succ \theta(1/2 - 1/p_1).$$

If $p_\theta > 2$, then

$$\pi_{q_\theta,q_\theta}(\mathrm{id} : \ell_{p_\theta}^n \hookrightarrow \ell_1^n) \asymp n^1,$$

which gives

$$d_\theta[q_0, q_1, \ell_{p_0}^n, \ell_{p_1}^n] \succ (1-\theta)(1/p_0 - 1/2).$$

(ii) Again by [17] we know that

$$\pi_{q_1,q_1}(\mathrm{id} : \ell_{p_1}^n \hookrightarrow \ell_1^n) = \pi_{q_1}(\mathrm{id} : \ell_{p_1}^n \hookrightarrow \ell_1^n) \asymp n^1,$$

and that for $r$ defined by $1/r = 1/q_0 - 1/2$

$$\pi_{r,q_0}(\mathrm{id} : \ell_{p_0}^n \hookrightarrow \ell_1^n) \asymp \|\mathrm{id} : \ell_{p_0}^n \hookrightarrow \ell_1^n\| \asymp n^{1-1/p_0}.$$

Thus, the above lemma with $r_\theta$ defined by $1/r_\theta = (1-\theta)/r + \theta/q_1$ gives

$$\pi_{r_\theta,q_\theta}(\mathrm{id} : \ell_{p_\theta}^n \hookrightarrow \ell_1^n) \prec d_\theta[q_0, q_1, \ell_{p_0}^n, \ell_{p_1}^n]\, n^{1-(1-\theta)/p_0}.$$

By [3] and after some elementary calculations,

$$\pi_{r_\theta,q_\theta}(\mathrm{id} : \ell_{p_\theta}^n \hookrightarrow \ell_1^n) \asymp n^{1-(1-\theta)/p_0+\theta(1-\theta)(1/p_0-1/p_1)},$$

which gives

$$d_\theta[q_0, q_1, \ell_{p_0}^n, \ell_{p_1}^n] \succ n^{\theta(1-\theta)(1/p_0-1/p_1)}.$$

(iii) We use again that

$$\pi_{q_1,q_1}(\mathrm{id} : \ell_{p_1}^n \hookrightarrow \ell_1^n) = \pi_q(\mathrm{id} : \ell_{p_1}^n \hookrightarrow \ell_1^n) \asymp n^1,$$

and that

$$\pi_{\infty,q_0}(\mathrm{id} : \ell_{p_0}^n \hookrightarrow \ell_1^n) = \|\mathrm{id} : \ell_{p_0}^n \hookrightarrow \ell_1^n\| \asymp n^{1-1/p_0}.$$

Thus, the above lemma gives

$$\pi_{q_1/\theta,q_\theta}(\mathrm{id} : \ell_{p_\theta}^n \hookrightarrow \ell_1^n) \prec d_\theta[q_0, q_1, \ell_{p_0}^n, \ell_{p_1}^n]\, n^{1-(1-\theta)/p_0}.$$

Again by [3]

$$\pi_{q_1/\theta,q_\theta}(\mathrm{id} : \ell_{p_\theta}^n \hookrightarrow \ell_1^n) \asymp n^{1-(1-\theta)/p_0+\theta(q_\theta/q_1p_\theta-1/p_1)},$$

which gives

$$d_\theta[q_0, q_1, \ell_{p_0}^n, \ell_{p_1}^n] \succ n^{\theta(q_\theta/q_1p_\theta-1/p_1)}.$$

The latter is unbounded since $q_\theta \ge q_1$ and $p_1 > p_\theta$. ∎

**Remark 1.** 1. Note that our lower estimates for $d_\theta[q_0, q_1, \ell^n_{p_0}, \ell^n_{p_1}]$ in the above proof for the case $1 \le q_0, q_1 \le 2$ and $p_\theta \le 2$ are asymptotically exact: together with the corresponding upper estimates given in [13] we obtain, e.g., that

$$d_\theta[q_0, q_1, \ell^n_1, \ell^n_\infty] = \|\mathrm{id} : [\ell_{q_0}, \ell_{q_1}]_\theta \otimes_\varepsilon [\ell^n_1, \ell^n_\infty]_\theta \to [\ell_{q_0} \otimes_\varepsilon \ell^n_1, \ell_{q_1} \otimes_\varepsilon \ell^n_\infty]_\theta\| \asymp n^{\frac{\theta}{2}}$$

for all $0 < \theta \le 1/2$.

2. In the above example, (ii) also shows that our one-sided interpolation formula cannot be generalized to a two-sided one (a logical choice of indices would have been $2 \le q_0 < q_1 < \infty$ and $1 \le p_0, p_1 < q_1'$).

3. Still left open are the following cases – we conjecture that again the corresponding formulas do not hold, for all $0 < \theta < 1$:

   - $2 < q_0 = q_1 < \infty$ and $q_0' \le p_0 < p_1 \le 2$;
   - $2 \le q_0 < q_1 < \infty$ and $2 \le p_0 < p_1 \le \infty$.

## References

[1] J. Bergh and J. Löfström, *Interpolation spaces.* Springer-Verlag, Berlin, 1978.

[2] B. Carl and A. Defant, *An inequality between the p- and $(p,1)$-summing norm of finite rank operators from $C(K)$-spaces*, Israel J. Math. **74** (1991), 323–335.

[3] B. Carl, B. Maurey and J. Puhl, *Grenzordnungen von absolut-$(r,p)$-summiernden Operatoren*, Math. Nachr. **82** (1978), 205–218.

[4] A. Defant, *Variants of the Maurey–Rosenthal Theorem for quasi Köthe function spaces*, Positivity **5** (2001), 153–175.

[5] A. Defant and K. Floret, *Tensor norms and operator ideals.* North-Holland, Amsterdam, 1993.

[6] A. Defant, M. Mastyło and C. Michels, *Summing inclusion maps between symmetric sequence spaces, a survey*, in: Recent progress in functional analysis (Valencia, 2000), 43–60, North-Holland Math. Stud., 189, North-Holland, Amsterdam, 2001.

[7] A. Defant, M. Mastyło and C. Michels, *Summing inclusion maps between symmetric sequence spaces*, Trans. Amer. Math. Soc. **354** (2002), 4473–4492.

[8] A. Defant, M. Mastyło and C. Michels, *On interpolation of tensor products of Banach spaces*, Rev. R. Acad. Cien. Serie A. Mat. **97** (2003), 209–221.

[9] A. Defant, M. Mastyło and C. Michels, *Eigenvalue estimates for operators on symmetric Banach sequence spaces*, Proc. Amer. Math. Soc. **132** (2004), 513–521.

[10] A. Defant, M. Mastyło and C. Michels, *Summing norms of identities between unitary ideals*, Math. Z. **252** (2006), 863–882.

[11] A. Defant and C. Michels, *A complex interpolation formula for tensor products of vector-valued Banach function spaces*, Arch. Math. **74** (2000), 441–451.

[12] J. Diestel, H. Jarchow, and A. Tonge, *Absolutely summing operators*, Cambridge Studies in Advanced Mathematics 43, 1995.

[13] O. Kouba, *On the interpolation of injective or projective tensor products of Banach spaces*, J. Funct. Anal. **96** (1991), 38–61.

[14] C. Le Merdy, *The Schatten space $S_4$ is a $Q$-algebra*, Proc. Amer. Math. Soc. **126** (1998), 715–719.

[15] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces II: Function spaces*, Springer-Verlag, 1979.

[16] C. Michels, *Absolutely $(r,p,q)$-summing inclusions*, Studia Math. **178** (2007), 19–45.

[17] A. Pietsch, *Operator ideals*, North-Holland, 1980.

[18] N. Tomczak-Jaegermann, *Banach–Mazur distances and finite-dimensional operator ideals*, Longman Scientific & Technical, 1989.

Department of Mathematics
University of Oldenburg
Postfach 2503
D-26111 Oldenburg
Germany
Email: michels@mathematik.uni-oldenburg.de