Extension of vector-valued functions*

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Abstract

We discuss problems of extension of vector-valued functions defined on subsets of a domain $\Omega \subset \mathbb{R}^N$ which have weak extensions belonging to a space $\mathscr{H}(\Omega)$ of smooth functions. We look for conditions which ensure that there exists an extension in the corresponding space $\mathscr{H}(\Omega, E)$ of vector-valued functions.

1 Introduction and notations

We study extension problems inspired by the work of Arendt-Nikolski [1] and Grosse-Erdmann [13] about holomorphic extension of vector-valued functions. Let E be a locally complete locally convex space, let $\mathscr{H}$ be a sheaf of smooth functions defined on a domain $\Omega \subset \mathbb{R}^N$, let $A \subset \Omega$ a subset, let $H \subset E'$ be a $\sigma(E', E)$-dense subspace and let $f : A \to E$ be a function such that $u \circ f$ admits an extension $g_u \in \mathscr{H}(\Omega)$ for each $u \in H$. We look for conditions on A, H and f which ensure that f admits an extension F to Ω which belongs to the space of vector-valued functions $\mathscr{H}(\Omega, E)$. This work continues [7]. We refer to [1, 5, 6, 9, 10, 11, 12, 13, 14, 16, 20] for literature related to this problem when $\mathscr{H}$ is the sheaf of holomorphic functions on $\Omega \subseteq \mathbb{C}$. Only [10] deals with vector-valued harmonic functions. Some of these papers discuss conditions to a function defined on the whole Ω and with values in E which permit to conclude that the function is in $\mathscr{H}(\Omega, E)$. In [7] most of these results are obtained as a particular case of extension results in the much more general setting of sheaves of vector-valued smooth functions.

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We refer the reader to [15, 17, 18, 19] for our standard notation about locally convex spaces. We recall briefly some notation which will be repeatedly used. All the locally convex spaces which we deal with are assumed to be Hausdorff. A Fréchet-Schwartz space is by definition a countable projective limit of Banach spaces with compact linking maps. Such a space is always reflexive. A subset W of E' is called *separating* if $u(x) = 0$ for each $u \in W$ implies $x = 0$. Clearly this is equivalent to the span of W being weak*-dense (or dense in the co-topology). We say that $W \subset E'$ *determines boundedness* if every $B \subset E$ is bounded whenever $u(B)$ is bounded in $\mathbb{C}$ for all $u \in W$. This holds if and only if every $\sigma(E,\text{span } W)$-bounded set is E-bounded. Given a Fréchet space E, an increasing sequence $(B_n)_n$ of bounded subsets of E' is said to *fix the topology* if the polars $(B_n^\circ)_n$ taken in E form a fundamental system of 0-neighbourhoods of E. A subspace H of the dual E' of a Banach space E is called *almost norming* if $B_H := \{u \in H : \|u\|' \leq 1\}$ fixes the topology (cf. [1, section 1]).

Throughout this paper Ω denotes a subset of $\mathbb{R}^N$ which is open and connected, i.e. a domain. We denote by $\mathscr{C}$ and $\mathscr{C}^\infty$ the sheaves of continuous functions and infinitely differentiable functions, respectively.
Let in the sequel $\mathscr{H}$ be a sheaf of smooth functions on Ω which is closed in $\mathscr{C}$, i.e. $\mathscr{H}(\omega)$ is a closed subspace of $\mathscr{C}(\omega)$ for each $\omega \subset \Omega$ open.
This yields that $\mathscr{C}(\omega)$ and $\mathscr{C}^\infty(\omega)$ induce the same topology on $\mathscr{H}(\omega)$. Hence $\mathscr{H}(\omega)$ is a Fréchet-Schwartz space endowed with the compact open topology for each $\omega \subset \Omega$ open. This kind of sheaves include the sheaves of harmonic and holomorphic functions on open subsets of $\mathbb{C}^N = \mathbb{R}^{2N}$. Given a sheaf $\mathscr{H}$, the sheaf $\mathscr{H}(\cdot, E)$ of vector-valued functions with values in a locally complete space E is defined by

$$\mathscr{H}(\omega, E) := \{x \mapsto T(\delta_x) : T \in L(\mathscr{H}(\omega)', E)\}, \quad \omega \subset \Omega \text{ open}.$$

Here δ_x denotes the evaluation at x. This definition agrees with the classical definitions of vector-valued holomorphic and harmonic functions. We refer the reader for more information, especially for the representation of spaces of vector-valued functions as spaces of linear mappings, and applications to extension problems, to [7] (see also [3, 4, 17, 20, 21]).

To formulate our results, we need to include some more definitions. A *set of uniqueness* for $\mathscr{H}(\Omega)$ is a subset $M \subseteq \Omega$ satisfying that each function which belongs to $\mathscr{H}(\Omega)$ and vanishes on M vanishes on the whole Ω. Therefore, $M \subset \Omega$ is a set of uniqueness if and only if $\{\delta_z : z \in M\} \subset \mathscr{H}(\Omega)'$ is separating. Notice that if $\mathscr{H}$ is the sheaf of holomorphic functions on $\Omega \subseteq \mathbb{C}$, then $M \subset \Omega$ is a set of uniqueness for $\mathscr{H}(\Omega)$ if and only if M has an accumulation point in Ω. An increasing sequence $(M_n)_n$ of relatively compact subsets of Ω is called *admissible* for $\mathscr{H}$ if
i) M_1 is a set of uniqueness for $\mathscr{H}(\omega)$ whenever $M_1 \subset \omega \subset \Omega$ is a domain, and
ii) $(\{\delta_x : x \in M_n\})_n$ fixes the topology of $\mathscr{H}(\Omega)$, i.e. the seminorms

$$p_n(f) = \sup_{z \in M_n} |f(z)|, \quad f \in \mathscr{H}(\Omega)$$

define the topology of $\mathscr{H}(\Omega)$.
If $(\omega_n)_n$ is a regular exhaustion of Ω by relatively compact open sets, D_n is a dense subset of $\partial\omega_n$, and $M_n := \cup_{1 \leq \nu \leq n} D_\nu$ then, in case of the sheaves of holomorphic or harmonic functions, the sequence $(M_n)_n$ is admissible.

2 Extension of functions

There are two types of extension theorems known: The first one guarantees extension of functions which are defined on sets of uniqueness, i.e. on very thin sets, by requesting that the extensions satisfy restrictive properties, the second one guarantees extension for bounded functions on sequences of sets which are admissible by requesting mild properties to the extensions. We formulate our new results together with the known ones in the following two theorems corresponding to the above mentioned types.

Theorem 1. *Let $f : M \to E$ be a function defined on a set of uniqueness $M \subset \Omega$ and assume that $u \circ f$ has an extension $f_u \in \mathscr{H}(\Omega)$ for all $u \in W \subset E'$. If*
i) W determines boundedness, or if
ii) E is Fréchet, $W = \cup_n B_n$, where $(B_n)_n$ fixes the topology of E, and $(f_u)_{u \in B_n}$ is bounded in $\mathscr{H}(\Omega)$ for all n,
then f has an (unique) extension $F \in \mathscr{H}(\Omega, E)$.

Remark 2. a) Case i) is [7, Theorem 9]. The result is sharp because of [1, Theorem 1.5]
b) Later we show (in Proposition 5) that also condition ii) cannot be considerably weaker.

Theorem 3. *Let $f : \cup_n M_n \to E$ be a function, where $(M_n)_n$ is admissible for $\mathscr{H}$, and assume that $f(M_n)$ is bounded for each n and $u \circ f$ has an extension $f_u \in \mathscr{H}(\Omega)$ for all u contained in a separating subset $W \subset E'$. If one of the following conditions is satisfied, then f has an (unique) extension $F \in \mathscr{H}(\Omega, E)$:*
i) span W is strongly dense in E',
ii) E is Fréchet (or more generally, B_r-complete),
iii) $f(M_n)$ is bounded in a continuously embedded quasireflexive (i.e. with finite codimension in the bidual) Banach space $E_n \subset E$, $n \in \mathbb{N}$,
iv) M_n is closed and $f(M_n)$ is bounded in a continuously embedded separable Banach space $E_n \subset E$, $n \in \mathbb{N}$.

Remark 4. a) Case i) is [12, Satz 3.3] and case ii) is obtained in [13, Theorem 2] for the sheaf of one variable holomorphic functions. Both are simultaneously obtained in [7, Theorem 17] for sheaves of smooth functions.
b) In case of the test functions space $\mathscr{D}(X)$, any bounded set is bounded in a continuously embedded reflexive Banach space and then iii) can be applied. This space is reflexive and then also i) can applied because in a semireflexive space E (i.e. $E = E''$ algebraically) each separating subset $W \subseteq E'$ has strongly dense span (see also [16, Theorem 6]).
c) Condition iv) is satisfied if E is a regular inductive limit (and then if it is a direct sum) of separable Banach spaces and M_n is closed for each n. In [7, Example 20 a)] it is shown the existence of an admissible sequence $(M_n)_n$ for the space of entire functions $\mathscr{H}(\mathbb{C})$ formed by non closed subsets of $\mathbb{C}$, a countable direct sum of separable Banach spaces $E = \oplus_n E_n$, and a separating subspace $W \subset E'$ for which there exists a non continuous function $f : \cup_n M_n \to E$ bounded on each M_n such that $u \circ f$ admits an entire extension f_u for each $u \in W$. [7, Example 20 b)] gives a similar function taking the sequence $(M_n)_n$ formed by closed subsets of $\mathbb{C}$ but taking

$E = \oplus E_n$ as a direct sum of non separable Banach spaces. Consequently Theorem 3 is sharp.
d) Theorem 3 allows to formulate the following result when $\mathscr{H}(\Omega)$ is the sheaf of holomorphic or harmonic functions on Ω. Let E, F be locally complete spaces such that $E \hookrightarrow F$ is continuously embedded. If $f \in \mathscr{H}(\Omega, F)$ and there exists a compact subset K of Ω such that (the restriction of) $f \in \mathscr{H}(\Omega \setminus K, E)$ then $f \in \mathscr{H}(\Omega, E)$.

The following result and its proof should be compared with [1, Theorem 1.5] and [2, Theorem 2]. It shows that in Theorem 1 the condition ii) is also sharp.

Proposition 5. *Let E be a Banach space, and let H be a subspace of E' such that B_H does not fix the topology of E (i.e. H is not almost norming). There exists a function $f : \mathbb{C} \to E$ which is not continuous at 0 and such that $u \circ f$ is entire for each $u \in H$ and $(u \circ f)_{u \in B_H}$ is bounded in the space $\mathscr{H}(\mathbb{C})$ of entire functions.*

Proof. Consider the compact subsets $Q_n := ([-2^n, 1/2^{n+1}] \cup [1/2^{n-1}, 2^n] \times [-2^n, 2^n]$ of $\mathbb{C} = \mathbb{R}^2$. For each $n \in \mathbb{N}$ we apply the Runge's Theorem to get polynomials q_n such that $q_n(1/2^n) = 2^n$ and $q_n(z) < 1/2^n$ if $z \in Q_n$. We set $b_n := \sup\{|q_n(z)| : |Im(z)|, |Re(z)| \leq 2^n\}$.

Since H is not almost norming, there exists a sequence $(x_n)_n \subset E$ such that $\sup_{u \in B_H} |u(x_n)| \leq 1$ and $\|x_n\| \geq 2^n b_n$ for each $n \in \mathbb{N}$. We define

$$f(z) := \sum_{n=1}^{\infty} q_n(z) \frac{x_n}{\|x_n\|} \quad z \in \mathbb{C}.$$

For each $z \in \mathbb{C}$, there exists $n_0 \in \mathbb{N}$ such that $z \in Q_n$ for each $n \geq n_0$. This yields that f is well defined. Now let $u \in H$ such that $\|u\| = 1$, and let $K \subset \mathbb{C}$ compact. Let $l \in \mathbb{N}$ such that $K \subset [-2^l, 2^l] \times [-2^l, 2^l]$. Let $C(l)$ be the maximum of the function $\sum_{n=1}^{l-1} |q_n(\cdot)|$ on $[-2^l, 2^l] \times [-2^l, 2^l]$. Then

$$\sup_{z \in K} \|u \circ f(z)\| \leq \sum_{n=1}^{l-1} |q_n(z)| + \sum_{n=l}^{\infty} b_n \frac{|u(x_n)|}{\|x_n\|} \leq C(l) + 1.$$

From these estimates it follows that $u \circ f$ is holomorphic for each $u \in H$ and $(u \circ f)_{u \in B_H}$ is bounded in $\mathscr{H}(\mathbb{C})$. However, for each $j \in \mathbb{N}$,

$$\left\| f\left(\frac{1}{2^j}\right) \right\| \geq \left| q_j\left(\frac{1}{2^j}\right) - \sum_{n \neq j} q_n\left(\frac{1}{2^j}\right) \right| \geq 2^j - 1,$$

hence f is not continuous at 0. ■

3 Auxiliary results and proofs

Let us discuss shortly the concepts of locally convex theory which appear in our theorems. We present below a first relation between fixing the topology and determining boundedness. *In what follows in this section, E denotes a Fréchet space and E' denotes its dual, endowed with its strong topology if nothing else is specified.*

Proposition 6. *Let $(B_n)_n$ be a sequence of bounded subsets of E' fixing the topology of E. Set $H := span\{B_n : n \in \mathbb{N}\}$. If $H \subset E'$ is a subspace such that there exists a barrelled locally convex topology τ in H which is stronger than $\sigma(H, E)$ and B_n is $\tau-$bounded for each $n \in \mathbb{N}$, then H determines boundedness in E.*

Proof. Since $(B_n)_n$ fixes the topology of E it follows that $\langle H, E\rangle$ is a dual pair. Let $B \subset E$ be a $\sigma(E, H)$-bounded subset. $B^\circ \cap H$ is an absolutely convex, $\sigma(H, E)$-(and then $\tau-$) closed and absorbing subset of (H, τ), i.e. $B^\circ \cap H$ is a barrel, and then a 0-neighbourhood, in (H, τ). Consequently $B^\circ \cap H$ absorbs τ-bounded sets of H. Hence, for each $n \in \mathbb{N}$ there exists $\lambda_n > 0$ such that

$$B_n \subset \lambda_n B^\circ \cap H \subset \lambda_n B^\circ.$$

This yields

$$B \subset B^{\circ\circ} \subset \lambda_n B_n^\circ.$$

Since $(B_n)_n$ fixes the topology of E, we get that B is bounded in E. ■

Let $(B_n)_n \subset E'$ be an increasing sequence of bounded subsets fixing the topology of E. If each B_n is a Banach disc, then $\text{ind}_n E'_{B_n}$ satisfies the hypothesis of Proposition 6. We can consider even smaller subspaces of E' satisfying the hypothesis of Proposition 6. Given a bounded subset B in a locally complete space E we define

$$E(B) := \{\sum_j \alpha_j b_j : (\alpha_j)_j \in l_1, (b_j)_j \subset B\}$$

endowed with the Banach space topology which makes it isomorphic to a quotient of $l_1(I)$, I being an index set with the same cardinality as B. Then, if $(B_n)_n$ is an increasing sequence of bounded sets in E' one can always consider the (LB)-space $E'((B_n)_{n\in\mathbb{N}}) := \text{ind}_n E'(B_n)$ continuously embedded in E'.

Proposition 7. *Let $(B_n)_n$ be an increasing sequence of bounded subsets of E'. The following assertions are equivalent*

(a) $(B_n)_n$ *fixes the topology of* E*.*

(b) $E'((B_n)_{n\in\mathbb{N}})$ *determines boundedness in* E*.*

(c) $span \cup_n \overline{B_n}^{\sigma(E',E)}$ *determines boundedness in* E*.*

Proof. (a) implies (b) follows directly from Proposition 6. (b) implies (c) because $E'(B_n) \subset span\overline{B_n}^{\sigma(E',E)}$ for each $n \in \mathbb{N}$. To see (c) implies (a) we consider the semimetrisable topology τ on E which has a fundamental system of 0-neighbourhoods defined by $(B_n^\circ)_n$. The topology τ is coarser than the topology of E and (c) implies that each τ-bounded set is bounded in E. In fact, if $B \subset E$ is τ-bounded then, for each $n \in \mathbb{N}$, there exits $\lambda_n > 0$ such that $B \subset \lambda_n B_n^\circ$ and this is equivalent to $\lambda_n B^\circ \supset \overline{B_n}^{\sigma(E',E)}$. Thus, if we denote by H the space defined in (c), we have that B is $\sigma(E, H)$ bounded and then bounded by the assumption. Therefore τ is metrisable and $(E, \tau) = E$ since both topologies have the same bounded sets. ■

Remark 8. In case E is a Banach space and $H \subset E'$, the unit ball B_H of H fixes the topology of E (H is almost norming) is equivalent to $\overline{H}^{\|\cdot\|_{E'}}$ determining boundedness in E by Proposition 7, since B_H fixes the topology in E if and only if the unit ball of $\overline{H}^{\|\cdot\|_{E'}}$ does, and it is easy to check that, if $B_{\overline{H}}$ denotes the unit ball of the norm closure of H, then $E'(B_{\overline{H}}) = \overline{H}^{\|\cdot\|_{E'}}$. Therefore, a strongly closed almost norming subspace $H \subset E'$ determines boundedness, and we obtain Theorem 3.5 of [1] as a corollary of Theorem 1, case i).

We obtain now a result about the extension of operators densely defined which will be used to prove Theorem 1 case ii).

Lemma 9. *Let Y be a Fréchet-Schwartz space, let $X \subset Y'$ a dense subspace, let E be a Fréchet space and let $(B_n)_n$ be an increasing sequence of bounded subsets of E' fixing the topology of E. Set $H := span \cup_n B_n$. Let $T : X \to E$ a linear map which is $\sigma(X,Y)$-$\sigma(E,H)$ continuous and satisfies that $T^t(B_n)$ is bounded in Y for each $n \in \mathbb{N}$. Then there exists a continuous linear extension $\widehat{T} : Y' \to E$.*

Proof. Notice that Y can be considered as a proper subspace of the algebraic dual X^* of X since X is dense in Y'. Let $(a_i)_i \in l_1$ and $(c_i)_i \in B_n$. If $u = \sum_i a_i c_i \in E'(B_n)$, then $\sum_i a_i(c_i \circ T) \in Y(T^t(B_n)) \subset Y$. Hence T is $\sigma(X,Y)$-$\sigma(E, E'((B_n)_{n\in\mathbb{N}}))$ continuous, and the result follows from [7, Proposition 7] and Proposition 7. ■

Due to the symmetry of the ε-product [17, I, 43.3.(3)], this result can be formulated in the following way: *If $T : H \to Y$ is a $\sigma(H,E) - \sigma(Y,X)$ continuous linear mapping bounded on each B_n then T admits a continuous linear extension* $\hat{T} : E'_{co} \to Y$. By [7, Theorem 16], the result is completely symmetric in the sense that we can interchange the roles of Fréchet Schwartz and Fréchet between E and Y.

Proof of Theorem 1, case ii). Let $Y := \mathscr{H}(\Omega)$, and let $X := \text{span}\{\delta_z : z \in M\}$. The linear map $T : X \to E$ determined by $T(\delta_x) := f(x)$ is well defined since $\cup_n B_n$ is separating. If we define $H := \text{span} \cup_n B_n$, then T is $\sigma(X,Y)$-$\sigma(E,H)$ continuous and $T^t(B_n)$ is bounded in Y for every n. By Lemma 9, T has an extension $\widehat{T} \in L(Y', E)$. Setting $F(x) := \widehat{T}(\delta_x)$ we obtain our extension $F \in \mathscr{H}(\Omega, E)$ of f.□

By the above remark, Theorem 1 ii) and Theorem 3 ii) are symmetric, because they are direct consequences of Lemma 9 and [7, Theorem 16] respectively. To prove the cases iii) and iv) of Theorem 3 we need another lemma:

Lemma 10. *Let $\omega, A \subset \Omega$ be relatively compact and let, in addition, ω be open such that $A \cap \omega$ is a set of uniqueness for $\mathscr{H}(\omega)$ and for $\mathscr{H}(\Omega)$, and such that*

$$\sup_{x\in\omega} |g(x)| \leq C \sup_{x\in A} |g(x)|$$

for all $g \in \mathscr{H}(\Omega)$. If $f : A \to E$ is a bounded function with values in a Banach space E such that $u \circ f$ has an extension $f_u \in \mathscr{H}(\Omega)$ for all u contained in a separating subspace H of E', then $f|_{A\cap\omega}$ has an extension $F \in \mathscr{H}(\omega, E)$ whenever one of the following conditions is satisfied:
i) E is quasireflexive,
ii) A is closed and E is separable.

Proof. i) Due to a result of Davis and Lindenstrauss [8] (cf also [1, Remark 1.2 h)]) every separating subspace of the dual of a quasireflexive Banach space is almost norming. Hence B_H fixes the topology of E. The estimate in the assumption allows us to apply Theorem 1, case ii).
ii) $\mathscr{H}(\Omega)$ is continuously embedded into the Banach space $(\mathscr{C}(A), \|\cdot\|_A)$ since A is a set of uniqueness for $\mathscr{H}(\Omega)$. From the estimate we obtain that, for all $x \in \omega$, δ_x belongs to the dual of $(\mathscr{H}(\Omega), \|\cdot\|_A)$ and $\|\delta_x\|'_A \leq C$. Hence, by the Hahn-Banach theorem and the Riesz theorem there is a family $(\mu_x)_{x\in\omega}$ of complex valued measures μ_x whose total variations do not exceed C such that

$$g(x) = \int_A g d\mu_x$$

for all $g \in \mathscr{H}(\Omega)$ and all $x \in \omega$.
Let W be the subspace of all $u \in E'$ satisfying the following two conditions:
a) $u \circ f$ is measurable,
b) $u \circ f$ has an extension f_u to $A \cup \omega$ such that $f_u|_\omega \in \mathscr{H}(\omega)$ and

$$f_u(x) = \int_A f_u d\mu_x$$

for all $x \in \omega$. Then W is weak*-dense, since it contains H. We will show that $W = E'$. By the Krein-Smulyan theorem we only have to prove that any u belonging to the weak*-closure of $W \cap B_{E'}$ is contained in W. The set $B_{E'}$ equipped with the weak*-topology is metrizable, because E is separable, hence there is a sequence $(u_n)_n$, contained in $W \cap B_{E'}$, converging to u. The sequence $(u_n \circ f)_n$ is measurable and uniformly bounded, hence the pointwise limit $u \circ f$ is also measurable (and bounded). We apply Lebesgue's theorem to obtain that f_{u_n} converges pointwise to a function $f_u : A \cup \omega \to \mathbb{C}$ with $f_u|_A = u \circ f$ and $f_u(x)$ satisfies the integral representation of b) for each $x \in \omega$. Moreover, from the integral representations it follows that $(f_{u_n}|_\omega)_n$ is a bounded sequence in the Fréchet-Schwartz space $\mathscr{H}(\omega)$, hence it possesses a convergent subsequence. Denote its limit by g. In particular, the subsequence converges pointwise to g. Hence $g = f_u|_\omega$. We get $u \in W$.
In particular, we proved that the maps $u \circ f|_{A\cap\omega}$ possess (unique) extensions $g_u \in \mathscr{H}(\omega)$ for all $u \in E'$, so we may apply [7, Theorem 9] (Theorem 1 i)) to get an extension $F \in \mathscr{H}(\omega, E)$ of $f|_{A\cap\omega}$. ■

Proof of Theorem 3, cases iii), iv). Let $(\omega_n)_n$ be a regular exhaustion of Ω consisting of open and relatively compacts sets. Without loss of generality we may assume that

$$\sup_{x\in\omega_n} |g(x)| \leq C_n \sup_{x\in M_n} |g(x)|$$

for all $g \in \mathscr{H}(\Omega)$ and all n. The subspaces $H_n := \{u|_{E_n} : u \in \text{span } W\}$ are separating. From the previous Lemma we get extensions $F_n \in \mathscr{H}(\omega_n, E_n)$ of $f|_{M_n\cap\omega_n}$. Since $\mathscr{H}(\cdot, E)$ is a sheaf, there is a unique $F \in \mathscr{H}(\Omega, E)$ extending f. □
As a final remark, we notice that Theorem 3 iii) can also be obtained from Theorem 3 ii). In fact, under the hypothesis of theorem 3 iii) one can construct a Hausdorff countable inductive limit $F := \text{ind}_n F_n$ of quasireflexive Banach spaces continuously embedded in E such that $f : \cup_n M_n \to F$ is bounded on each M_n, and the restrictions of the elements of W to F form a separating subset of F'. The conclusion follows since F is B_r-complete due to [19, 7.2.3, 7.5.12].

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