On conjugate harmonic pairs (U_r, V_{r-1}) of multi-vector valued functions

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Dedicated to my dear friend Professor Jean Schmets
on the occasion of his 65-th birthday

Abstract

Let $\mathbb{R}_{0,m}$ be the real Clifford algebra constructed over the real quadratic space $\mathbb{R}^{0,m}$ with signature $(0, m)$ and let U_r be an $\mathbb{R}^{+}_{0,m}$-valued harmonic function in an appropriate open domain Ω of $\mathbb{R}^{m+1}$ $(0 < r \leq m; m \geq 2)$. Then a necessary and sufficient condition is given upon U_r for the existence of an $\mathbb{R}^{r-1}_{0,m}$-valued harmonic function in Ω which is conjugate to U_r.

1 Introduction

Clifford analysis, a function theory for the Dirac operator ∂_x in Euclidean space $\mathbb{R}^{m+1}(m \geq 2)$, generalizes classical complex analysis in the plane to higher dimensional space and refines the theory of harmonic functions.
If $\mathbb{R}^{0,m+1}$ denotes the space $\mathbb{R}^{m+1}$ provided with a real quadratic form of signature $(0, m+1)$ and $e = (e_0, e_1, ..., e_m)$ is an orthogonal basis for $\mathbb{R}^{0,m+1}$, then ∂_x is given by

$$\partial_x = \sum_{i=0}^{m} e_i \partial_{x_i}.$$

Taking into account the basic multiplication rules

$$\begin{aligned} e_i^2 = -1 \quad &, \quad i = 0, 1, ..., m \\ e_i e_j + e_j e_i = 0 \quad &, \quad i \neq j, 0 \leq i, j \leq m, \end{aligned}$$

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in the Clifford algebra $\mathbb{R}_{0,m+1}$ constructed over $\mathbb{R}^{0,m+1}$, then for an $\mathbb{R}^{0,m+1}$-valued $\mathcal{C}_1$-function $F = \sum_{i=0}^{m} F_i e_i$ in $\Omega \subset \mathbb{R}^{m+1}$ open, are equivalent:

$$\partial_x F = 0 \Longleftrightarrow \left\{ \begin{array}{rcl} \sum_{i=0}^{m} \frac{\partial F_i}{\partial x_i} & = & 0 \\ \frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i} & = & 0 \end{array} \right.$$

As is well known, the latter system of equations is called the Riesz system. It clearly generalizes the classical Cauchy-Riemann system for the plane. In [3] E. Stein and G. Weiss called an $(m+1)$-tuple $(u_0, u_1, ..., u_m)$ of $\mathbb{R}$-valued harmonic functions in Ω conjugate harmonic in Ω if it satisfies the Riesz system.

More generally, using the decomposition $\mathbb{R}_{0,m+1} = \mathbb{R}_{0,m} \oplus \overline{e}_0 \mathbb{R}_{0,m}$ where $\overline{e}_0 = -e_0$ and where $\mathbb{R}_{0,m}$ is the Clifford algebra constructed over $\mathbb{R}^{0,m}$ with orthogonal basis $(e_1, e_2, ..., e_m)$, a pair (U, V) of $\mathbb{R}_{0,m}$-valued harmonic functions in Ω is called conjugate harmonic in Ω if $F = U + \overline{e}_0 V$ satisfies $\partial_x F = 0$ in Ω (see [1]).

Now let $0 < r \leq m$ be fixed and consider the subspace $\mathbb{R}^r_{0,m+1}$ of $\mathbb{R}_{0,m+1}$ consisting of so-called r-vectors, i.e. $\mathbb{R}^r_{0,m+1} = \text{span}_{\mathbb{R}}(e_A : |A| = r)$, where for $A = \{i_1, i_2, ..., i_r\} \subset \{0, ..., m\}$ with $0 \leq i_1 < i_2 < ... < i_r \leq m$, $e_A = e_{i_1} ... e_{i_r}$.

In section 3 of this paper, we give an answer to the following problem: Given U_r, a harmonic and $\mathbb{R}^r_{0,m}$-valued function in Ω, under which conditions upon U_r does there exist V_{r-1}, $\mathbb{R}^{r-1}_{0,m}$-valued and harmonic in Ω, such that the pair (U_r, V_{r-1}) is conjugate harmonic in Ω, i.e. the $\mathbb{R}^r_{0,m+1}$-valued function $F_r = U_r + \overline{e}_0 V_{r-1}$ satisfies $\partial_x F_r = 0$ in Ω. Such functions F_r are also called monogenic r-vector valued functions in Ω.
For convenience of the reader, in section 2 we briefly recall some notions and results concerning monogenic r-vector valued functions.

2 Monogenic r-vector valued functions

Let again $\mathbb{R}^{0,m+1} (m \geq 2)$ be the space $\mathbb{R}^{m+1}$ provided with a quadratic form of signature $(0, m+1)$ and let $\mathbb{R}_{0,m+1}$ be the universal real Clifford algebra constructed over $\mathbb{R}^{0,m+1}$. If $e = (e_0, e_1, ..., e_m)$ is an orthogonal basis for $\mathbb{R}^{0,m+1}$, then the basic multiplication rules in $\mathbb{R}_{0,m+1}$ are governed by

$$\left\{ \begin{array}{rcl} e_i^2 = -1 & , & i = 0, 1, ..., m \\ e_i e_j + e_j e_i = 0 & , & i \neq j, 0 \leq i, j \leq m \end{array} \right. .$$

A basis for $\mathbb{R}_{0,m+1}$ is given by the set of elements $e_A = e_{i_1} e_{i_2} ... e_{i_r}$ where for $A = \{i_1, ..., i_r\} \subset \{0, 1, ..., m\}, 0 \leq i_1 < i_2 < ... < i_r \leq m$ and where $e_\phi = 1$, the identity element in $\mathbb{R}_{0,m+1}$.
Putting for $r \in \{0, 1, ..., m+1\}$ fixed, $\mathbb{R}^r_{0,m+1} = \text{span}_{\mathbb{R}}\ (e_A : |A| = r)$, it is clear that

$$\mathbb{R}_{0,m+1} = \sum_{r=0}^{m+1} \bigoplus \mathbb{R}^r_{0,m+1.}$$

The space $\mathbb{R}^r_{0,m+1}$ is called the space of r-vectors and the projection operator from $\mathbb{R}^r_{0,m+1}$ is denoted by $[\ \ ]_r$.

Notice that $\mathbb{R}$ and $\mathbb{R}^{m+1}$ may thus be identified with $\mathbb{R} \cong \mathbb{R}^0_{0,m+1}$ and $\mathbb{R}^{m+1} \cong \mathbb{R}^{0,m+1} \cong \mathbb{R}^1_{0,m+1}$.
The product of a 1-vector u and an r-vector $v_r (r \geq 1)$ splits into the sum of an $(r-1)$-vector $u \bullet v_r$ and an $(r+1)$-vector $u \wedge v_r$, i.e.

$$uv_r = u \bullet v_r + u \wedge v_r$$

where

$$u \bullet v_r = [uv_r]_{r-1} = \frac{1}{2}\left(uv_r - (-1)^r v_r u\right)$$

and (2.1)

$$u \wedge v_r = [uv_r]_{r+1} = \frac{1}{2}\left(uv_r + (-1)^r v_r u\right)$$

Now decompose $\mathbb{R}^{m+1}$ into $\mathbb{R}^{m+1} = \mathbb{R} \times \mathbb{R}^m$; denote an arbitrary element $x = (x_0, x_1, ..., x_m) \in \mathbb{R}^{m+1}$ as $x = (x_0, \underline{x})$; identify $\mathbb{R}^{m+1}$ and $\mathbb{R}^m$ with the subspaces $\text{span}_{\mathbb{R}}(e_0, e_1, ..., e_m)$ and $\text{span}_{\mathbb{R}}(e_1, ..., e_m)$ in $\mathbb{R}_{0,m+1}$ and put $x = \sum_{i=0}^m x_i e_i$ and $\underline{x} = \sum_{j=1}^m x_j e_j$. Then inside $\mathbb{R}_{0,m+1}$, the Clifford algebra $\mathbb{R}_{0,m}$ is generated by $\underline{e} = (e_1, ..., e_m)$ and obviously

$$\mathbb{R}_{0,m+1} = \mathbb{R}_{0,m} \bigoplus \overline{e}_0 \mathbb{R}_{0,m}$$

Clearly, for $0 < r \leq m$ fixed,

$$\mathbb{R}^r_{0,m+1} = \mathbb{R}^r_{0,m} \bigoplus \overline{e}_0 \mathbb{R}^{r-1}_{0,m} \tag{2.2}$$

The Dirac operators ∂_x and $\partial_{\underline{x}}$ in $\mathbb{R}^m$ are defined by

$$\partial_x = \sum_{i=0}^m e_i \partial_{x_i}$$

and

$$\partial_{\underline{x}} = \sum_{j=1}^m e_j \partial_{x_j}$$

whence

$$\partial_x = e_0 \partial_{x_0} + \partial_{\underline{x}}.$$

The Cauchy-Riemann operator D_x in $\mathbb{R}^{m+1}$ is determined by

$$D_x = \overline{e}_0 \partial_x = \partial_{x_0} + \overline{e}_0 \partial_{\underline{x}}.$$

Now let $\Omega \subset \mathbb{R}^{m+1}$ be open, let $\tilde{\Omega}$ be its orthogonal projection onto $\mathbb{R}^m$ and let $0 < r \leq m$ be fixed. Then the space of C_∞-functions from Ω into $\mathbb{R}^r_{0,m+1}$, respectively from $\tilde{\Omega}$ into $\mathbb{R}^r_{0,m}$, is denoted by $\mathcal{E}_r(\Omega)$, respectively $\mathcal{E}_r(\tilde{\Omega})$.

An element $F_r \in \mathcal{E}_r(\Omega)$ is said to be left monogenic in Ω if $\partial_x F_r = 0$ in Ω. Taking into account the relations (2.1) we thus have that the action of ∂_x on F_r splits into

$$\begin{aligned} \partial_x F_r &= [\partial_x F_r]_{r-1} + [\partial_x F_r]_{r+1} \\ &= \partial_x \bullet F_r + \partial_x \wedge F_r \end{aligned} \tag{2.3}$$

We put $\partial_x^+ F_r = \partial_x \wedge F_r$ and $\partial_x^- F_r = \partial_x \bullet F_r$. Clearly, on $\mathcal{E}_r(\Omega)$,

$$\partial_x = \partial_x^+ + \partial_x^-$$

Moreover,

$$\partial_x^2 = -\triangle_x; {\partial_x^+}^2 = 0; {\partial_x^-}^2 = 0, \tag{2.4}$$

$\triangle_x$ being the Laplacian in $\mathbb{R}^{m+1}$.
Consequently, on $\mathcal{E}_r(\Omega)$,

$$\triangle_x = -(\partial_x^+ \partial_x^- + \partial_x^- \partial_x^+) \tag{2.5}$$

Notice also that, as

$$[\partial_x F_r]_{r-1} = (-1)^{r+1} [F_r \partial_x]_{r-1}$$

and

$$[\partial_x F_r]_{r+1} = (-1)^r [F_r \partial_x]_{r+1},$$

$$\partial_x F_r = 0 \Longleftrightarrow F_r \partial_x = 0,$$

i.e. for $F_r \in \mathcal{E}_r(\Omega)$, the notions of left and right monogenicity coincide. Applying the decomposition (2.2) to $F_r \in \mathcal{E}_r(\Omega)$, we may write F_r as

$$F_r = U_r + \overline{e}_0 V_{r-1}$$

where U_r and V_{r-1} are $\mathbb{R}_{0,m}$-valued r- and $(r-1)$-vector functions in Ω.
In what follows, for $0 < s \leq m$ fixed, $\mathcal{E}_s(\Omega; \mathbb{R}_{0,m})$ denotes the space of $\mathbb{R}_{0,m}^s$-valued smooth functions in Ω.
Clearly, for $F_r \in \mathcal{E}_r(\Omega)$,

$$\partial_x F_r = 0 \Longleftrightarrow D_x F_r = 0 \Longleftrightarrow \begin{cases} \partial_{x_0} U_r + \partial_{\underline{x}} V_{r-1} = 0 \\ \partial_{\underline{x}} U_r + \partial_{x_0} V_{r-1} = 0 \end{cases} \tag{2.6}$$

We put

$$\ker^r \partial_x = \{F_r \in \mathcal{E}_r(\Omega) : \partial_x F_r = 0 \text{ in } \Omega\}$$

$$\ker^r \partial_x^+ = \{F_r \in \mathcal{E}_r(\Omega) : \partial_x^+ F_r = 0 \text{ in } \Omega\}$$

and

$$\ker^r \partial_x^- = \{F_r \in \mathcal{E}_r(\Omega) : \partial_x^- F_r = 0 \text{ in } \Omega\}.$$

Let us recall that if Ω is contractible to a point, then

$$\partial_x^+ \partial_x^- \quad : \quad \ker^r \partial_x^+ \longrightarrow \ker^r \partial_x^+$$

and (2.7)

$$\partial_x^- \partial_x^+ \quad : \quad \ker^r \partial_x^- \longrightarrow \ker^r \partial_x^-$$

are surjective.
Of course, similar definitions may be given for the operator $\partial_{\underline{x}}$ acting on $\mathcal{E}_r(\tilde{\Omega})$ and relations analogous to (2.3), (2.4), (2.5) and (2.7) may then be formulated.
Let us also point out the relationship between r-vector valued functions F_r and smooth differential forms ω^r in Ω.
Denoting by $\wedge^r(\Omega)$, $0 \leq r \leq m+1$, the algebra of smooth differential forms ω^r on Ω, then a natural isomorphism Θ between $\mathcal{E}_r(\Omega)$ and $\wedge^r(\Omega)$ considered as real vector spaces may be defined in the following way.
Let $\omega^r = \sum_{|A|=r} \omega_A^r dx^A \in \wedge^r(\Omega)$ and $F_r = \sum_{|A|=r} F_A^r e_A \in \mathcal{E}_r(\Omega)$ where for $A = \{i_1, ..., i_r\} \subset \{0, ..., m\}, dx^A = dx^{i_1} \wedge dx^{i_2} \wedge ... \wedge dx^{i_r}$.
Then $\Theta F_r = \omega^r$ if and only if $\omega_A^r = F_A^r$ for all A.
Finally notice that through this isomorphism, $\partial_x^+ \longleftrightarrow d$ and $\partial_x^- \longleftrightarrow d^*$, d and d^* being the exterior derivative and co-derivative on $\wedge^r(\Omega)$. It thus follows that for $F_r \in \mathcal{E}_r(\Omega)$ and $\omega^r = \Theta F_r \in \wedge^r(\Omega)$ are equivalent $(0 < r < m+1)$

$$\partial_x F_r = 0 \Longleftrightarrow \begin{cases} d\omega^r &= 0 \\ d^*\omega^r &= 0 \end{cases}$$

i.e. F_r monogenic in Ω is equivalent to saying that $\omega^r = \Theta F_r$ is a harmonic r-form in Ω.
For more details concerning the notions and results mentioned in this section, we refer the reader to [2].

3 Conjugate harmonic pairs (U_r, V_{r-1})

In this section $\Omega \subset \mathbb{R}^{m+1}$ open is supposed to satisfy the following conditions:

(i) Ω is normal w.r.t. the e_0-direction, i.e. there exists $x_0^* \in \mathbb{R}$ such that for each $\underline{x} \in \tilde{\Omega}$, $\Omega \cap \{\underline{x} + t\overline{e}_0 : t \in \mathbb{R}\}$ is connected and it contains the element $(x_0^*, \underline{x})$

(ii) Ω is contractible to a point.

Condition (i) is sufficient for ensuring the existence of a conjugate harmonic function V to a given U, $\mathbb{R}_{0,m}$-valued and harmonic in Ω, while condition (ii) is sufficient to guarantee the validity of the properties (2.7).
We wish to solve the following

Problem: Let $U_r \in \mathcal{E}_r(\Omega; \mathbb{R}_{0,m})(0 < r \leq m)$ be harmonic. Give necessary and sufficient condition(s) upon U_r such that there exists a $V_{r-1} \in \mathcal{E}_{r-1}(\Omega; \mathbb{R}_{0,m})$ which is conjugate harmonic to U_r, i.e.

(C1) $\triangle_x V_{r-1} = 0$ in Ω

(C2) $F_r = U_r + \overline{e}_0 V_{r-1}$ is monogenic in Ω.

If such V_{r-1} exists, then condition (C2) together with the second equation in (2.6) readily imply the following condition upon U_r to be satisfied in Ω:

$$\partial_{\underline{x}}^{+} U_r = 0 \tag{3.1}$$

We now claim that condition (3.1) is also sufficient.
To this end, let us first recall that the general form of a function V conjugate harmonic to U_r reads (see [1])

$$V = -\partial_{\underline{x}} H$$

where

$$H(x_0, \underline{x}) = \int_{x_0^*}^{x_0} U_r(t, \underline{x}) dt - \tilde{h}(\underline{x}) - h(\underline{x}).$$

Hereby

(i) $\triangle_{\underline{x}} \tilde{h}(\underline{x}) = \partial_{x_0} U_r(x_0^*, \underline{x})$ in $\tilde{\Omega}$ with $\tilde{h}$ $\mathbb{R}_{0,m}$-valued in $\tilde{\Omega}$

(ii) $\triangle_{\underline{x}} h(\underline{x}) = 0$ in $\tilde{\Omega}$.

Let us also recall that for any $\mathbb{R}_{0,m}$-valued solution $\tilde{h}(\underline{x})$ to $\triangle_{\underline{x}} \tilde{h}(\underline{x}) = \partial_{x_0} U_r(x_0^*, \underline{x})$,

$$\tilde{H}(x_0, \underline{x}) = \int_{x_0^*}^{x_0} U_r(t, \underline{x}) dt - \tilde{h}(\underline{x}) \tag{3.2}$$

is harmonic and $\mathbb{R}_{0,m}$-valued in Ω.
Now assume that $\partial_{\underline{x}}^{+} U_r = 0$ in Ω. Then from

$$\partial_{\underline{x}} H(x_0, \underline{x}) = \int_{x_0^*}^{x_0} \partial_{\underline{x}} U_r(t, \underline{x}) dt - \partial_{\underline{x}}(\tilde{h}(x) + h(\underline{x}))$$

it follows that, as $\partial_{\underline{x}} U_r = \partial_{\underline{x}}^{+} U_r + \partial_{\underline{x}}^{-} U_r$,

$$\int_{x_0^*}^{x_0} \partial_{\underline{x}} U_r(t, \underline{x}) dt = \int_{x_0^*}^{x_0} \partial_{\underline{x}}^{-} U_r(t, \underline{x}) dt \in \mathcal{E}_{r-1}(\Omega; \mathbb{R}_{0,m}).$$

Moreover, as $\triangle_{\underline{x}} : \mathcal{E}_r(\tilde{\Omega}) \longrightarrow \mathcal{E}_r(\tilde{\Omega})$ is surjective, there exists $\tilde{h}_r \in \mathcal{E}_r(\tilde{\Omega})$ such that $\triangle_{\underline{x}} \tilde{h}_r(\underline{x}) = \partial_{x_0} U_r(x_0^*, \underline{x})$.
Take such $\tilde{h}_r$ fixed. Then we claim that we can find $h_r \in \mathcal{E}_r(\tilde{\Omega})$ such that in $\tilde{\Omega}$

$$\left\{ \begin{array}{rcl} \triangle_{\underline{x}} h_r &=& 0 \\ \partial_{\underline{x}}^{+}(\tilde{h}_r + h_r) &=& 0 \end{array} \right. \tag{3.3}$$

To this end first notice that, as by assumption $\partial_{\underline{x}}^{+} U_r(x_0, \underline{x}) = 0$ in Ω, we also have that $\partial_{\underline{x}}^{+} \partial_{x_0} U_r(x_0, \underline{x}) = 0$ in Ω, whence $\partial_{\underline{x}}^{+} \partial_{x_0} U_r(x_0^*, \underline{x}) = 0$ in $\tilde{\Omega}$.
It thus follows that $\partial_{x_0} U_r(x_0^*, \underline{x}) \in \ker^r \partial_{\underline{x}}^{+}$.
Consequently, by virtue of (2.7), there ought to exist $W_r \in \ker^r \partial_{\underline{x}}^{+}$ such that $\partial_{\underline{x}}^{+} \partial_{\underline{x}}^{-} W_r = -\partial_{x_0} U_r(x_0^*, \underline{x})$, i.e. W_r satisfies in $\tilde{\Omega}$ the equations

$$\left\{ \begin{array}{rcl} \partial_{\underline{x}}^{+} W_r &=& 0 \\ \partial_{\underline{x}}^{+} \partial_{\underline{x}}^{-} W_r &=& -\partial_{x_0} U_r(x_0^*, \underline{x}) \end{array} \right. \tag{3.4}$$

Define $h_r \in \mathcal{E}_r(\tilde{\Omega})$ by

$$h_r + \tilde{h}_r = W_r$$

Then clearly $\partial^+_{\underline{x}}(\tilde{h}_r + h_r) = 0$ in $\tilde{\Omega}$.
Moreover, as

$$\triangle_{\underline{x}}(\tilde{h}_r + h_r) = \triangle_{\underline{x}} W_r,$$

we obtain on the one hand that

$$\begin{aligned}\triangle_{\underline{x}}(\tilde{h}_r + h_r) &= -(\partial^-_{\underline{x}}\partial^+_{\underline{x}} + \partial^+_{\underline{x}}\partial^-_{\underline{x}})W_r \\ &= -\partial^+_{\underline{x}}\partial^-_{\underline{x}} W_r \\ &= \partial_{x_0} U_r(x_0^*, \underline{x})\end{aligned}$$

while on the other hand

$$\begin{aligned}\triangle_{\underline{x}}(\tilde{h}_r + h_r) &= \triangle_{\underline{x}}\tilde{h}_r + \triangle_{\underline{x}} h_r \\ &= \partial_{x_0} U_r(x_0^*, \underline{x}) + \triangle_{\underline{x}} h_r.\end{aligned}$$

Consequently $\triangle_{\underline{x}} h_r = 0$ in $\tilde{\Omega}$.
We have thus proved the existence of $h_r \in \mathcal{E}_r(\tilde{\Omega})$ satisfying (3.3).
Putting

$$H_r(x_0, \underline{x}) = \int_{x_0^*}^{x_0} U_r(t, \underline{x})dt - \tilde{h}_r(\underline{x}) - h_r(\underline{x}),$$

we thus have that

(i) H_r is harmonic and $\mathbb{R}^r_{0,m}$-valued in Ω

(ii) $V_{r-1} = -\partial_{\underline{x}} H_r$ is harmonic and $\mathbb{R}^{r-1}_{0,m}$-valued in Ω

(iii) $F_r = \overline{D}_x H_r = U_r + \overline{e}_0 V_{r-1} \in \ker^r \partial_x$,

where $\overline{D}_x = \partial_{x_0} - \overline{e}_0\partial_{\underline{x}}$.

Summarizing we get

Theorem 3.1. *Let $U_r \in \mathcal{E}_r(\Omega; \mathbb{R}_{0,m})$ $(0 < r \leq m)$ be harmonic in Ω. Then U_r admits a conjugate harmonic function $V_{r-1} \in \mathcal{E}_{r-1}(\Omega; \mathbb{R}_{0,m})$ if and only if $\partial^+_{\underline{x}} U_r = 0$ in Ω.*

Now let $V^*_{r-1} \in \mathcal{E}_{r-1}(\Omega; \mathbb{R}_{0,m})$ also be conjugate harmonic to U_r, or equivalently, let $F^*_r = U_r + \overline{e}_0 V^*_{r-1}$ be monogenic in Ω. Then clearly $F^*_r - F_r = \overline{e}_0(V^*_{r-1} - V_{r-1})$ is monogenic in Ω, which implies that $V^*_{r-1} - V_{r-1}$ is independent of x_0 and satisfies $\partial_{\underline{x}}(V^*_{r-1} - V_{r-1}) = 0$ in $\tilde{\Omega}$.
Putting

$$V^*_{r-1} = V_{r-1} + W_{r-1}$$

we have that

$$F^*_r = U_r + \overline{e}_0(V_{r-1} + W_{r-1})$$

We so obtain

Theorem 3.2. *Let* $U_r \in \mathcal{E}_r(\Omega; \mathbb{R}_{0,m})$ $(0 < r \leq m)$ *be harmonic in* Ω *such that* $\partial_{\underline{x}}^+ U_r = 0$ *in* Ω. *Then the most general harmonic function* $V_{r-1}^* \in \mathcal{E}_{r-1}(\Omega; \mathbb{R}_{0,m})$ *conjugate to* U_r *in* Ω *has the form*

$$V_{r-1}^* = -\partial_{\underline{x}} H_r + W_{r-1}$$

Hereby

(i) $H_r(x_0, \underline{x}) = \int_{x_0^*}^{x_0} U_r(t, \underline{x})dt - \tilde{h}_r(\underline{x}) - h_r(\underline{x})$

where

(i.1) $\tilde{h}_r \in \mathcal{E}_r(\tilde{\Omega})$ satisfies $\triangle_{\underline{x}} \tilde{h}_r(\underline{x}) = \partial_{x_0} U_r(x_0^*, \underline{x})$ in $\tilde{\Omega}$

(i.2) $h_r \in \mathcal{E}_r(\tilde{\Omega})$ is harmonic in $\tilde{\Omega}$ such that $\partial_{\underline{x}}^+(\tilde{h}_r + h_r) = 0$ in $\tilde{\Omega}$

(ii) $W_{r-1} \in \mathcal{E}_{r-1}(\tilde{\Omega})$ satisfies $\partial_{\underline{x}} W_{r-1} = 0$ in $\tilde{\Omega}$

Remark 3.1. *As Dr. D. Eelbode pointed out to us,* $F_r \in \ker^r \partial_x$ *is equivalent to saying that* $F_M^r = F_r e_M \in \ker^{m+1-r} \partial_x$. *Hereby* e_M *is the pseudo-scalar in* $\mathbb{R}_{0,m+1}$, *i.e.* $e_M = e_0 e_1 ... e_m$.
It thus follows that a pair (U_r, V_{r-1}) *with* $U_r \in \mathcal{E}_r(\Omega; \mathbb{R}_{0,m})$ *and* $V_{r-1} \in \mathcal{E}_{r-1}(\Omega; \mathbb{R}_{0,m})$ *is conjugate harmonic in* Ω *if and only if the pair* $(V_{r-1} \overset{\circ}{e}_M, U_r \overset{\circ}{e}_M)$ *is conjugate harmonic in* Ω. *Hereby* $\overset{\circ}{e}_M = \overline{e}_0 e_M = e_1 e_2 ... e_m$, *the pseudoscalar in* $\mathbb{R}_{0,m}$.
Notice that $F_M^r = (-1)^{r-1}[V_{r-1} \overset{\circ}{e}_M + \overline{e}_0 U_r \overset{\circ}{e}_M]$.
In the case $m = 1$, *this remark expresses the well known property stating that a pair* (u, v) *of* $\mathbb{R}$*-valued harmonic functions in* $\Omega \subset \mathbb{C}$ *open is conjugate harmonic if and only if* $(-v, u)$ *is conjugate harmonic in* Ω.

Example. The case $r = 1$
Let $U_1 = \sum_{j=1}^m e_j U_1^j \in \mathcal{E}_1(\Omega; \mathbb{R}_{0,m})$ be harmonic in Ω and suppose U_1 satisfies condition (3.1) in Ω, or equivalently

$$\partial_{x_i} U_1^j - \partial_{x_j} U_1^i = 0 \text{ in } \Omega, i \neq j, i, j = 1, ..., m.$$

Define the harmonic potential field H_1 by

$$H_1(x_0, \underline{x}) = \int_{x_0^*}^{x_0} U_1(t, \underline{x})dt - \tilde{h}_1(\underline{x}) - h_1(\underline{x})$$

with

(i) $\tilde{h}_1$ $\mathbb{R}_{0,m}^1$-valued in $\tilde{\Omega}$ such that $\triangle_{\underline{x}} \tilde{h}_1(\underline{x}) = \partial_{x_0} U_1(x_0^*, \underline{x})$.

(ii) h_1 $\mathbb{R}_{0,m}^1$-valued and harmonic in $\tilde{\Omega}$ such that $\partial_{\underline{x}}^+(\tilde{h}_1 + h_1) = 0$ in $\tilde{\Omega}$.
Then $H_1 \in \mathcal{E}_1(\Omega; \mathbb{R}_{0,m})$ satisfies $\triangle_x H_1 = 0$ in Ω and $F = \overline{D}_x H_1 = U_1 + \overline{e}_0 V_0 \in \ker^1 \partial_x$, where $V_0 = -\partial_{\underline{x}} H_1$ is $\mathbb{R}$-valued.

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