Menchoff-Rademacher type theorems in vector-valued Banach function spaces

Andreas Defant Marius Junge

Dedicated to Professor J. Schmets on the occasion of his 65th birthday

Abstract

We present a simple procedure which transfers classical coefficient tests on summation of orthonormal series in $L_2(\mu)$ into theorems on summation for unconditionally convergent series in vector-valued Banach function spaces $E(X)$ (where E is a Banach function space over a measure space (Ω, μ) and X a Banach space).

1 Introduction

The fundamental theorem of Menchoff [12] and Rademacher [16] is the most important coefficient test for almost everywhere summation of general orthonormal series. It states that whenever a sequence (α_k) of coefficients satisfies the "test" $\sum_k |\alpha_k \log(k+2)|^2 < \infty$, then every orthonormal series $\sum_k \alpha_k x_k$ in $L_2(\mu)$ converges almost everywhere – moreover, by a result of Kantorovitch [8] its maximal function is square-integrable,

$$\Big\| \sup_j \Big| \sum_{k=0}^{j} \alpha_k x_k \Big| \Big\|_2 \leq C \|(\log(k+2)\alpha_k)\|_2 \,. \tag{1.1}$$

There is a long list of analogs of this result for various summation methods as Cesaro, Riesz, or Abel summation.

Recall that a *summation method* formally is a matrix $S = (s_{jk})$ with positive entries such that for each convergent series $s = \sum_k x_k$ of scalars we have

$$s = \lim_j \sum_k s_{jk} \sum_{\ell=0}^{k} x_\ell \tag{1.2}$$

Bull. Belg. Math. Soc. Simon Stevin 14 (2007), 473–482

(see e.g. [1] and [19]).

For example, the coefficient test $\sum_k |\alpha_k \log\log(k+2)|^2 < \infty$ assures that all orthonormal series $\sum_k \alpha_k x_k$ are almost everywhere Cesaro-summable, and moreover for the Cesaro means of its partial sums we have $\sup_j \left|\frac{1}{j+1}\sum_{k=0}^{j}\sum_{\ell=0}^{k}\alpha_\ell x_\ell\right| \in L_2(\mu)$, a result of Kaczmarz [7] and Menchoff [13].

Given a summation method $S = (s_{kj})$ and a scalar sequences (ω_k), we speak of a *coefficient test* whenever for each orthonormal series $\sum_k \alpha_k x_k$ in $L_2(\mu)$ with coefficients (α_k) satisfying the test $\sum_k |\alpha_k \omega_k|^2 < \infty$ we have

$$\sum_k \alpha_k x_k = \lim_j \sum_k s_{jk} \sum_{\ell=0}^{k} \alpha_\ell \, x_\ell \quad \mu\text{-a.e.};$$

such test sequences (ω_k) are then usually called *Weyl sequences* with respect to S. Obviously, $(\log(k+2))_k$ is a Weyl sequence for ordinary summation, and $(\log\log(k+2))_k$ for Cesaro-summation.

It is well-known that some fundamental coefficient tests transfer to almost everywhere summation theorems for unconditionally convergent series $\sum_k x_k$ in arbitrary $L_p(\mu)$-spaces (see e.g. [2], [3], [5], [6], [11], [15], and [17]). Following the approach from [5], the aim of this article is to extend several of these classical tests even to the vector-valued situation. We show a couple of results reflecting the following general philosophy:

Given a coefficient test with respect to a summation method S and a Weyl sequence ω, then for every unconditionally convergent series $\sum_k x_k$ in a Banach-space-valued Banach function space $E(X)$ we have

$$\sum_k \frac{x_k}{\omega_k} = \lim_j \sum_k s_{jk} \sum_{\ell=0}^{k} \frac{x_\ell}{\omega_\ell} \quad \mu\text{-a-e.},$$

and moreover for its maximal function

$$\sup_j \Big\| \sum_k s_{jk} \sum_{\ell=0}^{k} \frac{x_\ell}{\omega_\ell} \Big\|_X \in E.$$

Since any orthonormal series $\sum_k \alpha_k x_k$ in L_2 is unconditionally convergent, such results then still contain the original test as a special case ($E = L_2$ and $X = \mathbb{K}$).

Applied to ordinary summation we obtain a Menchoff-Rademacher type theorem for unconditionally convergent series $\sum_k x_k$ in spaces $E(X)$ (which needs no further assumption on the underlying Banach function space E and Banach space X), and similar results for Cesaro, Abel, or Riesz summation. Part of this article will be implicitly contained in [5].

2 Preliminaries

We shall use standard notation and notions from Banach space theory as presented e.g in [4], [6], [10], or [17]. We will need Grothendieck's notion of integral and Hilbertian operators in Banach spaces, and denote the integral norm of a (bounded and linear) operator u in Banach spaces by $\iota(u)$ and its Hilbertian norm by $\gamma_2(u)$. Grothendieck's "fundamental theorem of the metric theory of tensor products" states that for every operator u from ℓ_1 into ℓ_∞ we have

$$\gamma_2(u) \leq \iota(u) \leq K_G\, \gamma_2(u)\,. \tag{2.1}$$

The projective norm $\|\cdot\|_\pi$ for an element z in the tensor product $X \otimes Y$ of two Banach spaces is given by $\|z\|_\pi = \inf \sum_k \|x_k\|\,\|y_k\|$, where the infimum is taken over all finite representation $z = \sum_k x_k \otimes y_k$. Dually, the injective norm $\|\cdot\|_\varepsilon$ for $z = \sum_k x_k \otimes y_k$ (a fixed finite representation) is defined by $\|z\|_\varepsilon = \sup_{\|x'\|_{X'},\,\|y'\|_{Y'} \leq 1} \left|\sum_k x'(x_k)\, y'(y_k)\right|$. We will need the fact that for each integral operator $u \in \mathcal{L}(X, Y)$

$$\iota(u) = \sup \|\operatorname{id} \otimes u : Z \otimes_\varepsilon X \longrightarrow Z \otimes_\pi Y\|\,, \tag{2.2}$$

where the supremum is taken over all Banach spaces Z (see e.g., [4]).

Recall that the vector space of all unconditionally summable sequences $x = (x_k)$ in a Banach space X (i.e., the series $\sum_k x_k$ are unconditionally convergent) together with the norm $w_1(x) := \sup_{\|\alpha\|_\infty \leq 1} \left\|\sum_{k=0}^\infty \alpha_k x_k\right\|$ forms the Banach space $\ell_1^{\mathrm{unc}}(X)$. There is a canonical way to identify $\ell_1^{\mathrm{unc}}(X)$ with the completion of $\ell_1 \otimes_\varepsilon X$.

For the definition of Banach function spaces (= Köthe function spaces) E over measure spaces (Ω, μ) see [10]. Recall that a function $f : \Omega \to X$ is μ-measurable whenever it is an almost everywhere limit of a sequence of vector-valued step functions. The vector-valued Banach function space $E(X)$ consists of all (μ-equivalence classes of) μ-measurable functions $f : \Omega \to X$ such that $\|f\|_X \in E$, a vector space which together with the norm $\|f\|_{E(X)} = \|\|f\|_X\|_E$ forms a Banach space. For $E = L_p(\mu)$ this construction leads to the well-known space $L_p(\mu, X)$ of Bochner-integrable functions. For $\Omega = \{0, 1, \ldots, n\}$ with the discrete measure we as usual write ℓ_p^n instead of $L_p(\mu)$ (here $\dim \ell_p^n = n + 1$).

3 Summation processes and maximizing matrices

By a result of Toeplitz, summation methods S (see the introduction for a definition) can be characterized through the following two conditions

$$\lim_j \sum_k s_{jk} = 1 \quad \text{and} \quad \lim_j s_{jk} = 0 \quad \text{for all} \quad k\,, \tag{3.1}$$

and it can be shown easily that these conditions even assure that (1.2) holds for all convergent series $\sum_k x_k$ in a Banach space X. A sequence (x_k) in a Banach space X for which the sequence $(\sum_k s_{jk} x_k)_j$ converges, is said to be *S-summable*. The identity matrix is the first important example of a summation process. Let us recall three other fundamental methods – Cesaro, Riesz and Abel summation.

The *Cesaro matrix* C with

$$c_{jk} := \begin{cases} \dfrac{1}{j+1} & k \leq j \\ 0 & k > j \end{cases} \tag{3.2}$$

defines Cesaro-summation, and more generally for $\alpha > 0$ the *Cesaro matrix* C^α defined by

$$c^\alpha_{jk} := \begin{cases} \dfrac{A^{\alpha-1}_{j-k}}{A^\alpha_j} & k \leq j \\ 0 & k > j \end{cases} \tag{3.3}$$

gives Cesaro-summation of order α; here $A^\alpha_j := \binom{j+\alpha}{j}$ for $\alpha \in \mathbb{R}$. Next, if λ is a positive, strictly increasing and unbounded sequence of scalars, then the *Riesz matrices* R^λ are given by

$$r^\lambda_{jk} := \begin{cases} \dfrac{\lambda_{k+1} - \lambda_k}{\lambda_{j+1}} & k \leq j \\ 0 & k > j \end{cases} \tag{3.4}$$

Note that for $\lambda_j = j$ Riesz$^\lambda$-summation means Cesaro-summation, and for $\lambda_j = 2^j$ a sequence is Riesz$^\lambda$-summable iff it is summable. Finally, we define the *Abel matrices* A^ρ by

$$a^\rho_{jk} := \rho^k_j (1 - \rho_j)\ , \tag{3.5}$$

where ρ is a positive sequence which increases to 1. Recall that a scalar sequence (x_k) is said to be Abel-summable whenever the limit $\lim_{r\to 1} \sum_{k=0}^\infty x_k r^k$ exists. For $0 < r < 1$ we have $\sum_{k=0}^\infty x_k r^k = \sum_{k=0}^\infty (1-r) r^k \sum_{\ell=0}^k x_\ell$ which justifies our name for A^ρ. For more information on these summation methods see e.g. [1] and [19].

Now we define maximizing matrices, a definition crucial for our purpose. Let $A = (a_{jk})_{j,k\in\mathbb{N}_0}$ be a matrix satisfying $\|A\|_\infty := \sup_{jk} |a_{jk}| < \infty$, or equivalently A defines an operator from ℓ_1 into ℓ_∞ with norm $\|A\|_\infty$. We say that A is *maximizing* whenever for each orthonormal series $\sum_k \alpha_k x_k$ in $L_2(\mu)$ we have

$$\sup_j \Big| \sum_{k=0}^\infty a_{jk}\, \alpha_k\, x_k \Big| \in L_2(\mu)\,.$$

Clearly, by a closed graph argument A is maximizing if and only if the following maximal inequality holds: There is a constant $C > 0$ such that for all orthonormal series $\sum_k \alpha_k x_k$ in $L_2(\mu)$ we have

$$\Big\| \sup_j \Big| \sum_{j=0}^\infty a_{jk}\, \alpha_k\, x_k \Big| \Big\|_2 \leq C \|\alpha\|_2\,,$$

and the best of these constants is denoted by

$$\mathrm{m}(A) := \inf C\,.$$

In [5, section 1.7] it is proved that A is maximizing if and only if A as an operator from ℓ_1 into ℓ_∞ factorizes through a Hilbert space (is a Hilbertian operator), and in this case

$$\gamma_2(A) = \mathrm{m}(A)\,. \tag{3.6}$$

Let us finally list some examples of maximizing matrices. Note first that all matrix products of the form

$$S\,\Sigma\,D_{(1/\log(k+2))} \;=\; \Big(\frac{1}{\log(k+2)}\sum_{\ell=k}^{\infty} s_{j\ell}\Big)_{j,k} \tag{3.7}$$

are maximizing where S is some summation method, Σ the sum matrix defined by

$$\sigma_{jk} := \begin{cases} 1 & k \leq j \\ 0 & k > j\ , \end{cases}$$

and $D_{(1/\log(k+2))}$ the diagonal matrix with diagonal $(1/\log(k+2))_k$. Indeed, S by (3.1) defines a (bounded and linear) operator on ℓ_∞ which implies that

$$\sup_j \Big|\sum_{k=0}^{\infty} s_{jk} \sum_{\ell=0}^{k} \frac{\alpha_\ell}{\log(\ell+2)}\, x_\ell\Big| \leq \|S : \ell_\infty \to \ell_\infty\| \,\sup_k \Big|\sum_{\ell=0}^{k} \frac{\alpha_\ell}{\log(\ell+2)}\, x_\ell\Big|\,,$$

and hence the conclusion is an immediate consequence of the Menchof-Rademacher-Kantorovitch inequality (1.1).

In the particular case of Cesaro, Riesz and Abel summation the above diagonal sequence $(1/\log(k+2))_k$ (or Weyl sequence $(\log(k+2))_k$) can be improved. This follows from a careful analysis of the proofs for the famous coefficient tests for almost everywhere convergence of orthonormal series due to Kaczmarz [7], Kantorovitch [8], Menchoff [12], [13], Rademacher [16], and Zygmund [18]. For a detailed presentation of the following three facts see [5, section 2].

Example 3.1. The following matrices generated (as in (3.7)) by the Cesaro matrices C^α, Riesz matrices R^λ and Abel matrices A^ρ are maximizing:

$$R^\lambda\,\Sigma\,D_{(\frac{1}{\log\log\lambda_k})} = \left(\begin{cases} (1-\dfrac{\lambda_k}{\lambda_{j+1}})\dfrac{1}{\log\log\lambda_k} & k \leq j \\ 0 & k > j \end{cases}\right)_{jk} \tag{3.8}$$

$$C^\alpha\,\Sigma\,D_{(\frac{1}{\log\log(k+2)})} = \left(\begin{cases} \dfrac{A^\alpha_{j-k}}{A^\alpha_j}\dfrac{1}{\log\log(k+2)} & k \leq j \\ 0 & k > j \end{cases}\right)_{jk} \tag{3.9}$$

$$A^\rho\,\Sigma D_{(\frac{1}{\log\log(k+2)})} = \Big(\frac{\rho_j^k}{\log\log(k+2)}\Big)_{jk}\,. \tag{3.10}$$

4 The main result

It is remarkable that most of the classical almost everywhere summation theorems for orthonormal series in $L_2(\mu)$ can be extended without any further assumptions to spaces $L_p(\mu, X)$ of Bochner-integrable functions with values in a Banach space X, or even more generally, to vector-valued Banach function spaces $E(X)$ (where E is a Banach function space over a measure space (Ω, μ) and X a Banach space).

Theorem 4.1. *Let $E(X)$ be a vector-valued Banach function space. Let S be a summation method and ω a Weyl sequence with the additional property that for each orthonormal series $\sum_k \alpha_k x_k$ in $L_2(\mu)$ we have*

$$\sup_J \Big| \sum_{k=0}^{\infty} s_{jk} \sum_{\ell=0}^{k} \frac{\alpha_k}{\omega_\ell} x_\ell \Big| \in L_2(\mu) \ .$$

Then for each unconditionally convergent series $\sum_k x_k$ in $E(X)$ the following two statements hold:

(1) $\sup_j \Big\| \sum_{k=0}^{\infty} s_{jk} \sum_{\ell=0}^{k} \frac{x_\ell}{\omega_\ell} \Big\|_X \in E$

(2) $\sum_{k=0}^{\infty} \frac{x_k}{\omega_k} = \lim_j \sum_{k=0}^{\infty} s_{jk} \sum_{\ell=0}^{k} \frac{x_\ell}{\omega_\ell}$ *μ-a.e.*

Proof. By $E(X)[\ell_\infty(I)]$ (I some countable partially ordered index set) we denote all families $(x_i)_{i\in I}$ in $E(X)$ such that $\sup_i \|x_i\|_X \in E$. This vector space together with the norm $\|(x_i)\| := \big\| \sup_i \|x_i\|_X \big\|_E$ forms a Banach space. Note that (x_i) belongs to $E(X)[\ell_\infty(I)]$ if and only if there is a *factorization* $x_i = z_i\, f$ where (z_i) is a uniformly bounded sequence in $L_\infty(X)$ and $f \in E$. We will also need the closed subspace $E(X)[c_0(I)] \subset E(X)[\ell_\infty(I)]$ of all families (x_i) allowing a factorization $x_i = z_i\, f$ for which the z_i even form a zero sequence in $L_\infty(X)$.

We start with the proof of (1): Define the matrix

$$A = S\,\Sigma\, D_{1/\omega} = \Big(\frac{1}{\omega_k} \sum_{\ell=k}^{\infty} s_{j\ell} \Big)_{j,k} ,$$

and note that

$$\sum_{k=0}^{\infty} a_{jk}\, x_k = \sum_{k=0}^{\infty} s_{jk} \sum_{\ell=0}^{k} \frac{x_\ell}{\omega_\ell} \, .$$

We know from (3.6) that the maximizing matrix A as an operator from ℓ_1 into ℓ_∞ factorizes through a Hilbert space. By the identification of $\ell_1^{\mathrm{unc}}(E(X))$ with the completed ε-tensor product of $E(X)$ and ℓ_1 and a density argument (all finite sequences in $E(X)$ form a dense subspace of $\ell_1^{\mathrm{unc}}(E(X))$), all we have to show is that for each n

$$\| \operatorname{id}_{E(X)} \otimes A : E(X) \otimes_\varepsilon \ell_1^n \longrightarrow E(X)[\ell_\infty^n] \| \le K_G\, \gamma_2(A) < \infty \, ; \tag{4.1}$$

indeed, for $x = (x_k)_{k=0}^n \in E(X)^{n+1}$

$$w_1(x) = \Big\| \sum_{k=0}^{n} x_k \otimes e_k \Big\|_{E(X)\otimes_\varepsilon \ell_1^n}$$

and

$$\begin{aligned} \big(\operatorname{id}_{E(X)} \otimes A \big) \Big(\sum_k x_k \otimes e_k \Big) &= \sum_k x_k \otimes A e_k \\ &= \sum_k x_k \otimes \sum_j a_{jk} e_j = \sum_j \Big(\sum_k a_{jk} x_k \Big) \otimes e_j \, , \end{aligned}$$

therefore

$$\Big\| \sup_j \Big\| \sum_k a_{jk} x_k \Big\|_X \Big\|_E = \Big\| \mathrm{id}_{E(X)} \otimes A \Big(\sum_k x_k \otimes e_k \Big) \Big\|_{E(X)[\ell_\infty^n]} \le K_G \, \gamma_2(A) \, w_1(x) \,.$$

Let us now prove (4.1). From (2.1) we know that

$$\iota(A) \le K_G \, \gamma_2(A)$$

hence we conclude from (2.2)

$$\| \mathrm{id}_{E(X)} \otimes A : E(X) \otimes_\varepsilon \ell_1^n \longrightarrow E(X) \otimes_\pi \ell_\infty^n \| \le K_G \, \gamma_2(A) \,.$$

But since

$$\| \mathrm{id} : E(X) \otimes_\pi \ell_\infty^n \longrightarrow E(X)[\ell_\infty^n] \| \le 1 \,,$$

we obtain as desired (4.1). This completes the proof of (1).

It remains to prove (2): Note first that (1) and an easy closed graph argument yield that the linear mapping

$$\begin{aligned} \Phi : \ell_1^{\mathrm{unc}}(E(X)) &\longrightarrow E(X)[\ell_\infty(\mathbb{N}_0 \times \mathbb{N}_0)] \\ (x_k)_{k=0}^\infty &\rightsquigarrow \Big(\sum_k a_{ik} x_k - \sum_k a_{jk} x_k \Big)_{(i,j)} \end{aligned}$$

is well-defined and bounded.

Our aim is to show that Φ has its values in the closed subspace $E(X)[c_0(\mathbb{N}_0 \times \mathbb{N}_0)]$. By continuity it suffices to prove that, given a finite sequence $x = (x_0, \dots, x_k, 0, \dots)$ of functions in $E(X)$, the sequence $\Phi x \in E(X)[c_0(\mathbb{N}_0 \times \mathbb{N}_0)]$. Clearly, $(x_k)_{0 \le k \le k_0} \in E(X)[\ell_\infty]$, and hence there are $z_k \in L_\infty(X)$ with $\|z_k\|_\infty \le 1$ and $f \in E$ satisfying $x_k = z_k f$ for all k. But then for all i, j

$$\sum_{k=0}^{k_0} a_{ik} x_k - \sum_{k=0}^{k_0} a_{jk} x_k = \sum_{k=0}^{k_0} (a_{ik} - a_{jk}) x_k = \Big(\sum_{k=0}^{k_0} (a_{ik} - a_{jk}) z_k \Big) f \,,$$

and moreover

$$\Big\| \sum_{k=0}^{k_0} (a_{ik} - a_{jk}) z_k \Big\|_\infty \le \sum_{k=0}^{k_0} |a_{ik} - a_{jk}| \|z_k\|_\infty \le \sum_{k=0}^{k_0} |a_{ik} - a_{jk}| \,.$$

On the other hand, we have that each column of A viewed as a sequence converges $\Big($ by (3.1) we know for each k that $\lim_j a_{jk} = \lim_j \frac{1}{\omega_k} \sum_{\ell=k}^\infty s_{j\ell} = \lim_j \Big(\frac{1}{\omega_k} \sum_{\ell=0}^\infty s_{j\ell} - \frac{1}{\omega_k} \sum_{\ell=0}^{k-1} s_{j\ell} \Big) = \frac{1}{\omega_k} \Big)$. Hence, $\Big(\sum_k a_{ik} x_k - \sum_k a_{jk} x_k \Big)_{(i,j)}$ is contained in $E(X)[c_0(\mathbb{N}_0 \times \mathbb{N}_0)]$.

As a consequence, for every unconditionally convergent series $\sum_k x_k$ in $E(X)$ the sequence $(y_{(i,j)}) = \Big(\sum_k a_{ik} x_k - \sum_k a_{jk} x_k \Big)_{(i,j)}$ in fact belongs to $E(X)[c_0(\mathbb{N}_0 \times \mathbb{N}_0)]$, i.e., there is a factorization $y_{(i,j)} = z_{(i,j)} f$ where $(z_{(i,j)})$ is a zero sequence in $L_\infty(X)$ and $f \in E$. Clearly, this implies that the sequence $\Big(\sum_k a_{jk} x_k \Big)_j$ in $E(X)$ is an almost everywhere Cauchy sequence. This completes the proof. ∎

We illustrate the preceding theorem by the following collection of results on summation of unconditionally convergent series $\sum_k x_k$ in vector-valued Banach function spaces $E(X)$.

Corollary 4.2. *Let $\sum_k x_k$ be an unconditionally convergent series in a vector-valued Banach function space $E(X)$. Then*

(1) $\sup_j \Big\| \sum_{k=0}^j \dfrac{x_k}{\log(k+2)} \Big\|_X \in E$

(2) $\sup_j \Big\| \sum_{k=0}^j \dfrac{\lambda_{k+1}-\lambda_k}{\lambda_{j+1}} \sum_{\ell=0}^k \dfrac{x_\ell}{\log\log\lambda_\ell} \Big\|_X \in E$ *for every strictly increasing, unbounded and positive sequence (λ_k) of scalars*

(3) $\sup_j \Big\| \sum_{k=0}^j \dfrac{A_{j-k}^{r-1}}{A_j^r} \sum_{\ell=0}^k \dfrac{x_\ell}{\log\log\ell} \Big\|_X \in E$ *for every $r>0$*

(4) $\sup_j \Big\| \sum_{k=0}^\infty \rho_j^k \dfrac{x_k}{\log\log k} \Big\|_X \in E$ *for every positive strictly increasing sequence (ρ_j) converging to 1.*

Moreover, in each of these cases

$$\sum_{k=0}^{\infty} \frac{x_k}{\omega_k} = \lim_j \sum_{k=0}^{\infty} s_{jk} \sum_{\ell=0}^{k} \frac{x_\ell}{\omega_\ell} \quad \mu - a.e.\,,$$

where the summation method S is either given by the identity, Riesz$^\lambda$, Cesaro^r, or Abel$^\rho$ matrix, and ω is the related Weyl sequence from (1) *up to* (4).

Let us give some references for this result whenever $E = L_p(\mu)$ and $X = \mathbb{K}$: In this case the origin of 4.2 (1) is in the article of Kwapien and Pelczynski [9] where a slightly weaker result is shown, and its final form is due to Bennett [2] and independently Maurey-Nahoum [11]. The result was then reproved by Orno in [15] with a method related to ours. But we do not see how the method in Orno should be modified in order to yield (1) in the above more general (vector-valued) formulation. Statement (2) for $E = L_p(\mu)$ and $X = \mathbb{K}$ is again a result due to Bennett from [2], and it is important to remark that he gives a in a sense elementary and direct argument. The statements (3) and (4) are new, even for $E(X) = L_p(\mu)$. Recall that the underlying models for all four results are well-known almost everywhere coefficient tests for orthogonal series due to Kaczmarz [7], Menchoff [12], [13], Rademacher [16], and Zygmund [18]; as already mentioned, the required maximal inequality for orthonormal series needed for the proof of (1) is a result of Kantorovitch [8], whereas for (2) up to (4) this inequality follows from a careful inspection of the classical results (for details see again [5, Chapter 1]).

References

[1] G. Alexits, *Convergence problems of othogonal series*, Pergamon Press, 1961.

[2] G. Bennett, *Unconditional convergence and almost everywhere convergence*, Z. Wahrscheinlichkeitstheorie und verw. Gebiete **34** (1976), 135-155.

[3] G. Bennett, *Schur multipliers*, Duke Mathematical Journal **44,3** (1977), 603-639.

[4] A. Defant and K . Floret, *Tensor Norms and Operator Ideals*, North–Holland Math. Studies **176**, 1993.

[5] A. Defant and M. Junge, *Classical summation methods in non-commutative probability*, in preparation

[6] J. Diestel, H. Jarchow, and A. Tonge, *Absolutely summing operators*, Cambridge Studies in Advanced Mathematics 43, 1995.

[7] S. Kaczmarz, *Über die Summierbarkeit der Orthogonalreihen*, Math. Zeitschrift **26** (1927), 99-105.

[8] L. Kantorovitch, *Some theorems on the almost everywhere convergence*, Dokl. Akad. Nauk USSR, **14** (1937), 537-540.

[9] S. Kwapień and A. Pełczyński, *The main triangle projection in matrix spaces and its applications*, Studia Math. **34** (1974), 43-68.

[10] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I and II*, Springer-Verlag, 1996.

[11] B. Maurey and A. Nahoum, *Applications radonifiantes dans l'espaces des séries convergentes*, C. R. Acad. Sci. Paris Sér.A-B **276** (1973), 751–754.

[12] D. Menchoff, *Sur les séries de fonctions orthogonales I*, Fund. Math. **4** (1923), 82-105.

[13] D. Menchoff, *Sur les séries de fonctions orthogonales II*, Fund. Math. **8** (1926), 56-108.

[14] D. Menchoff, *Sur la sommation des séries orthogonales*, Comptes Rendus Acad. Sci. Paris **180** (1925), 2011-2013.

[15] P. Orno, *A note on unconditionally converging series in L_p*, Proc. Amer. Math. Soc. **59,2** (1976), 251-254.

[16] H. Rademacher, *Einige Sätze über Reihen von allgemeinen Orthogonalfunktionen*, Math. Ann. **87** (1922), 112-138.

[17] P. Wojtaszczyk, *Banach spaces for analysts*, Cambridge Studies in Advanced Mathematics **25**, 1991.

[18] A. Zygmund, *Sur la sommation de séries de fonctions orthogonales*, Bulletin Intern. Acad. Polonaise Sci. Lettres (Cracovie), série **A** (1927), 293-308.

[19] A. Zygmund, *Trigonometric series I and II*, second edition, Cambridge University Press, 1927.

Institute of mathematics, Carl von Ossietzky University,
D–26111 Oldenburg, Germany
email:defant@mathematik.uni-oldenburg.de

Department of Mathematics, University of Illinois at Urbana-Champaign,
Urbana,IL 61801, USA
email:junge@math.uiuc.edu