

Isomorphism of spaces of analytic functions on n -circular domains

P. Chalov V. Zahariuta

Abstract

The space $A(D)$ of all analytic functions in a complete n -circular domain D in \mathbb{C}^n , $n \geq 2$, is considered with a natural Fréchet topology. Some sufficient conditions for the isomorphism of such spaces are obtained in terms of certain subtle geometric characteristic of domains D . This investigation complements essentially the second author's result [8] on necessary geometric conditions of such isomorphisms.

1 Introduction

By $A(D)$ we denote the Fréchet space of all analytic functions in a domain $D \in \mathcal{C}^n$ with the natural topology of the uniform convergence on compact subsets of D . We study the isomorphic classification of the spaces $A(D)$ with D from the class \mathcal{R}^n of all *complete logarithmically convex n -circular* (Reinhardt) domains in \mathbb{C}^n , $n \geq 2$ (see also, [1, 7, 8, 10]). We represent the system of monomials $z^k := z_1^{k_1} \dots z_n^{k_n}$, $k = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$, which forms an absolute basis in each space $A(D)$, $D \in \mathcal{R}^n$, as a sequence

$$e_i(z) := z^{k(i)}, \quad i \in \mathbb{N}, \quad (1)$$

so that $|k(i)| := k_1(i) + \dots + k_n(i)$ does not decrease. The characteristic function of a domain $D \in \mathcal{R}^n$: $h_D(\theta) := \sup \left\{ \sum_{\nu=1}^n \theta_\nu \ln |z_\nu| : z = (z_\nu) \in D \right\}$, defined on the simplex $\Sigma := \left\{ \theta = (\theta_\nu) \in \mathbb{R}_+^n : \sum_{k=1}^n \theta_k = 1 \right\}$, is convex (hence continuous) on the convex set $\pi(D) := \{ \theta \in \Sigma : h_D(\theta) < \infty \}$. It turns out that invariant properties of spaces $A(D)$ depend essentially on the topological behavior of the set $\pi(D)$, for example, $A(D)$ is not isomorphic to $A(G)$ if $\pi(D)$ is relatively open in Σ but $\pi(G) \neq \Sigma$ is closed. In what follows we restrict ourselves to the class \mathcal{R}_o^n of domains D for

which $\pi(D)$ is relatively open in Σ , $\pi(D) \neq \Sigma$ (if $\pi(D) = \Sigma$, then $A(D) \simeq A(\mathbb{U}^n)$ [1, 7]). In order to investigate the isomorphic classification for this class it is convenient to introduce the following geometric characteristic of those domains:

$$g(\alpha) := g_D(\alpha) := \left(\frac{n! \operatorname{mes} \Sigma}{\operatorname{mes} \pi(D)} \right)^{1/n} \chi^{-1}(\alpha), \quad 0 < \alpha \leq 1, \tag{2}$$

where $\chi(t) := \frac{\operatorname{mes} \{\theta \in \pi(D) : h_D(\theta) \geq t\}}{\operatorname{mes} \pi(D)}$, $t \geq t_0 := \min_{\theta \in \pi(D)} \{h_D(\theta)\}$ and mes is the Lebesgue measure on Σ .

Using this characteristic, the following necessary condition for the isomorphism of spaces from the class $\mathcal{A}_o^n := \{A(D) : D \in \mathcal{R}_o^n\}$ was obtained in [8].

Proposition 1. *Given domains $D, \widetilde{D} \in \mathcal{R}_o^n$ and $A(D) \simeq A(\widetilde{D})$, then*

$$\exists c : \frac{1}{c} g_D(c\alpha) \leq g_{\widetilde{D}}(\alpha) \leq c g_D\left(\frac{\alpha}{c}\right), \quad 0 < \alpha \leq \frac{1}{c}.$$

As a corollary, it was proved in [8] that there is a *continuum of pairwise nonisomorphic spaces* in \mathcal{A}_o^n . Here we represent, *in terms of the same characteristic* (2), some sufficient conditions for the isomorphism of those spaces. A distinction must be made between two types of domains from \mathcal{A}_o^n , described by one of the conditions:

$$(a) \operatorname{mes}(\Sigma \setminus \pi(D)) = 0; \quad (b) \operatorname{mes}(\Sigma \setminus \pi(D)) > 0. \tag{3}$$

It turns out that the spaces $A(D)$ and $A(\widetilde{D})$ are not isomorphic for domains of different type (see, Proposition 5 and Remark 6).

Theorem 2. *Suppose $D, \widetilde{D} \in \mathcal{R}_o^n$, $g(\alpha) := g_D(\alpha)$, $\tilde{g}(\alpha) := g_{\widetilde{D}}(\alpha)$ and $\sigma : [0, q] \rightarrow [0, 1]$, $0 < q < 1$, is the continuous increasing function, which is continuously differentiable on $(0, q]$ and satisfies the differential equation*

$$\sigma'(\alpha) = \left(\frac{\tilde{g}(\sigma(\alpha))}{g(\alpha)} \right)^n, \quad 0 < \alpha \leq q, \tag{4}$$

with the initial condition $\sigma(0) = 0$. If there is a constant $L > 0$ such that

$$\frac{1}{L} \leq \sigma'(\alpha) \leq L, \quad 0 < \alpha \leq q, \tag{5}$$

and both domains are of the same type (3), then $A(D)$ is isomorphic to $A(\widetilde{D})$; moreover, there is an isomorphism $T : A(D) \rightarrow A(\widetilde{D})$ such that $Te_i = t_i e_{\rho(i)}$, $i \in \mathbb{N}$, where e_i is the monomial basis (1), t_i a scalar sequence and $\rho : \mathbb{N} \rightarrow \mathbb{N}$ a bijection.

This theorem will be an immediate consequence of some more general result about the isomorphic classification on a certain class of Köthe spaces (see, Theorem 7 below).

2 Modeled Köthe spaces

Köthe space $K(A)$ defined by a Köthe matrix $A = (a_{i,p})_{i,p \in \mathbb{N}}$ (see, e.g., [4]) is the Fréchet space of all sequences $x = (\xi_i)_{i \in \mathbb{N}}$ such that $|x|_p := \sum_{i=1}^{\infty} |\xi_i| a_{i,p} < \infty$ for all $p \in \mathbb{N}$, equipped with the topology generated by these seminorms. An operator $T : K(A) \rightarrow K(\tilde{A})$ is called *quasidiagonal* (with respect to the canonical bases $e_i := (\delta_{i,j})_{j=1}^{\infty}$, $i \in \mathbb{N}$) if $Te_i = t_i e_{\sigma(i)}$, where (t_i) is a scalar sequence, $\sigma : \mathbb{N} \rightarrow \mathbb{N}$; if T is an isomorphism we say that the spaces $K(A)$ and $K(\tilde{A})$ are *quasidiagonally isomorphic*. Given $(a_i) \in \omega^+$ (where ω^+ is the set of all positive scalar sequences) and $\lambda = (\lambda_i)$, $\lambda_i \geq 1$, the space

$$F(\lambda, a) := K\left(\exp\left(\min\left\{p, \lambda_i - \frac{1}{p}\right\} a_i\right)\right), \tag{6}$$

is called *power Köthe space of second type* (in contrast to those spaces of first type [8, 9]); it is *Montel* if and only if $a_i \rightarrow +\infty$.

A. Grothendieck considered ([3], II,p.122) the important special classes of Köthe spaces:

$$E_{\alpha}(a) := K(\exp(\alpha_p a_i)), \tag{7}$$

where $a = (a_i)_{i \in \mathbb{N}} \in \omega^+$, $\alpha_p \uparrow \alpha$, $-\infty < \alpha \leq +\infty$. We will call them *power Köthe spaces of finite type* (if $\alpha < \infty$) or *infinite type* (if $\alpha = \infty$) (*centers of Riesz scales* in [5] or *power series spaces* in [6]).

The space (6) is quasidiagonally isomorphic to (i) *the space (7) of finite type* if λ_i is bounded, (ii) *the space (7) of infinite type* if $\lambda_i \rightarrow \infty$. Otherwise the space (6) is called *mixed power Köthe space of second type*; it is *essentially mixed* if it is not isomorphic quasidiagonally to a Cartesian product $E_0(b) \times E_{\infty}(c)$.

Proposition 3. ([9], Lemma 2.3). *Let (t_i) be a scalar sequence and $\rho : \mathbb{N} \rightarrow \mathbb{N}$ a bijection. Then the rule $Te_i = t_i e_{\rho(i)}$, $i \in \mathbb{N}$, defines a quasidiagonal isomorphism from a Montel space $F(\lambda, a)$ onto a space $F(\tilde{\lambda}, \tilde{a})$ if and only if the following assertions are valid: **(a)** $a_i \asymp \tilde{a}_{\rho(i)}$, i.e. $a_i/c \leq \tilde{a}_{\rho(i)} \leq ca_i$, $i \in \mathbb{N}$, with some constant $c > 1$; **(b)** $-\Delta \leq \frac{\ln |t_i|}{a_i} \leq \Delta$, $i \in \mathbb{N}$, with some constant $\Delta > 0$; **(c)** for any subsequence $I \subset \mathbb{N}$, such that $\lambda_i \rightarrow l \in [1, \infty]$, $\tilde{\lambda}_{\rho(i)} \rightarrow \tilde{l} \in [1, \infty]$, $\frac{\tilde{a}_{\rho(i)}}{a_i} \rightarrow \gamma$ as $i \rightarrow \infty$, $i \in I$, either $l = \tilde{l} = \infty$ or both of l and \tilde{l} are finite and $\lim \frac{\ln |t_i|}{a_i} = l - \tilde{l}\gamma$.*

The following fact (see, e.g., [9], Proposition 3.3) will be useful later.

Proposition 4. *Let $m_a(t) := |\{k : a_k \leq t\}|$, $m_b(t) := |\{k : b_k \leq t\}|$ be the counting functions of non-decreasing positive sequences $a = (a_i)$ and $b = (b_i)$. If $m_a(t) \leq m_b(Ct)$, $t > 0$, with some constant C , then $b_k \leq Ca_k$, $k \in \mathbb{N}$.*

With an eye to spaces from the class \mathcal{A}_0^n we deal with the following quite narrow subclass of power Köthe spaces of the second type dealing only with "thickly distributed" sequences $\lambda : \Phi^{(n)}(\varphi, g) := F\left((g(\varphi(i))), (i^{1/n})\right)$, where $g : (0, 1] \rightarrow \mathbb{R}_+$ is

a continuous function such that $\lim_{\xi \rightarrow 0} g(\xi) = \infty$ and $\varphi : \mathbb{N} \rightarrow (0, 1]$ is a function with *equidistributed values*, that is

$$\lim_{t \rightarrow \infty} \frac{|\{i \leq t : c < \varphi(i) \leq d\}|}{t} = d - c, \quad 0 \leq c < d \leq 1. \tag{8}$$

Given $D \in \mathcal{R}_o^n$ we divide the sequence $k(i)$ into two parts: the subsequence $l(i) = k(j_i)$ covering the set $\left\{k \in \mathbb{Z}_+^n : \frac{k}{|k|} \in \pi(D)\right\}$ and the complementary subsequence $m(i)$. By Lemma 2 from [8], certain asymptotics for the counting functions of the sequences $|k(i)|, |l(i)|, |m(i)|$ hold; from them, using Proposition 4, one can derive the asymptotics:

$$|k(i)| \sim (n!i)^{1/n}, \quad |l(i)| \sim \left(\frac{n! \operatorname{mes} \Sigma}{\operatorname{mes} \pi(D)} i\right)^{1/n}, \quad |m(i)| \asymp i^{1/d}, \quad i \rightarrow \infty, \tag{9}$$

where $d - 1 = \dim(\Sigma \setminus \pi(D))$. Define the function $\varphi = \varphi_D : \mathbb{N} \rightarrow (0, 1]$ by the formula

$$\varphi(i) := \chi(h_D(\theta(i))), \quad i \in \mathbb{N}, \tag{10}$$

where $\theta(i) := \frac{l(i)}{|l(i)|}$, $i \in \mathbb{N}$. To prove that φ is a function with equidistributed values we use the asymptotics ($\tau \rightarrow \infty$):

$$\left|\left\{i : |l(i)| \leq \tau, \chi^{-1}(d) \leq h_D(\theta(i)) \leq \chi^{-1}(c)\right\}\right| \sim \frac{(d - c) \operatorname{mes} \pi(D) \tau^n}{n! \operatorname{mes} \Sigma},$$

which follows from [8], Lemma 2. Then, taking into account (9), (10) and putting $t = \frac{\operatorname{mes} \pi(D) \tau^n}{n! \operatorname{mes} \Sigma}$, we arrive at (8).

A space $A(D) \in \mathcal{A}_o^n$ is represented as a direct sum of two closed basis subspaces $L(D) := \overline{\operatorname{span}}\{z^{l(i)} : i \in \mathbb{N}\}$ and $M(D) := \overline{\operatorname{span}}\{z^{m(i)} : i \in \mathbb{N}\}$. Due to the asymptotics (9) for $|m(i)|$, the space $M(D)$ is isomorphic to the space $E_\infty(i^{1/d})$. On the other hand, since by Proposition 3 $F(\lambda, ca) = F(c\lambda, a)$, $c > 0$, we obtain that the space $L(D)$ is isomorphic to the space $\Phi^{(n)}(\varphi, g)$ with φ and g defined in (10) and (2). Since the space $E_\infty(i^{1/d})$ is contained in $\Phi^{(n)}(\varphi, g)$ as a basic subspace if $d < n$ (what is the same, if $\operatorname{mes}(\Sigma \setminus \pi(D)) = 0$) we obtain the following statement.

Proposition 5. *Suppose $D \in \mathcal{R}_o^n$ and φ, g are defined in (10), (2). Then $A(D) \simeq \Phi^{(n)}(\varphi, g)$ if $\operatorname{mes}(\Sigma \setminus \pi(D)) = 0$ and $A(D) \simeq \Phi^{(n)}(\varphi, g) \times E_\infty\left(\left(i^{\frac{1}{n}}\right)\right)$, otherwise.*

Remark 6. *The spaces $\Phi^{(n)}(\varphi, g) \times E_\infty\left(\left(i^{\frac{1}{n}}\right)\right)$ and $\Phi^{(n)}(\varphi, \tilde{g})$ are not quasidiagonally isomorphic for any functions g, \tilde{g} , because the second space contains no basic subspace isomorphic to $E_\infty\left(\left(i^{\frac{1}{n}}\right)\right)$. In fact, these spaces are not isomorphic ([2]), but the proof of this fact is not the aim of the present paper.*

Proposition 5 reduces Theorem 2 to the following more general result which will be proved in section 4.

Theorem 7. *Suppose $g(\alpha), \tilde{g}(\alpha)$ are two continuous functions on $(0, 1]$ tending to ∞ as $\alpha \rightarrow 0$; $\varphi, \tilde{\varphi}$ are mappings from \mathbb{N} onto $(0, 1]$ with equidistributed values and $\sigma : [0, q] \rightarrow [0, 1], 0 < q < 1$, is the continuous increasing function, which satisfies the differential equation (4) with the initial condition $\sigma(0) = 0$. If the condition (5) holds, then the spaces $\Phi^{(n)}(\varphi, g)$ and $\Phi^{(n)}(\tilde{\varphi}, \tilde{g})$ are quasideagonally isomorphic.*

3 Main Lemma

Lemma 8. *Let α, β be two functions from \mathbb{N} to $(0, 1]$ with equidistributed values. Let $\sigma : [0, 1] \rightarrow [0, 1]$ be an increasing continuous function, continuously differentiable on $(0, 1]$, such that $\sigma(0) = 0, \sigma(1) = 1$. Suppose that the condition (5) is fulfilled with $q = 1$. Then there exists a bijection $\rho : \mathbb{N} \rightarrow \mathbb{N}$, satisfying the conditions: (i) $i \asymp \rho(i)$; (ii) $\beta(\rho(i_k)) \rightarrow \sigma(a), \frac{i_k}{\rho(i_k)} \rightarrow \sigma'(a)$ for each $a \in (0, 1]$ and any subsequence (i_k) such that $\alpha(i_k) \rightarrow a$.*

Proof. First we set $\alpha_\nu^{(s)} := \frac{\nu}{2^s}, \beta_\nu^{(s)} := \sigma(\alpha_\nu^{(s)}), \nu = \overline{0, 2^s}, s \in \mathbb{Z}_+$. By (5) we have

$$\frac{1}{L} \leq d_\nu^{(s)} := \frac{\beta_\nu^{(s)} - \beta_{\nu-1}^{(s)}}{\alpha_\nu^{(s)} - \alpha_{\nu-1}^{(s)}} \leq L, \nu = \overline{1, 2^s}, s \in \mathbb{N}. \tag{11}$$

Take any sequence $\varepsilon_s \downarrow 0$ with $\varepsilon_1 \leq 1/6$. Since the functions α and β are equidistributed, for each $s \in \mathbb{N}$ we find T_s such that for $t \geq T_s, \nu = \overline{1, 2^s}, s \in \mathbb{N}$, the counting functions $n_\nu^{(s)}(t) := \left| \left\{ i \leq t : \alpha_{\nu-1}^{(s)} < \alpha(i) \leq \alpha_\nu^{(s)} \right\} \right|, m_\nu^{(s)}(t) := \left| \left\{ i \leq t : \beta_{\nu-1}^{(s)} < \beta(i) \leq \beta_\nu^{(s)} \right\} \right|$ satisfy the estimates

$$\begin{aligned} t(1 - \varepsilon_s) \left(\alpha_\nu^{(s)} - \alpha_{\nu-1}^{(s)} \right) &\leq n_\nu^{(s)}(t) \leq t(1 + \varepsilon_s) \left(\alpha_\nu^{(s)} - \alpha_{\nu-1}^{(s)} \right), \\ t(1 - \varepsilon_s) \left(\beta_\nu^{(s)} - \beta_{\nu-1}^{(s)} \right) &\leq m_\nu^{(s)}(t) \leq t(1 + \varepsilon_s) \left(\beta_\nu^{(s)} - \beta_{\nu-1}^{(s)} \right) \end{aligned} \tag{12}$$

Now introduce the sets $N_\nu^{(s)} = \left\{ i \in \mathbb{N} : \alpha_{\nu-1}^{(s)} < \alpha(i) \leq \alpha_\nu^{(s)}, a_s < i \leq a_{s+1} \right\}, \nu = \overline{1, 2^{s-1}}, s \in \mathbb{Z}_+$, where the sequence a_s is chosen so that

$$a_0 = 0, 2LT_s \leq a_s \leq \frac{\varepsilon_s a_{s+1}}{8L^2}, s \in \mathbb{N}, \tag{13}$$

and the sets $M_\nu^{(s)} = \left\{ i \in \mathbb{N} : \beta_{\nu-1}^{(s)} < \beta(i) \leq \beta_\nu^{(s)}, b_{\zeta(\nu)}^{(s)} < i \leq b_\nu^{(s+1)} \right\}, \nu = \overline{1, 2^{s-1}}, s \in \mathbb{Z}_+$ where $\zeta(\nu)$ is equal to the integral part of $\frac{\nu+1}{2}$ and the parameters $b_1^{(0)} = 0, b_\nu^{(s)}, \nu = \overline{1, 2^{s-1}}, s \in \mathbb{N}$, are chosen so that

$$\left| N_\nu^{(s)} \right| = \left| M_\nu^{(s)} \right| =: K(\nu, s), \nu = \overline{1, 2^{s-1}}, s \in \mathbb{Z}_+. \tag{14}$$

Represent the sets $N_\nu^{(s)}, M_\nu^{(s)}$ in the form of increasing finite sequences: $i_k^{(\nu, s)}$ and $j_k^{(\nu, s)}$ with $k = \overline{1, K(\nu, s)}$ and construct the bijection $\rho : \mathbb{N} \rightarrow \mathbb{N}$ by the rule $\rho \left(i_k^{(\nu, s)} \right) := j_k^{(\nu, s)}, k = \overline{1, K(\nu, s)}, \nu = \overline{1, 2^{s-1}}, s \in \mathbb{Z}_+$. Let us show that this is

the desired mapping. Using (13), (14), (12), (11), one can easily check by induction that

$$b_\nu^{(s)} \geq \frac{a_s}{2L}, \quad \nu = \overline{1, 2^{s-1}}, \quad s \in \mathbb{N}. \tag{15}$$

Let us check the conditions (i), (ii). Setting $r_s := \frac{1 + \varepsilon_s}{1 - 2\varepsilon_s}$, and applying (14), (12), we obtain the inequalities

$$\frac{a_s}{r_{s-1}d_{\zeta(\nu)}^{(s-1)}} \leq b_\nu^{(s)} \leq \frac{r_{s-1}a_s}{d_{\zeta(\nu)}^{(s-1)}}, \quad \nu = \overline{1, 2^{s-1}}, \quad s \in \mathbb{N}. \tag{16}$$

The counting functions for the finite sequences $i_k^{(\nu,s)}$ and $j_k^{(\nu,s)}$, $k = \overline{1, K(\nu, s)}$ can be written in the following form

$$\begin{aligned} p_\nu^{(s)}(t) &= \max \left\{ 0, \min \left\{ n_\nu^{(s)}(t) - n_\nu^{(s)}(a_s), K(\nu, s) \right\} \right\} \\ q_\nu^{(s)}(t) &= \max \left\{ 0, \min \left\{ m_\nu^{(s)}(t) - m_\nu^{(s)}(b_{\zeta(\nu)}^{(s)}), K(\nu, s) \right\} \right\} \end{aligned} \tag{17}$$

Due to (17), (12), (16), we obtain, for $a_s < t \leq a_{s+1}$, the estimates

$$\begin{aligned} p_\nu^{(s)}(t) &\leq ((1 + \varepsilon_s)t - (1 - \varepsilon_s)a_s) (\alpha_\nu^{(s)} - \alpha_{\nu-1}^{(s)}) \\ &\leq \left(\frac{(1 + \varepsilon_s)t}{d_\nu^{(s)}} - \frac{(1 - \varepsilon_s)b_{\zeta(\nu)}^{(s)}d_{\zeta(\nu)}^{(s-1)}}{r_{s-1}d_\nu^{(s)}} \right) (\beta_\nu^{(s)} - \beta_{\nu-1}^{(s)}) \\ &\leq m_\nu^{(s)} \left(\frac{h_\nu^{(s)}t}{d_\nu^{(s)}} \right) - m_\nu^{(s)}(b_{\zeta(\nu)}^{(s)}) = q_\nu^{(s)} \left(\frac{h_\nu^{(s)}t}{d_\nu^{(s)}} \right), \end{aligned} \tag{18}$$

where

$$h_\nu^{(s)} = \frac{(1 + \varepsilon_s)r_{s-1} + 2L \left| (1 - \varepsilon_s)d_{\zeta(\nu)}^{(s-1)} - (1 + \varepsilon_s)r_{s-1}d_\nu^{(s)} \right|}{(1 - \varepsilon_s)r_{s-1}d_\nu^{(s)}}. \tag{19}$$

Analogously, we obtain the estimate:

$$q_\nu^{(s)}(t) \leq p_\nu^{(s)}(g_\nu^{(s)}d_\nu^{(s)}t), \tag{20}$$

with

$$g_\nu^{(s)} = \frac{(1 + \varepsilon_s)r_{s-1}d_{\zeta(\nu)}^{(s-1)}d_\nu^{(s)} + 2L \left| (1 - \varepsilon_s)d_\nu^{(s)} - (1 + \varepsilon_s)r_{s-1}d_{\zeta(\nu)}^{(s-1)} \right|}{(1 - \varepsilon_s)r_{s-1}d_{\zeta(\nu)}^{(s-1)}}. \tag{21}$$

By Lemma 4 and (18), (20) we have

$$\frac{i_k^{(\nu,s)}}{g_\nu^{(s)}} \leq d_\nu^{(s)}j_k^{(\nu,s)} \leq h_\nu^{(s)}i_k^{(\nu,s)} \tag{22}$$

for $k = \overline{1, K(\nu, s)}$; $\nu = \overline{1, 2^{s-1}}$; $s \in \mathbb{N}$. Taking into account (11), the definitions of the numbers $h_\nu^{(s)}$ and $g_\nu^{(s)}$ and (22), we obtain that there is a constant M independent of ν and s such that $p_\nu^{(s)}(t) \leq q_\nu^{(s)}(Mt)$, $q_\nu^{(s)}(t) \leq p_\nu^{(s)}(Mt)$, $t > 0$. Thus, the mapping $\rho : \mathbb{N} \rightarrow \mathbb{N}$ is constructed so that the condition (i) is fulfilled.

It remains to check the condition (ii). Take any subsequence (i_n) such that $\alpha(i_n) \rightarrow a \in (0, 1]$. For every n we find $s = s(n)$, $\nu = \nu(n)$ and $k = k(n)$ such

that $i_n = i_{k(n)}^{(\nu(n),s(n))} \in N_{\nu(n)}^{(s(n))}$. Then $\alpha_{\nu(n)-1}^{(s(n))} < \alpha(i_n) \leq \alpha_{\nu(n)}^{(s(n))}$ and $\alpha_{\nu(n)}^{(s(n))} \rightarrow a$. By the construction, $\rho(i_n) \in M_{\nu(n)}^{(s(n))}$, therefore $\beta_{\nu(n)-1}^{(s(n))} < \beta(\rho(i_n)) \leq \beta_{\nu(n)}^{(s(n))}$. Hence, by smoothness of σ , we have

$$\lim_{n \rightarrow \infty} \beta(\rho(i_n)) = \sigma(a), \quad \lim_{n \rightarrow \infty} d_{\nu(n)}^{(s(n))} = \lim_{n \rightarrow \infty} d_{\nu(n)}^{(s(n)-1)} = \sigma'(a). \tag{23}$$

Then, taking into account (19), (21), (23), we conclude that $\lim_{n \rightarrow \infty} h_{\nu(n)}^{(s(n))} = \lim_{n \rightarrow \infty} g_{\nu(n)}^{(s(n))} = 1$. Combining this with (22), (23), we obtain that $i_{k(n)}^{(\nu(n),s(n))} \sim \sigma'(a) j_{k(n)}^{(\nu(n),s(n))}$. Hence the condition (ii) is also proved. The proof is complete. \blacksquare

4 Proof of Theorem 7

Lemma 9. *Let $\varphi, \tilde{\varphi}$ be two functions from \mathbb{N} to $(0, 1]$ with equidistributed values and $g : (0, 1] \rightarrow \mathbb{R}_+$ a decreasing continuous function such that $g(\xi) \rightarrow +\infty$ as $\xi \rightarrow 0$. Then $\Phi^{(n)}(\varphi, g) = F(g(\varphi(i)), (i^{\frac{1}{n}}))$ is quasidiagonally isomorphic to $\Phi^{(n)}(\tilde{\varphi}, g) = F(g(\tilde{\varphi}(i)), (i^{\frac{1}{n}}))$.*

Proof. Assume that the mapping σ in Lemma 8 is the identity. Then the bijection $\rho : \mathbb{N} \rightarrow \mathbb{N}$, constructed there, satisfies the condition $i \asymp \rho(i)$ and for any subsequence i_k such that $\varphi(i_k) \rightarrow \alpha \neq 0$ the conditions $\tilde{\varphi}(\rho(i_k)) \rightarrow \alpha$ and $i_k \sim \rho(i_k)$ hold. Then, by Proposition 3, the operator $T : \Phi^{(n)}(\varphi, g) \rightarrow \Phi^{(n)}(\tilde{\varphi}, g)$ defined by $Te_i = e_{\rho(i)}$, $i \in \mathbb{N}$, is a required isomorphism. \blacksquare

Proof of Theorem 7. By Lemma 9, we assume that $\tilde{\varphi} = \varphi$. Let us introduce the functions $G(\alpha) := \int_0^\alpha \frac{d\lambda}{(g(\lambda))^\pi}$, $\tilde{G}(\alpha) := \int_0^\alpha \frac{d\lambda}{(\tilde{g}(\lambda))^\pi}$ and choose $q \in (0, 1)$ so that $G(q) < \tilde{G}(1)$. Then $\tilde{q} := \tilde{G}^{-1}(G(q)) < 1$ and the function $\sigma := \tilde{G}^{-1} \circ G : [0, q] \rightarrow [0, \tilde{q}]$ is continuous on $[0, q]$, continuously differentiable on $(0, q]$ and satisfies the equation (4) and the condition $\sigma(0) = 0$. We extend the function σ to a bijection of the interval $[0, 1]$ onto itself preserving continuous differentiability and denote this mapping by the same symbol σ . The constructed mapping meets the conditions of Lemma 8, hence there is a bijection $\rho : \mathbb{N} \rightarrow \mathbb{N}$ satisfying the conditions (i), (ii) of this lemma. Applying Proposition 3, one can easily check that a required isomorphism can be realized as the quasidiagonal operator defined by $Te_i := e_{\rho(i)}$ for $0 < \varphi(i) \leq q$, and by $Te_i := \left(\exp\left(g(\varphi(i)) i^{1/n} - \tilde{g}(\varphi(\rho(i)))\right) (\rho(i))^{1/n} \right) e_{\rho(i)}$ for the rest of i 's.

References

- [1] Aizenberg L.A. and Mityagin B.S., Spaces of analytic functions in multicircular domains. *Sib. Mat. Zh.* **1** (1960), 153–170 (Russian).
- [2] Chalov P., Zahariuta V., The structure of basic subspaces for power Köthe spaces of the second type. Preprint.
- [3] Grothendieck A., Produits tensoriels topologiques et espaces nucléaires. *Mem. Amer. Math. Soc.* **16**, Providence (1955).
- [4] Meise R. and Vogt D., *Introduction to Functional Analysis*. Clarendon Press, Oxford, 1997.
- [5] Mityagin B.S., Approximative dimension and bases in nuclear spaces. *Usp. Mat. Nauk* **16** (1961), 63–132 (Russian).
- [6] Pietsch A., *Nukleare Lokalkonvexe Räume*. Akademie-Verlag, Berlin, (1965).
- [7] Rolewicz S., On spaces of analytic functions, *Studia Math.* **21** (1962), 135–160.
- [8] Zahariuta V.P., Generalized Mityagin invariants and a continuum of pairwise non-isomorphic spaces of analytic functions. *Funct. Anal. and Appl.* **11** (1977), 24–30 (Russian).
- [9] Zahariuta V.P., Linear topological invariants and their applications to isomorphic classification of generalized power spaces. *Tr. J. Math.* **20** (1996), 237–288.
- [10] Zahariuta V.P. and Shubarin M. A., On isomorphism of spaces of analytic functions in unbounded multicircular domains, in: *Differential, Integral Equations and Complex Analysis*, Elista, Kalmuk State University (1988), 34–44 (Russian).

Department of Mathematics, Rostov State University,
344090 Rostov-on-Don, Russia
Sabanci University, Orhanli, 34956 Tuzla/Istanbul, Turkey