A metric dependent Hilbert transform in Clifford analysis

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Abstract

In earlier research generalized multidimensional Hilbert transforms have been constructed in $\mathbb{R}^m$ in the framework of Clifford analysis, a generalization to higher dimension of the theory of holomorphic functions in the complex plane. These Hilbert transforms, obtained as part of the boundary value of an associated Cauchy transform in $\mathbb{R}^{m+1}$, might be characterized as isotropic, since the metric in the underlying space is the standard Euclidean one. In this paper we adopt the idea of a so–called anisotropic Clifford setting, leading to the introduction of a metric dependent Hilbert transform in $\mathbb{R}^m$, which formally shows similar properties as the isotropic one, but allows to adjust the co-ordinate system to preferential directions. A striking fact is that the associated Cauchy transform in $\mathbb{R}^{m+1}$ is no longer uniquely determined, but may correspond to various $(m+1)$–dimensional metrics.

1 Introduction

In one–dimensional signal processing, the Hilbert transform has become an indispensable tool for both global and local descriptions of a signal, yielding information on various independent signal properties. The instantaneous amplitude, phase and frequency are estimated by means of so–called quadrature filters, which allow to distinguish different frequency components and therefore locally refine the structure analysis. They are essentially based on the notion of "analytic signal", which consists of the linear combination of a bandpass filter, selecting a small part of the spectral information, and its Hilbert transform, the latter basically being the result of a phase shift by $\frac{\pi}{2}$ on the original filter (see e.g. [1]). Mathematically, if

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$f(x) \in L_2(\mathbb{R})$ is a real valued signal of finite energy, and $\mathcal{H}[f]$ denotes its Hilbert transform, i.e.

$$\mathcal{H}[f](x) = \frac{1}{\pi}\ Pv \int_{-\infty}^{+\infty} \frac{f(y)}{x-y}\, dy$$

then the corresponding analytic signal is the function $\frac{1}{2}f + \frac{i}{2}\mathcal{H}[f]$, which belongs to the Hardy space $H^2(\mathbb{R})$, and arises as the L_2 non-tangential boundary value for $y \to 0+$ of the holomorphic Cauchy transform of f in the upper half of the complex plane.

The Hilbert transform has been generalized to higher dimension by embedding $\mathbb{R}^m$ in $\mathbb{R}^{m+1}$ (see e.g. [2], [3]), by considering Lipschitz hypersurfaces in $\mathbb{R}^{m+1}$ (see e.g. [4]) and by considering the boundary of smooth closed surfaces (see [5], [6]). Notice that higher dimensional applications have already been addressed as well, for instance in [7], where the concept of "analytic signal" was generalized to two dimensions in order to design appropriate quadrature filters, using a two–dimensional Hilbert transform, also referred to as Riesz transform.

Several of these generalizations were established in Clifford analysis, a comprehensive theory offering an elegant and powerful generalization to higher dimension of the theory of holomorphic functions in the complex plane. In its most simple yet still useful setting, flat $(m+1)$–dimensional Euclidean space, Clifford analysis focusses on monogenic functions, i.e. null solutions of the Clifford-vector valued Dirac operator $\partial = \sum_{j=0}^{m} e_j \partial_{x_j}$ where $(e_0, \ldots, e_m)$ forms an orthogonal basis for the quadratic space $\mathbb{R}^{m+1}$ underlying the construction of the Clifford algebra $\mathbb{R}_{0,m+1}$ (see e.g. [8],[9]). Monogenic functions are actually refining the properties of harmonic functions of several variables, since the rotation-invariant Dirac operator factorizes the m–dimensional Laplace operator, as does the Cauchy-Riemann operator in the complex plane.

The above described form of Clifford analysis may be referred to as isotropic, since the metric in the underlying space is the standard Euclidean one. In this paper however, we adopt the idea of a metric dependent, also called anisotropic or metrodynamical, Clifford setting (see e.g. [10], [11], [12]), which offers the possibility of adjusting the co-ordinate system to preferential, and not necessarily mutually orthogonal, directions in the m–dimensional signal to be analyzed. This leads to the introduction of a metric dependent generalized Hilbert transform in $\mathbb{R}^m \subset \mathbb{R}^{m+1}$, a special case of which was already introduced and used for two–dimensional image processing in [7].

2 The Clifford toolbox in a metric dependent setting

Let $\widetilde{G} = (g_{kl})_{k,l=0,\ldots,m} \in \mathbb{R}^{(m+1)\times(m+1)}$ be a real symmetric and positive definite tensor, which we will refer to as the metric tensor, and let $G = (g_{kl})_{k,l=1,\ldots,m} \in \mathbb{R}^{m\times m}$, i.e.

$$\widetilde{G} = \begin{pmatrix} g_{00} & \cdots & g_{0m} \\ \vdots & & \\ & G & \\ g_{0m} & & \end{pmatrix}$$

Notice that G is a metric tensor as well, being simply the restriction of $\widetilde{G}$ to $\mathbb{R}^m$, the latter being identified with the hyperplane $x^0 = 0$ of $\mathbb{R}^{m+1}$. Furthermore, let $\widetilde{G}^{-1} = (g^{kl})_{k,l=0,\ldots,m}$ denote the reciprocal, or inverse, tensor of $\widetilde{G}$, i.e.

$$\sum_{s=0}^{m} g_{ks}\, g^{sl} = \delta_k^l$$

In $\mathbb{R}^{m+1}$ we will consider a covariant basis $(e_l) = (e_0, \ldots, e_m)$ and a contravariant basis $(e^k) = (e^0, \ldots, e^m)$, corresponding to each other through the metric, i.e.

$$e_l = \sum_{k=0}^{m} g_{lk}\, e^k, \qquad e^k = \sum_{l=0}^{m} g^{kl}\, e_l$$

We construct the universal Clifford algebra $\mathbb{R}_{0,m+1}$ over $(\mathbb{R}^{m+1}, \widetilde{G})$, with a non–commutative multiplication governed by

$$\begin{array}{rcll} e_j\, e_k + e_k\, e_j &=& -2\, g_{jk}, & j,k = 0,\ldots,m \\ e^j\, e^k + e^k\, e^j &=& -2\, g^{jk}, & j,k = 0,\ldots,m \\ e_j\, e^k + e^k\, e_j &=& -2\, \delta_j^k, & j,k = 0,\ldots,m \end{array}$$

For a set $A = \{i_1, \ldots, i_h\} \subset \{0, \ldots, m\}$ with $0 \leq i_1 < i_2 < \ldots < i_h \leq m$, we put $e_A = e_{i_1} e_{i_2} \ldots e_{i_h}$. Moreover, $e_\emptyset = 1$ is the identity element. In this way a basis for the Clifford algebra $\mathbb{R}_{0,m+1}$ is constructed by means of which any $a \in \mathbb{R}_{0,m+1}$ may be written as $a = \sum_A a_A\, e_A$ with $a_A \in \mathbb{R}$, or still as $a = \sum_{k=0}^{m+1} [a]_k$ where $[a]_k = \sum_{|A|=k} a_A\, e_A$ is a so-called k-vector ($k = 0, 1, \ldots, m+1$). A vector $x = (x^0, \ldots, x^m) \in \mathbb{R}^{m+1}$ will be identified with the Clifford (1-)vector $\sum_{k=0}^{m} e_k x^k$. The above multiplication rules then lead in a natural way to the replacement of the classical scalar product

$$\langle x, y \rangle = \sum_{k=0}^{m} x^k\, y^k \tag{1}$$

by the symmetric bilinear form

$$\langle x, y \rangle_g = \sum_{k=0}^{m} \sum_{l=0}^{m} g_{kl}\, x^k\, y^l \tag{2}$$

Obviously, when $\widetilde{G} = \mathbb{E}$ (unity matrix), we recover the traditional Clifford algebra stemming from the standard Euclidean metric, and (2) reduces to (1).

In this metric dependent context, we introduce the anisotropic Dirac operator

$$\partial_g = \sum_{k=0}^{m} e^k \, \partial_{x^k}$$

with fundamental solution

$$E_g(x) = \frac{1}{a_{m+1}} \frac{\overline{x}}{|x|^{m+1}}$$

as well as the anisotropic Laplace operator

$$\Delta_g = -\partial_g \, \partial_g = \sum_{k=0}^{m} \sum_{l=0}^{m} g^{kl} \partial_{x^k} \partial_{x^l}$$

with fundamental solution

$$F_g(x) = -\frac{1}{(m-1)a_{m+1}} \frac{1}{|x|^{m-1}}$$

In the above, $\overline{\cdot}$ denotes the usual conjugation in $\mathbb{R}_{0,m+1}$, defined as the main anti-involution for which $\overline{e}_k = -e_k$, $k = 0, \ldots, m$. In particular for a vector x we have $\overline{x} = -x$. Additionally, a_{m+1} stands for the area of the unit sphere S^m in $\mathbb{R}^{m+1}$, and the norm $|.|$ is deduced from the inner product (2).

A function f defined on $\mathbb{R}^{m+1}$ and taking values in $\mathbb{R}_{0,m+1}$, is called g–monogenic in the open region Ω of $\mathbb{R}^{m+1}$ if f is continuously differentiable in Ω and satisfies in Ω the equation $\partial_g \, f = 0$. As the Dirac operator ∂_g factorizes the Laplace operator Δ_g, a g–monogenic function in Ω is g–harmonic, and so are its components.

In what follows, we will also refer to the anisotropic Cauchy–Riemann operator

$$D_g = \overline{e}_0 \, \partial_g = \overline{e}_0 \, e^0 \, \partial_{x^0} + \overline{e}_0 \, \underline{\partial}_g$$

and the corresponding function

$$C_g(x) \;=\; \frac{1}{a_{m+1}} \frac{\overline{x} \, e^0}{|x|^{m+1}}$$

almost, but not entirely, covering the notion of fundamental solution of D_g, since

$$D_g C_g(x) = \overline{e}_0 \, e^0 \, \delta(x)$$

Here, in an obvious notation, $\underline{x} = \sum_{k=1}^{m} x^k e_k$ is a vector in $\mathbb{R}^m$ and $\underline{\partial}_g = \sum_{k=1}^{m} e^k \partial_{x^k}$ is the m–dimensional Dirac operator in the G–metric. Note that, as $D_g = \overline{e}_0 \, \partial_g$, g–monogenicity may equally be expressed w.r.t. the Cauchy–Riemann operator.

3 The anisotropic Plemelj-Sokhotski formulae

The function $C_g(x)$ above is easily seen to split into

$$C_g(x) = \frac{1}{2}\left(\overline{e}_0\, e^0\, P_g(x) + \overline{e}^0\, Q_g(x)\right), \qquad x^0 \neq 0$$

with

$$P_g(x) = P_g(x^0, \underline{x}) = \frac{2}{a_{m+1}}\,\frac{x^0}{|x|^{m+1}}, \qquad x^0 \neq 0$$

$$Q_g(x) = Q_g(x^0, \underline{x}) = \frac{2}{a_{m+1}}\,\frac{\overline{\underline{x}}}{|x|^{m+1}}, \qquad x^0 \neq 0$$

It then readily follows from the g–monogenicity of $C_g(x)$ in $\mathbb{R}^{m+1}_+$ that $P_g(x)$ and $Q_g(x)$ are g–harmonic in $\mathbb{R}^{m+1}_+$ (and similarly in $\mathbb{R}^{m+1}_-$). In accordance with previous definitions (see e.g. [13]) we will call them g–conjugate harmonic functions.

The above functions may be used as the kernels for metric dependent counterparts of well–known integral transforms. Indeed, for an appropriate function belonging to $L_2(\mathbb{R}^m)$, or a tempered distribution f, we may define its metrodynamical Cauchy integral by

$$\mathcal{C}_g[f] = C_g * f$$

which is g–monogenic in $\mathbb{R}^{m+1}_+$ (and in $\mathbb{R}^{m+1}_-$). Analogously we introduce its metrodynamical Poisson and (g–)conjugate Poisson transforms as the g–harmonic functions

$$\mathcal{P}_g[f] = P_g * f, \qquad \mathcal{Q}_g[f] = Q_g * f$$

so that

$$\mathcal{C}_g[f] = \frac{\overline{e}_0\, e^0}{2}\,\mathcal{P}_g[f] + \frac{\overline{e}^0}{2}\,\mathcal{Q}_g[f]$$

either in $\mathbb{R}^{m+1}_+$ or in $\mathbb{R}^{m+1}_-$.

Taking limits in distributional sense for $x^0 \to 0+$ gives, through careful calculation,

$$\lim_{x^0\to 0+} P_g(x^0, \underline{x}) = \frac{1}{\sqrt{\det \widetilde{G}}}\,\delta(\underline{x}), \qquad \lim_{x^0\to 0+} Q_g(x^0, \underline{x}) = \frac{1}{\sqrt{\det \widetilde{G}}}\,H_g(\underline{x})$$

with

$$H_g(\underline{x}) = \sqrt{\det \widetilde{G}}\,\left(\frac{2}{a_{m+1}}\ \mathrm{Pv}\,\frac{\overline{\underline{x}}}{|\underline{x}|^{m+1}}\right)$$

leading to

$$\lim_{x^0\to 0+} \mathcal{P}_g[f] = \frac{1}{\sqrt{\det \widetilde{G}}}\,f, \qquad \lim_{x^0\to 0+} \mathcal{Q}_g[f] = \frac{1}{\sqrt{\det \widetilde{G}}}\,H_g * f$$

and thus

$$\lim_{x^0\to 0+} \mathcal{C}_g[f] = \frac{1}{\sqrt{\det \widetilde{G}}}\left(\ \frac{1}{2}\,\overline{e}_0\, e^0\, f + \frac{1}{2}\,\overline{e}^0\, H_g * f\right)$$

Similarly, for $x^0 \to 0-$, we obtain

$$\lim_{x^0 \to 0-} \mathcal{C}_g[f] = \frac{1}{\sqrt{\det \widetilde{G}}} \left(-\frac{1}{2}\, \overline{e}_0\, e^0\, f + \frac{1}{2}\, \overline{e}^0\, H_g * f \right)$$

In the isotropic case the above results are known as the Plemelj–Sokhotski formulae and give rise to the definition of the Hilbert transform.

4 The anisotropic Hilbert transform

For a function $f \in L_2(\mathbb{R}^m)$ (or a tempered distribution), we define its anisotropic Hilbert transform as

$$\mathcal{H}_g[f] = \overline{e}^0\, H_g * f$$

such that the anisotropic Plemelj–Sokhotski formulae take the form

$$\lim_{x^0 \to 0\pm} \mathcal{C}_g[f] = \frac{1}{\sqrt{\det \widetilde{G}}} \left(\pm\frac{1}{2}\, \overline{e}_0\, e^0\, f + \frac{1}{2}\, \mathcal{H}_g[f] \right) \tag{3}$$

For $m = 2$, such an anisotropic Hilbert transform was considered in [7], however for the special case where the e_0–direction in $\mathbb{R}^3$ is perpendicular to the plane spanned by (e_1, e_2). This corresponds to a $\widetilde{G}$–matrix of order 3 in which $g_{01} = g_{02} = 0$, whence $\overline{e}_0\, e^0 = 1$.

In order to study the properties of the linear operator $\mathcal{H}_g$, we will also have to pass to frequency space, so we need to introduce a proper definition for the Fourier transform on $\mathbb{R}^m$ in the present metric dependent setting. In the isotropic case we had

$$\mathcal{F}[f](\underline{x}) = \int_{\mathbb{R}^m} \exp(-2\pi i \langle \underline{x}, \underline{y} \rangle)\, f(\underline{y})\, dV(\underline{y}) = \int_{\mathbb{R}^m} \exp(-2\pi i\ \underline{x}^T \underline{y})\, f(\underline{y})\, dV(\underline{y})$$

where $\langle \underline{x}, \underline{y} \rangle$ denotes the restriction of the classical scalar product (1) to $\mathbb{R}^m$ (identified with $x^0 = 0$) and, in the last equality, the vectors $\underline{x}$ and $\underline{y}$ are interpreted as column matrices. In a natural way, this leads to the following anisotropic Fourier transform:

$$\mathcal{F}_g[f](\underline{x}) = \int_{\mathbb{R}^m} \exp(-2\pi i \langle \underline{x}, \underline{y} \rangle_g)\, f(\underline{y})\, dV(\underline{y}) = \int_{\mathbb{R}^m} \exp\left(-2\pi i\ \underline{x}^T G \underline{y}\right)\, f(\underline{y})\, dV(\underline{y})$$

where the restriction of the scalar product (2) to $\mathbb{R}^m$ comes into play. One immediately finds

$$\mathcal{F}_g[f](\underline{x}) = \mathcal{F}[f](G\underline{x})$$

due to the symmetric character of G.

The following properties of the Hilbert transform $\mathcal{H}_g$ may then be proven:

(P1) $\mathcal{H}_g$ is translation invariant, i.e.

$$\mathcal{H}_g[f(\underline{y} - \underline{t})](\underline{x}) = \mathcal{H}_g[f](\underline{x} - \underline{t})$$

(P2) $\mathcal{H}_g$ is dilation invariant, i.e.

$$\mathcal{H}_g[f(a\underline{y})](\underline{x}) = \mathcal{H}_g[f](a\underline{x}), \qquad \forall a > 0$$

which is equivalent to its kernel H_g being a homogeneous distribution of degree $-m$

(P3) $\mathcal{H}_g$ is a bounded operator on $L_2(\mathbb{R}^m)$, which is equivalent to its Fourier symbol

$$\mathcal{F}_g[H_g](\underline{x}) = \sqrt{\frac{\det \widetilde{G}}{\det G}}\, i\, \frac{\underline{x}}{|\underline{x}|} \tag{4}$$

being a bounded function

(P4) Up to a metric related constant, $\mathcal{H}_g$ squares to unity, i.e.

$$(\mathcal{H}_g)^2 = g^{00}\, \frac{\det \widetilde{G}}{\det G}\, \mathbf{1}$$

(P5) $\mathcal{H}_g$ is selfadjoint, or $\mathcal{H}_g^* = \mathcal{H}_g$

(P6) $\mathcal{H}_g$ arises in a natural way by considering non-tangential boundary values of the Cauchy transform $\mathcal{C}_g$ in $\mathbb{R}^{m+1}$ of an appropriate function or distribution in $\mathbb{R}^m$.

The proof of properties (P1)–(P2) and (P5) is rather straightforward, starting from the definition of $\mathcal{H}_g$ and taking into account the anisotropic setting. Next, the calculation of the Fourier symbol in (P3) is established by invoking the factorization of the positive definite tensor G as $B^T B$, with $B \in \mathrm{GL}(m, \mathbb{R})$, while (P4) then results from a conversion to frequency space. Finally, (P6) is a direct consequence of the results in previous section and was, in fact, already contained in (3).

Notice that, due to the properties (P4)–(P5), the operator

$$\widetilde{\mathcal{H}_g} = \sqrt{\frac{\det G}{g^{00}\, \det \widetilde{G}}}\, \mathcal{H}_g$$

is unitary.

5 Concluding remark

As is the case in the isotropic setting, the present Hilbert kernel $H_g(\underline{x})$ has been obtained in a constructive way, by taking distributional limits of a g–harmonic function in $\mathbb{R}^{m+1}_+$, which is one of two g–conjugate harmonic parts in which the g–monogenic Cauchy kernel $C_g(x)$ splits. The resulting Hilbert transform $\mathcal{H}_g[f] = \overline{e}^0\, H_g * f$ shows the influence of the underlying metric in two different ways:

(1) the determinant of the "mother" metric $\widetilde{G}$ on $\mathbb{R}^{m+1}$ arises as an explicit factor in the expression for the kernel,

and

(2) the induced metric G on $\mathbb{R}^m$ implicitly comes into play through the denominator of the kernel, since $|\underline{x}|^{m+1}$ can be rewritten as $\left[\underline{x}^T\, G\underline{x}\right]^{\frac{m+1}{2}}$.

The particularity of this metric dependence may also be seen in frequency space, where the metric G not only arises in the Fourier symbol (4) of $\mathcal{H}_g$, but is also hidden in the definition of the Fourier transform itself, while the "mother" metric $\widetilde{G}$ again only pops up through its determinant.

The above observations do raise the question whether there exists a one–to–one correspondence between a given Hilbert transform on $(\mathbb{R}^m, G)$ and the associated Cauchy transform on $(\mathbb{R}^{m+1}, \widetilde{G})$ from which it originates, or in other words: does the Hilbert transform contain enough geometrical information to completely determine the "mother" metric $\widetilde{G}$?

To answer this question, notice that

$$\det \widetilde{G} = \det G \left(g_{00} - \underline{u}^T\, G^{-1}\, \underline{u} \right)$$

where $\underline{u}^T$ denotes the row $(g_{01}, \ldots, g_{0m})$. Hence, given G and $\det \widetilde{G}$, the equation

$$g_{00} - \underline{u}^T\, G^{-1}\, \underline{u} = \frac{\det \widetilde{G}}{\det G}$$

should have a unique solution $(g_{00}, \underline{u}^T)$, which clearly is not the case, since

$$g_{00} = \frac{\det \widetilde{G}}{\det G}, \qquad \underline{u}^T = \underline{0}^T \qquad \text{and} \qquad g_{00} = \frac{\det \widetilde{G}}{\det G} + (G^{-1})_{11}, \qquad \underline{u}^T = (1, 0, \ldots, 0)$$

already constitute two different solutions, and many others may be found straightaway.

We conclude that, given a Hilbert kernel

$$H_g(\underline{x}) = c \left(\frac{2}{a_{m+1}}\, \mathrm{Pv}\, \frac{\overline{x}}{|\underline{x}|^{m+1}} \right)$$

being dependent on the m–dimensional metric G and on the strictly positive constant c, it is part of the boundary value of a Cauchy kernel in $(\mathbb{R}^{m+1}, \widetilde{G})$, with

$$\widetilde{G} = \begin{pmatrix} g_{00} & \underline{u}^T \\ \underline{u} & G \end{pmatrix}$$

where $(g_{00}, \underline{u}^T)$ are characterized, but not uniquely determined, by the equation

$$g_{00} - \underline{u}^T\, G^{-1}\, \underline{u} = \frac{c^2}{\det G}$$

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