Gevrey and analytic hypoellipticity on the torus for non-linear operators constructed from rigid vector fields

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Abstract

We give a result of global Gevrey and analytic regularity on the torus for non-linear operators constructed from rigid vector fields.

Let $\mathbb{T}^N$ be the N-dimensional torus and split $\mathbb{T}^N_z \simeq \mathbb{T}^m_t \times \mathbb{T}^n_x$. Let us then consider, for $u \in C^\infty(\mathbb{T}^N)$ and for some integer $n' \geq n$, the operator

$$P = P_u = P(x,u,D) = \sum_{i,j=1}^{n'} a_{ij}(u(t,x))X_iX_j + \sum_{j=1}^{n'} b_j(u(t,x))X_j + X_0 + c(u(t,x)) \quad (1)$$

defined for $z=(t,x) \in \mathbb{T}^m \times \mathbb{T}^n$, where the real analytic coefficients $a_{ij}(u)$, $b_j(u)$ and $c(u)$ are complex valued, but the real analytic rigid vector fields

$$X_j = \sum_{k=1}^{n} d_{jk}(x)\frac{\partial}{\partial x_k} + \sum_{k=1}^{m} e_{jk}(x)\frac{\partial}{\partial t_k}, \qquad j=0,\dots,n' \quad (2)$$

are real valued (*rigid* means that the coefficients d_{jk}, e_{jk} do not depend on t).

An example of operators of this type is the following:

$$P = \partial_x^2 + \partial_y^2 + \sin^2 x(1+a^2(u(t,x)))\partial_t^2, \quad (3)$$

for a real analytic function $a(u)$.

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The problem of regularity for the operator (3) in the C^∞, analytic or Gevrey classes on the torus is quite interesting since even in the linear case ($a \equiv 0$) we have a different behaviour on the torus and locally in $\mathbb{R}^3$. More precisely, the operator

$$P = \partial_x^2 + \partial_y^2 + \sin^2 x\, \partial_t^2 \tag{4}$$

is C^∞ hypoelliptic locally in $\mathbb{R}^3$ (cf. [H]), but it is not analytic hypoelliptic locally in $\mathbb{R}^3$ (cf. [BG]). On the contrary, it is C^∞ and analytic hypoelliptic globally on $\mathbb{T}^3$ (cf. [X], [CH], [T]). Moreover, Bove and Tartakoff obtained in [BT] a sharp result of non-isotropic Gevrey hypoellipticity for the operator (4), proving that it is $G^{3/2,1,2}$-hypoelliptic locally in $\mathbb{R}_x \times \mathbb{R}_y \times \mathbb{R}_t$. Finally, we proved in [BZ2] that (4) is G^s-hypoelliptic globally on $\mathbb{T}^3$ for all $s \geq 1$ (identifying $G^1(\mathbb{T}^N)$ with the real analytic class $\mathcal{A}(\mathbb{T}^N)$).

The next step is therefore the study of C^∞, analytic and Gevrey hypoellipticity for the non-linear operator (3), which is of the form (1). The C^∞-hypoellipticity on the torus for the non-linear operator (1) can be proved by the use of para-differential operators, following [X]. In [BZ2] we fixed then a solution $u \in C^\infty(\mathbb{T}^N)$ of the equation $P_u u = f$, for P_u defined by (1) and $f \in G^s(\mathbb{T}^N)$, and investigated, for all $s \geq 1$, G^s-hypoellipticity on the torus for the operator (1) and for its transposed operator ${}^tP_u$ defined by the relation

$$\langle {}^tP_u v, w\rangle = \langle v, P_u w\rangle \qquad \forall v, w \in G^s(\mathbb{T}^N),$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2(\mathbb{T}^N)$.

We write down here the computation of the transposed operator ${}^tP_u$ of P_u, that we omitted in [BZ2]:

Lemma 1.1. *If P_u is defined as in (1), then its transposed operator ${}^tP_u$ is given by*

$$\begin{aligned} {}^tP_u = \quad & \sum_{i,j=1}^{n'} \bar{a}_{ji}(u) X_i X_j + \sum_{j=1}^{n'} b_j^*(u, X_1 u, \ldots, X_{n'} u) X_j - X_o \\ & + c^*(u, X_1 u, \ldots, X_{n'} u) + \sum_{i,j=1}^{n'} \bar{a}'_{ji}(u)(X_i X_j u), \end{aligned}$$

where $\bar{a}'_{ij}(\cdot)$ is the complex conjugate of the first derivative of $a_{ij}(\cdot)$ and

$$\begin{aligned} b_j^*(u, X_1 u, \ldots, X_{n'} u) = \quad & \sum_{i=1}^{n'} (\bar{a}'_{ij}(u) + \bar{a}'_{ji}(u))(X_i u) \\ & + \sum_{i=1}^{n'} \left[(\bar{a}_{ij}(u) + \bar{a}_{ji}(u)) \left(\sum_{k=1}^{n} \frac{\partial d_{ik}}{\partial x_k} \right) \right] - \bar{b}_j(u) \end{aligned}$$

$$\begin{aligned} c^*(u, X_1u, \ldots, X_{n'}u) = & \sum_{i,j=1}^{n'} \bar{a}''_{ji}(u)(X_iu)(X_ju) \\ & + \sum_{j=1}^{n'} \left\{ \sum_{i=1}^{n'} \left[(\bar{a}'_{ij}(u) + \bar{a}'_{ji}(u)) \left(\sum_{k=1}^{n} \frac{\partial d_{ik}}{\partial x_k} \right) \right] - \bar{b}'_j(u) \right\} (X_ju) \\ & + \sum_{i,j=1}^{n'} \bar{a}_{ij}(u) \left[X_j \left(\sum_{k=1}^{n} \frac{\partial d_{ik}}{\partial x_k} \right) + \sum_{k=1}^{n} \frac{\partial d_{jk}}{\partial x_k} \sum_{k=1}^{n} \frac{\partial d_{ik}}{\partial x_k} \right] \\ & - \sum_{k=1}^{n} \frac{\partial d_{0k}}{\partial x_k} + \bar{c}(u) - \sum_{j=1}^{n'} \bar{b}_j(u) \left(\sum_{k=1}^{n} \frac{\partial d_{jk}}{\partial x_k} \right). \end{aligned}$$

Proof: Let us first compute ${}^tX_j$, for $j \in \{0, 1, \ldots, n'\}$. Since $u, v \in G^s(\mathbb{T}^N)$ and d_{jk}, e_{jk} are all real valued:

$$\begin{aligned} \langle {}^tX_jv, u \rangle = & \langle v, X_ju \rangle \\ = & \int_{\mathbb{T}^N} v(t,x) \left(\sum_{k=1}^{n} d_{jk}(x) \frac{\partial u(t,x)}{\partial x_k} + \sum_{k=1}^{m} e_{jk}(x) \frac{\partial u(t,x)}{\partial t_k} \right) dtdx \\ = & -\int_{\mathbb{T}^N} u(t,x) \left(\sum_{k=1}^{n} \frac{\partial}{\partial x_k}(v(t,x)d_{jk}(x)) + \sum_{k=1}^{m} \frac{\partial}{\partial t_k}(v(t,x)e_{jk}(x)) \right) dtdx \\ = & -\int_{\mathbb{T}^N} u(t,x) \left(X_jv(t,x) + v(t,x) \sum_{k=1}^{n} \frac{\partial d_{jk}(x)}{\partial x_k} \right) dtdx, \end{aligned}$$

i.e.

$${}^tX_j = -X_j - \left(\sum_{k=1}^{n} \frac{\partial d_{jk}}{\partial x_k} \right).$$

Let us now compute ${}^tP_u$:

$$\begin{aligned} & \langle w, {}^tP_uv \rangle = \langle P_uw, v \rangle \\ = & \sum_{i,j=1}^{n'} \langle a_{ij}(u)X_iX_jw, v \rangle + \sum_{j=1}^{n'} \langle b_j(u)X_jw, v \rangle + \langle X_ow, v \rangle + \langle c(u)w, v \rangle \\ = & \sum_{i,j=1}^{n'} \langle X_iX_jw, \bar{a}_{ij}(u)v \rangle + \sum_{j=1}^{n'} \langle X_jw, \bar{b}_j(u)v \rangle + \langle w, {}^tX_ov \rangle + \langle w, \bar{c}(u)v \rangle \\ = & \sum_{i,j=1}^{n'} \langle X_jw, {}^tX_i(\bar{a}_{ij}(u)v) \rangle + \sum_{j=1}^{n'} \langle w, {}^tX_j(\bar{b}_j(u)v) \rangle \\ & -\langle w, X_ov \rangle - \left\langle w, \left(\sum_{k=1}^{n} \frac{\partial d_{0k}}{\partial x_k} \right) v \right\rangle + \langle w, \bar{c}(u)v \rangle \\ = & -\sum_{i,j=1}^{n'} \langle X_jw, X_i(\bar{a}_{ij}(u)v) \rangle - \sum_{i,j=1}^{n'} \left\langle X_jw, \left(\sum_{k=1}^{n} \frac{\partial d_{ik}}{\partial x_k} \right) \bar{a}_{ij}(u)v \right\rangle \\ & -\sum_{j=1}^{n'} \langle w, X_j(\bar{b}_j(u)v) \rangle - \sum_{j=1}^{n'} \left\langle w, \left(\sum_{k=1}^{n} \frac{\partial d_{jk}}{\partial x_k} \right) \bar{b}_j(u)v \right\rangle \\ & -\langle w, X_ov \rangle - \left\langle w, \left(\sum_{k=1}^{n} \frac{\partial d_{0k}}{\partial x_k} - \bar{c}(u) \right) v \right\rangle \end{aligned}$$

$$
\begin{aligned}
= \quad & -\sum_{i,j=1}^{n'} \langle w, {}^t X_j[(X_i\bar{a}_{ij}(u))v + \bar{a}_{ij}(u)X_i v]\rangle - \sum_{i,j=1}^{n'} \left\langle w, {}^t X_j \left[\left(\sum_{k=1}^{n} \frac{\partial d_{ik}}{\partial x_k}\right)\bar{a}_{ij}(u)v\right]\right\rangle \\
& -\sum_{j=1}^{n'} \langle w, (X_j\bar{b}_j(u))v\rangle - \sum_{j=1}^{n'}\langle w, \bar{b}_j(u)X_j v\rangle - \langle w, X_o v\rangle \\
& -\left\langle w, \left[\sum_{k=1}^{n} \frac{\partial d_{0k}}{\partial x_k} - \bar{c}(u) + \sum_{j=1}^{n'}\left(\sum_{k=1}^{n}\frac{\partial d_{jk}}{\partial x_k}\right)\bar{b}_j(u)\right] v\right\rangle \\
= \quad & \sum_{i,j=1}^{n'} \langle w, (X_j X_i \bar{a}_{ij}(u))v + (X_i\bar{a}_{ij}(u))X_j v + (X_j\bar{a}_{ij}(u))X_i v + \bar{a}_{ij}(u)X_j X_i v\rangle \\
& + \sum_{i,j=1}^{n'} \left\langle w, \left[\left(\sum_{k=1}^{n}\frac{\partial d_{jk}}{\partial x_k}\right)(X_i\bar{a}_{ij}(u))\right] v\right\rangle + \sum_{i,j=1}^{n'}\left\langle w, \bar{a}_{ij}(u)\left(\sum_{k=1}^{n}\frac{\partial d_{jk}}{\partial x_k}\right) X_i v\right\rangle \\
& + \sum_{i,j=1}^{n'} \left\langle w, \left[X_j\left(\bar{a}_{ij}(u)\sum_{k=1}^{n}\frac{\partial d_{ik}}{\partial x_k}\right)\right] v\right\rangle + \sum_{i,j=1}^{n'}\left\langle w, \left(\bar{a}_{ij}(u)\sum_{k=1}^{n}\frac{\partial d_{ik}}{\partial x_k}\right) X_j v\right\rangle \\
& + \sum_{i,j=1}^{n'} \left\langle w, \bar{a}_{ij}(u)\left(\sum_{k=1}^{n}\frac{\partial d_{jk}}{\partial x_k}\right)\left(\sum_{k=1}^{n}\frac{\partial d_{ik}}{\partial x_k}\right) v\right\rangle - \sum_{j=1}^{n'}\langle w, \bar{b}_j(u)X_j v\rangle - \langle w, X_o v\rangle \\
& -\left\langle w, \left[\sum_{j=1}^{n'}(X_j\bar{b}_j(u)) + \sum_{k=1}^{n}\frac{\partial d_{0k}}{\partial x_k} - \bar{c}(u) + \sum_{j=1}^{n'}\left(\sum_{k=1}^{n}\frac{\partial d_{jk}}{\partial x_k}\right)\bar{b}_j(u)\right] v\right\rangle \\
= \quad & \sum_{i,j=1}^{n'} \langle w, \bar{a}_{ij}(u)X_j X_i v\rangle \\
& +\sum_{j=1}^{n'}\left\langle w, \left[\sum_{i=1}^{n'}(X_i\bar{a}_{ij}(u)) + \sum_{i=1}^{n'}(X_i\bar{a}_{ji}(u)) + \sum_{i=1}^{n'}\bar{a}_{ji}(u)\left(\sum_{k=1}^{n}\frac{\partial d_{ik}}{\partial x_k}\right)\right.\right. \\
& \qquad\qquad \left.\left. + \sum_{i=1}^{n'}\bar{a}_{ij}(u)\left(\sum_{k=1}^{n}\frac{\partial d_{ik}}{\partial x_k}\right) - \bar{b}_j(u)\right] X_j v\right\rangle - \langle w, X_o v\rangle \\
& +\left\langle w, \left[\sum_{i,j=1}^{n'} X_j X_i\bar{a}_{ij}(u) + \sum_{i,j=1}^{n'}\left(\sum_{k=1}^{n}\frac{\partial d_{jk}}{\partial x_k}\right)(X_i\bar{a}_{ij}(u)) + \sum_{i,j=1}^{n'} X_j\left(\bar{a}_{ij}(u)\sum_{k=1}^{n}\frac{\partial d_{ik}}{\partial x_k}\right)\right.\right. \\
& \qquad + \sum_{i,j=1}^{n'}\left(\sum_{k=1}^{n}\frac{\partial d_{jk}}{\partial x_k}\right)\left(\sum_{k=1}^{n}\frac{\partial d_{ik}}{\partial x_k}\right)\bar{a}_{ij}(u) - \sum_{j=1}^{n'} X_j\bar{b}_j(u) - \sum_{k=1}^{n}\frac{\partial d_{0k}}{\partial x_k} + \bar{c}(u) \\
& \qquad \left.\left. -\sum_{j=1}^{n'}\left(\sum_{k=1}^{n}\frac{\partial d_{jk}}{\partial x_k}\right)\bar{b}_j(u)\right] v\right\rangle.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
{}^tP_u = \quad & \sum_{i,j=1}^{n'} \bar{a}_{ji}(u)X_iX_j + \sum_{i,j=1}^{n'} (\bar{a}'_{ij}(u) + \bar{a}'_{ji}(u))(X_iu)X_j \\
& + \sum_{i,j=1}^{n'} (\bar{a}_{ij}(u) + \bar{a}_{ji}(u)) \left(\sum_{k=1}^{n} \frac{\partial d_{ik}}{\partial x_k} \right) X_j - \sum_{j=1}^{n'} \bar{b}_j(u)X_j - X_o \\
& + \sum_{i,j=1}^{n'} X_j(\bar{a}'_{ij}(u)(X_iu)) + \sum_{i,j=1}^{n'} \left(\sum_{k=1}^{n} \frac{\partial d_{jk}}{\partial x_k} \right) \bar{a}'_{ij}(u)(X_iu) \\
& + \sum_{i,j=1}^{n'} \bar{a}'_{ij}(u)(X_ju) \left(\sum_{k=1}^{n} \frac{\partial d_{ik}}{\partial x_k} \right) + \sum_{i,j=1}^{n'} \bar{a}_{ij}(u)X_j \left(\sum_{k=1}^{n} \frac{\partial d_{ik}}{\partial x_k} \right) \\
& + \sum_{i,j=1}^{n'} \bar{a}_{ij}(u) \left(\sum_{k=1}^{n} \frac{\partial d_{jk}}{\partial x_k} \sum_{k=1}^{n} \frac{\partial d_{ik}}{\partial x_k} \right) - \sum_{j=1}^{n'} \bar{b}'_j(u)(X_ju) - \sum_{k=1}^{n} \frac{\partial d_{0k}}{\partial x_k} \\
& + \bar{c}(u) - \sum_{j=1}^{n'} \bar{b}_j(u) \left(\sum_{k=1}^{n} \frac{\partial d_{jk}}{\partial x_k} \right).
\end{aligned}
$$

Since

$$X_j(\bar{a}'_{ij}(u)(X_iu)) = \bar{a}''_{ij}(u)(X_ju)(X_iu) + \bar{a}'_{ij}(u)(X_jX_iu),$$

the thesis follows. ∎

In order to obtain hypoellipticity results for both P_u and ${}^tP_u$ we can thus prove hypoellipticity for non-linear operators of the form

$$
\begin{aligned}
P = \quad & P_u = P(t,x,u,D) = \sum_{i,j=1}^{n'} a_{ij}(t,x,u,X_1u,\dots,X_{n'}u)X_iX_j \\
& + \sum_{j=1}^{n'} b_j(t,x,u,X_1u,\dots,X_{n'}u)X_j + X_o + c(t,x,u,X_1u,\dots,X_{n'}u)
\end{aligned} \tag{5}
$$

where all the coefficients a_{ij}, b_j, c are complex valued and real analytic.

Also for the operator (5) a result of C^∞-hypoellipticity on the torus can be proved, following [X], by the use of para-differential operators. We shall therefore assume, in the following, that $u \in C^\infty(\mathbb{T}^N)$ is a fixed solution of the equation $P_u u = f$, for P_u defined by (5) and $f \in G^s(\mathbb{T}^N)$, and that the following a-priori estimate is satisfied for some $0 < \delta \le \delta'$ and for all $v \in C^\infty(\mathbb{T}^N)$:

$$|||\, v \,|||_\mu := \sum_{i,j=1}^{n'} \|X_iX_jv\|_\mu + \sum_{j=1}^{n'} \|X_jv\|_{\mu+\delta} + \|v\|_{\mu+\delta'} \le C_u(\|P_uv\|_\mu + \|v\|_\mu), \tag{6}$$

where $C_u = C_u(u, X_1u, \dots, X_{n'}u) \le C$ is a positive bounded function and μ is a fixed integer with $\mu > N/2$, so that the Sobolev space $H^\mu(\mathbb{T}^N)$ is an algebra.

Let us also assume that, for every $x \in \mathbb{T}^n$, the fields

$$X'_j = \sum_{k=1}^{n} d_{jk}(x) \frac{\partial}{\partial x_k}, \qquad j = 1, \dots, n'$$

span the tangent space $T_x(\mathbb{T}^n)$,

Under these assumptions we proved in [BZ2] the following result of globally G^s-hypoellipticity on the torus, for all $s \geq 1$:

Theorem 1.2. *Let P be an operator of the form (5), with all the coefficients a_{ij}, b_j and c real analytic. Assume that the real analytic vector fields $\{X_j\}_{j=0,\dots,n'}$ are rigid and that for every fixed $x \in \mathbb{T}^n$ the $\{X'_j\}_{j=1,\dots,n'}$ span $T_x(\mathbb{T}^n)$.*

Assume moreover that $u \in C^\infty(\mathbb{T}^N)$ is a solution of the equation $P(t,x,u,D)u = f$, for some $f \in G^s(\mathbb{T}^N)$, with $s \geq 1$, and that the a-priori estimate (6) is satisfied. Then also $u \in G^s(\mathbb{T}^N)$.

Coming back to the operator (3) we can prove (cf. [BZ2]) that both $P = P_u$ and its transposed operator ${}^tP_u$ satisfy the a-priori estimate (6). Therefore, from Theorem 1.2, if $u \in C^\infty(\mathbb{T}^N)$ is a solution of $P_u u = f$ or of ${}^tP_u u = f$ for $f \in G^s(\mathbb{T}^N)$, with $s \geq 1$, then also $u \in G^s(\mathbb{T}^N)$.

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