Functionals on normed function spaces and exponential instability of linear skew-product semiflows

Mihail Megan Larisa Buliga

Abstract

The aim of this paper is to give necessary and sufficient conditions for uniform exponential instability of linear skew-product semiflows in terms of functionals on normed function spaces. We obtain the versions of some results due to Datko, Pazy, Neerven and Rolewicz for the case of instability of linear skew-product semiflows.

1 Introduction

In recent years, a significant progress has been made in the study of the asymptotic behavior of evolution equations, by treating them using the theory of linear skew-product semiflows. Thus, an important list of well-known results for the cases of C_0-semigroups and evolution operators has been extended for linear skew-product semiflows. One of the most remarkable result in stability theory of linear evolution operators in Banach spaces has been obtained by Datko in [3]. An extension of this result has been given by Pazy in [12]. An important generalization of Datko's result was proved by Rolewicz in [13]. A new and interesting idea has been presented by Neerven in [11], where an unified treatment of the preceding results is given and the exponential stability of C_0-semigroups has been characterized in terms of functionals on Banach function spaces. Some generalizations of these results for the case of linear skew-product semiflows have been presented in [1], [2], [3], [7], [8], [9] and [14].

Received by the editors April 2006.
Communicated by J. Mawhin.
2000 *Mathematics Subject Classification* : 34D05, 34D20.
Key words and phrases : Uniform exponential instability, linear-skew product semiflows.

In this paper, we shall present characterizations for uniform exponential instability of linear skew-product semiflows in the spirit of Neerven's approach. Thus, we obtain the versions of the theorems due to Datko, Pazy, Rolewicz and Neerven for the case of uniform exponential instability of linear skew-product semiflows. As consequences, we obtain some results presented in [9].

2 Normed function spaces

Let $\mathcal{M}(\mathbb{R}_+)$ be the space of all locally bounded and measurable functions $f : \mathbb{R}_+ \to \mathbb{R}$ with the topology of uniform convergence on compact sets. By $\mathcal{M}^+(\mathbb{R}_+)$ we denote the set of all $f \in \mathcal{M}(\mathbb{R}_+)$ with $f \geq 0$.

Definition 2.1. A function $N : \mathcal{M}(\mathbb{R}_+) \to [0, \infty]$ is called a *generalized norm* on $\mathcal{M}(\mathbb{R}_+)$, if it satisfies the following properties:

(i) $N(f) = 0$ if and only if $f = 0$ a.e;

(ii) $N(f + g) \leq N(f) + N(g)$, for all $f, g \in \mathcal{M}(\mathbb{R}_+)$;

(iii) $N(\alpha f) = |\alpha| N(f)$, for all $f \in \mathcal{M}(\mathbb{R}_+)$ with $N(f) < \infty$ and all $\alpha \in \mathbb{R}$;

(iv) if $|f| \leq |g|$ a.e then $N(f) \leq N(g)$.

Let $B = B_N$ be the set defined by

$$B := \{f \in \mathcal{M}(\mathbb{R}_+) : |f|_B = N(f) < \infty\}.$$

It is easy to see that $(B, |\cdot|_B)$ is a normed linear space, which is called *normed function space* generated by the generalized norm N.
Examples of normed function spaces are presented in [9].
We denote by $\mathcal{B}(\mathbb{R}_+)$ the set of all normed function spaces B with the properties:

(i) $\varphi_{[0,t)} \in B$, for all $t \geq 0$;

(ii) $\lim\limits_{t\to\infty} |\varphi_{[0,t)}|_B = \infty$;

(iii) there exists $\alpha > 0$ such that $|\varphi_{[t,t+1)}|_B \geq \alpha$, for all $t \geq 0$.

In what follows, we shall denote by $\mathcal{F}$ the set of all functions $F : \mathcal{M}^+(\mathbb{R}_+) \to [0, \infty]$ with the properties:

(f_1) if $f, g \in \mathcal{M}^+(\mathbb{R}_+)$ with $f \leq g$ then $F(f) \leq F(g)$;

(f_2) there exists $\alpha > 0$ such that $F(c\chi_{[t,t+1)}) \geq \alpha c$, for all $c > 0$ and $t \geq 0$;

(f_3) there exists a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim\limits_{t\to\infty} f(t) = \infty$ such that $F(c\chi_{[0,t)}) \geq cf(t)$, for all $c > 0$ and $t \geq 0$.

Here χ_A denotes the characteristic function of the set A.

Example 2.1. If $p \in [1, \infty)$ then the function $F : \mathcal{M}^+(\mathbb{R}_+) \to [0, \infty]$ defined by

$$F(f) = \int_0^\infty f^p(t)dt$$

has the property that $F \in \mathcal{F}$.

Example 2.2. If $B \in \mathcal{B}$ then the function $F : \mathcal{M}^+(\mathbb{R}_+) \to [0, \infty]$ defined by

$$F(f) = \sup_{t \geq 0} |f \cdot \chi_{[0,t)}|_B$$

belongs to $\mathcal{F}$.

3 Linear skew-product semiflows

Let X be a Banach space, let Θ be a metric space and let $E = \Theta \times X$. We denote by $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators from X into itself.

Definition 3.1. A continuous mapping $\sigma : \mathbb{R}_+ \times \Theta \to \Theta$ is called a *semiflow* on Θ, if it has the following properties:

(i) $\sigma(0, \theta) = \theta$, for all $\theta \in \Theta$;

(ii) $\sigma(t + s, \theta) = \sigma(t, \sigma(s, \theta))$, for all $t, s \geq 0$ and $\theta \in \Theta$.

Definition 3.2. An application $C : \mathbb{R}_+ \times \Theta \to \mathcal{B}(X)$ is called a *cocycle* on $E = \Theta \times X$, if it satisfies the following conditions:

(c_1) $C(0, \theta) = I$, the identity operator on X, for all $\theta \in \Theta$;

(c_2) $C(t + s, \theta) = C(t, \sigma(s, \theta))C(s, \theta)$, for all $(t, s, \theta) \in \mathbb{R}_+^2 \times \Theta$ (the cocycle identity);

(c_3) there are $M, \omega > 0$ such that $\|C(t, \theta)\| \leq Me^{\omega t}$, for all $(t, \theta) \in \mathbb{R}_+ \times \Theta$;

(c_4) for every $(\theta, x) \in \Theta \times X$, the function $C(\cdot, \theta)x$ is continuous.

Definition 3.3. A pair $S = (C, \sigma)$, where C is a cocycle on $E = \Theta \times X$ and σ is a semiflow on Θ is called a *linear skew-product semiflow* on E.

Examples of linear skew-product semiflows can be found in [1], [2], [3] and [14].

Definition 3.4. A linear skew-product semiflow $S = (C, \sigma)$ on E is said to be

(i) *injective*, if for every $(t, \theta) \in \mathbb{R}_+ \times \Theta$ the linear operator $C(t, \theta)$ is injective;

(ii) *uniformly exponentially instable*, if there are $N, \nu > 0$ such that

$$\|C(t, \theta)x\| \geq Ne^{\nu t}\|x\|, \quad \text{for all} \quad (t, \theta, x) \in \mathbb{R}_+ \times E.$$

An example of uniformly exponentially instable linear skew-product semiflow is given in [6].
We have:

Proposition 3.1. *A linear skew-product semiflow* $S = (C, \sigma)$ *on* $E = \Theta \times X$ *is uniformly exponentially instable if and only if there exists a function* $f : \mathbb{R}_+ \to \mathbb{R}_+^*$ *with* $\lim_{t\to\infty} f(t) = \infty$ *and*

$$\|C(t,\theta)x\| \geq f(t)\|x\|, \quad \text{for all} \quad (t,\theta,x) \in \mathbb{R}_+ \times E.$$

Proof. Necessity. It is obvious for $f(t) = Ne^{\nu t}$.
Sufficiency. Let $\delta > 0$ and $\nu > 0$ such that $1 < f(\delta) = e^{\nu\delta}$. Then for every $t \geq 0$, there is a natural number n, such that $n\delta \leq t < (n+1)\delta$. Then for all $(t,\theta,x) \in \mathbb{R}_+ \times E$, we have that

$$\begin{aligned} \|C(t,\theta)x\| &= \|C(t-n\delta, \sigma(n\delta,\theta))C(n\delta,\theta)x\| \geq f(t-n\delta)\|C(n\delta,\theta)x\| \\ &\geq f(0)f(\delta)\|C((n-1)\delta,\theta)x\| \geq \cdots \geq f(0)f(\delta)^n\|x\| \\ &= f(0)e^{\nu n\delta}\|x\| \geq f(0)e^{-\nu\delta}e^{\nu t}\|x\| = Ne^{\nu t}\|x\|, \end{aligned}$$

which shows that S is uniformly exponentially instable. ∎

4 Main results

Let $S = (C, \sigma)$ be an injective linear skew-product semiflow on $E = \Theta \times X$. We associate to S the function $c_{x,\theta}$ defined by

$$c_{x,\theta}(t) = \frac{1}{\|C(t,\theta)x\|}, \quad \text{for all} \quad (t,\theta,x) \in \mathbb{R}_+ \times E.$$

The main result of this paper is:

Proposition 4.1. *Let* $S = (C, \sigma)$ *be an injective and strongly measurable linear skew-product semiflow on* $E = \Theta \times X$. *Then* S *is uniformly exponentially instable if and only if there exists* $F \in \mathcal{F}$ *such that*

$$\sup_{\substack{\theta\in\Theta \\ \|x\|=1}} F(c_{x,\theta}) < \infty.$$

Proof. Necessity. It is sufficient to choose

$$F(f) = \int_0^\infty f(t)dt.$$

Sufficiency. Let $t, s \geq 0$ with $t \in [s, s+1)$. If M, ω are given by Definition 3.2, then

$$\|C(t,\theta)x\| \leq \|C(t-s, \sigma(s,\theta)C(s,\theta))x\| \leq Me^{\omega}\|C(s,\theta)x\|.$$

and hence

$$c_{x,\theta} \geq \frac{c_{x,\theta}(s)}{Me^{\omega}} \cdot \chi_{[s,s+1)}, \quad \text{for all} \quad s \geq 0 \quad \text{and} \quad (\theta,x) \in E \quad \text{with} \quad \|x\| = 1.$$

Using the hypothesis, we deduce that there exist $\alpha, M_1 > 0$ such that

$$M_1 \geq F(c_{x,\theta}) \geq F(\frac{c_{x,\theta}(s)}{Me^{\omega}} \cdot \chi_{[s,s+1)}) \geq \frac{\alpha c_{x,\theta}(s)}{Me^{\omega}},$$

and hence

$$c_{x,\theta}(s) \leq \frac{MM_1e^{\omega}}{\alpha} = M_2, \quad \text{for all} \quad s \geq 0 \quad \text{and} \quad (\theta, x) \in E \quad \text{with} \quad \|x\| = 1.$$

Thus, we obtain that

$$\|x\| \leq M_2 \|C(s,\theta)x\|, \quad \text{for all} \quad s \geq 0 \quad \text{and} \quad (\theta, x) \in E \quad \text{with} \quad \|x\| = 1.$$

It follows that

$$M_2 \cdot c_{x,\theta} \geq c_{x,\theta}(t)\chi_{[0,t)}, \quad \text{for all} \quad t \geq 0 \quad \text{and} \quad (\theta, x) \in E \quad \text{with} \quad \|x\| = 1,$$

which implies

$$M_1 \geq F(c_{x,\theta}) \geq F(\frac{c_{x,\theta}(t)}{M_2}\chi_{[0,t)}) \geq \frac{c_{x,\theta}(t)}{M_2} f(t),$$

where $f \in \mathcal{M}^+(\mathbb{R}_+)$ with $\lim_{t\to\infty} f(t) = \infty$.
From Proposition 3.1 we conclude that S is uniformly exponentially instable. ∎

Let $S = (C, \sigma)$ be an injective linear skew-product semiflow on $E = \Theta \times X$. For all $s \geq 0$ and $(\theta, x) \in E$ with $\|x\| = 1$, we associate to S the function

$$c^s_{x,\theta}(t) = c_{x,\theta}(t+s) = \frac{1}{\|C(t+s,\theta)x\|}.$$

Corollary 4.1. *An injective and strongly measurable linear skew-product semiflow $S = (C, \sigma)$ on $E = \Theta \times X$ is uniformly exponentially instable if and only if there exists $N > 0$ and $F \in \mathcal{F}$ such that*

$$F(c^s_{x,\theta}) \leq N c_{x,\theta}(s),$$

for all $s \geq 0$, $(\theta, x) \in E$ with $\|x\| = 1$.

Proof. Necessity. It is a simple verification for

$$F(f) = \int_0^\infty f(t)dt.$$

Sufficiency. We observe that

$$\sup_{\substack{\theta\in\Theta \\ \|x\|=1}} F(c_{x,\theta}) = \sup_{\substack{\theta\in\Theta \\ \|x\|=1}} F(c^0_{x,\theta}) \leq N \sup_{\substack{\theta\in\Theta \\ \|x\|=1}} c_{x,\theta}(0) = N < \infty$$

and by Proposition 4.1 it follows that S is uniformly exponentially instable. ∎

For an injective linear skew-product semiflow $S = (C, \sigma)$ on $E = \Theta \times X$, we associate the function

$$v^s_{x,\theta}(t) = \frac{c^s_{x,\theta}(t)}{c_{x,\theta}(t)} = \frac{\|C(t,\theta)x\|}{\|C(t+s,\theta)x\|}.$$

Corollary 4.2. *Let $S = (C, \Theta)$ be an injective and strongly measurable linear skew-product semiflow on $E = \Theta \times X$. Then S is uniformly exponentially instable if and only if there exists $F \in \mathcal{F}$ such that*

$$\sup\{F(v^s_{x,\theta}) : s \geq 0, \theta \in \Theta, x \in X : \|x\| = 1\} < \infty.$$

Proof. Necessity. It is sufficient to choose

$$F(f) = \int_0^\infty f(t)dt.$$

Sufficiency. We have that

$$\sup_{\substack{\theta\in\Theta \\ \|x\|=1}} F(c_{x,\theta}) = \sup_{\substack{\theta\in\Theta \\ \|x\|=1}} F(v^0_{x,\theta}) \leq \sup_{\substack{\theta\in\Theta \\ \|x\|=1}} F(v^s_{x,\theta}) < \infty,$$

and by Proposition 4.1 it follows that S is uniformly exponentially instable. ∎

Remark 4.1. The preceding results are versions of a Neerven's theorem ([11]) for the case of instability property.

We shall denote by Φ the set of all nondecreasing functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(0) = 0$ and $\varphi(t) > 0$, for every $t > 0$.

Corollary 4.3. *Let $S = (C, \sigma)$ be an injective and strongly measurable linear skew-product semiflow on $E = \Theta \times X$. Then S is uniformly exponentially instable if and only if there exists $\varphi \in \Phi$ such that*

$$\sup_{\substack{\theta\in\Theta \\ \|x\|=1}} \int_0^\infty \varphi(c_{x,\theta}(t))dt < \infty.$$

Proof. Necessity. It is trivial for $\varphi(t) = t$. ∎

Sufficiency. It results from Proposition 4.1 for

$$F(f) = \int_0^\infty \varphi(f(t))dt.$$

Remark 4.2. The preceding corollary extends Rolewicz's theorem ([13]) for the case of exponential instability.

For the particular case $\varphi(t) = t^p$, we obtain:

Corollary 4.4. *Let $S = (C, \sigma)$ be an injective and strongly measurable linear skew-product semiflow on $E = \Theta \times X$. Then S is uniformly exponentially instable if and only if there exists $p \in [1, \infty)$ such that*

$$\sup_{\substack{\theta \in \Theta \\ \|x\|=1}} \int_0^\infty \frac{dt}{\|C(t,\theta)x\|^p} < \infty.$$

Remark 4.3. The preceding corollary is an extension of the Datko's theorem ([4]) for the case of exponential instability.

Corollary 4.5. *Let $S = (C, \sigma)$ be an injective and strongly measurable linear skew-product semiflow on $E = \Theta \times X$. Then S is uniformly exponentially instable if and only if there exists a Banach function space $B \in \mathcal{B}(\mathbb{R}_+)$ such that for all $\theta \in \Theta$ and $x \in X$ with $\|x\| = 1$, we have that*

$$c_{x,\theta} \in B \quad \textit{and} \quad \sup_{\substack{\theta \in \Theta \\ \|x\|=1}} |c_{x,\theta}|_B < \infty.$$

Proof. *Necessity.* It results for $B = L^1(\mathbb{R}_+)$.
Sufficiency. Let $F : \mathcal{M}^+(\mathbb{R}_+) \to [0, \infty]$ be the function defined by

$$F(f) = \sup_{t \in \mathbb{R}_+} |f \cdot \chi_{[0,t)}|_B.$$

Then $F \in \mathcal{F}$ and

$$\sup_{\substack{\theta \in \Theta \\ \|x\|=1}} F(c_{x,\theta}) \leq \sup_{\substack{\theta \in \Theta \\ \|x\|=1}} |c_{x,\theta}|_B < \infty.$$

By Proposition 4.1 it results that S is uniformly exponentially instable. ■

Remark 4.4. The Corollary 4.5 is a version for exponential instability of Theorem 3.1.5. from [10].

As a particular case for the Banach function space

$$B = \{f \in \mathcal{M}^+(\mathbb{R}_+) : \beta f \in L^p(\mathbb{R}_+)\},$$

we obtain:

Corollary 4.6. *Let $S = (C, \sigma)$ be an injective and strongly measurable linear skew-product semiflow on $E = \Theta \times X$. Then S is uniformly exponentially instable if and only if there exists $p \in [1, \infty)$ and $\beta \in L^p_{loc}(\mathbb{R}_+) \setminus L^p(\mathbb{R}_+)$ with $\beta > 0$ such that*

$$\sup_{\substack{\theta \in \Theta \\ \|x\|=1}} \int_0^\infty \beta^p(t)[c_{x,\theta}(t)]^p dt < \infty.$$

Remark 4.5. The preceding corollary is an extension of Corollary 3.1.6 from [10].

References

[1] C. Chicone, Y. Latushkin, *Evolution Semigroups in Dynamical Systems and Differential Equations*, Math. Surveys Monogr., **70** (1999), Amer. Math. Soc.

[2] S. N. Chow, H. Leiva, *Two definitions of exponential dichotomy for skew-product semiflows in Banach spaces*, Proc. Amer. Math. Sc. **124** (1996), 1071-1081.

[3] R. Datko, *Uniform asimptotic stability of evolutionary processes in Banach spaces*, SIAM J. Math. Anal. **3** (1972), 428-445.

[4] R. Datko, *Extending a theorem of Liapunov to Hilbert spaces*, J. Math. Anal. Appl. **32** (1970), 610-616.

[5] Y. Latushkin, R. Schnaubelt, *Evolution semigroups, translation algebras and exponential dichotomy of cocycles*, J. Differential Equations **159** (1999), 321-369.

[6] M. Megan, A.L. Sasu, B. Sasu, A. Pogan, *Exponential stability and instability of semigroups of linear operators in Banach spaces*, Math. Ineq. and Appl. **5(3)** (2002), 557-567.

[7] M. Megan, B. Sasu, A.L. Sasu, *On uniform exponential stability of linear skew-product semiflows in Banach spaces*, Bull. Belg. Math. Soc. Simon Stevin **9** (2002), 143-154.

[8] M. Megan, B. Sasu, A.L. Sasu, *Theorems of Perron type for uniform exponential dichotomy of linear skew-product semiflows*, Bull. Belg. Math. Soc. Simon Stevin **10** (2003), 1-21.

[9] M. Megan, B. Sasu, A.L. Sasu, *Exponential instability of linear skew-product semiflows in terms of Banach function spaces*, Results in Mathematics **45** (2004), 309-318.

[10] J.M.A.M Van Neerven, *The Asymptotic Behaviour of Semigroups of Linear Operators*, Theory, Advances and Applications, Vol. **88** (1996) Birkhaüser.

[11] J.M.A.M Van Neerven, *Lower semicontinuity and the theorem of Datko and Pazy*, Integral Equations and Operator Theory **42** (2002), 482-492.

[12] A. Pazy, *On the applicability of Liapunov's theorem in Hilbert spaces*, SIAM. J. Math. Anal. Appl. **3** (1972), 291-294.

[13] R. Rolewicz, *On uniform N-equstability*, J. Math. Anal. Appl. **115** (1986), 434-441.

[14] R. S. Sacker, G. R. Sell, *Dichotomies for linear evolutionary equations in Banach spaces*, J. Differential Equations **12** (1994), 721-735.

Department of Mathematics,
Faculty of Mathematics and Computer Science,
West University of Timişoara,
Bul. V. Pârvan Nr. 4, 300223 Timişoara,
Romania.
e-mail: mmegan@rectorat.uvt.ro, buliga@math.uvt.ro.