Almost Kenmotsu manifolds and local symmetry

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Abstract

We consider locally symmetric almost Kenmotsu manifolds showing that such a manifold is a Kenmotsu manifold if and only if the Lie derivative of the structure, with respect to the Reeb vector field ξ, vanishes. Furthermore, assuming that for a $(2n+1)$-dimensional locally symmetric almost Kenmotsu manifold such Lie derivative does not vanish and the curvature satisfies $R_{XY}\xi = 0$ for any X, Y orthogonal to ξ, we prove that the manifold is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant curvature -4 and a flat n-dimensional manifold. We give an example of such a manifold.

Introduction

An *almost contact structure* on a differentiable manifold M^{2n+1} is given by a tensor field φ of type $(1,1)$, a vector field ξ and a 1-form η satisfying $\varphi^2 = -I + \eta \otimes \xi$ and $\eta(\xi) = 1$, which imply that $\varphi(\xi) = 0$ and $\eta \circ \varphi = 0$.

Furthermore, on the product manifold $M^{2n+1} \times \mathbb{R}$ one can define an almost complex structure J by $J\left(X, f\frac{d}{dt}\right) = \left(\varphi X - f\xi, \eta(X)\frac{d}{dt}\right)$, where X is a vector field tangent to M^{2n+1}, t is the coordinate of $\mathbb{R}$ and f is a $\mathcal{C}^\infty$ function on $M^{2n+1} \times \mathbb{R}$. If J is integrable, the almost contact structure is said to be *normal* and it is known that this is equivalent to the vanishing of the tensor field $N = [\varphi, \varphi] + 2d\eta \otimes \xi$, where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ ([3]).

An *almost contact metric structure* (φ, ξ, η, g) is given by an almost contact structure and a Riemannian metric g satisfying $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$ for any vector fields X and Y. Then, the *fundamental 2-form* Φ is defined by

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$\Phi(X,Y) = g(X,\varphi Y)$ for any vector fields X and Y. For more details, we refer to Blair's books [3], [5].

A *contact metric structure* (φ,ξ,η,g) is an almost contact metric structure such that $\Phi = d\eta$ and if the structure is normal, then it is a *Sasakian structure*. In [14], Z. Olszak proved that in dimension $2n+1 \geq 5$ any contact metric manifold of constant sectional curvature has sectional curvature equal to 1 and is a Sasakian manifold. In [4], D.E. Blair proved that if the Riemannian curvature of a contact metric manifold M^{2n+1} satisfies $R_{XY}\xi = 0$ for all vector fields X and Y, then M^{2n+1} is locally the product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of constant curvature 4. In particular, the tangent sphere bundle of a flat Riemannian manifold admits such a structure. More recently, in [6] E. Boeckx and J.T. Cho proved that a locally symmetric contact metric space is either Sasakian of constant curvature 1 or locally isometric to $\mathbb{R}^{n+1} \times S^n(4)$.

In this paper, we consider the class of almost contact metric manifolds called *almost Kenmotsu manifolds*. In [15], Olszak proved that if such a manifold has constant sectional curvature K and dimension $2n+1 \geq 5$, then it is a Kenmotsu manifold and $K = -1$. We give another proof of the same result without restrictions on the dimension. We also study locally symmetric almost Kenmotsu manifolds M^{2n+1} showing that such a manifold is a Kenmotsu manifold if and only if the operator $h = \frac{1}{2}\mathcal{L}_\xi\varphi$ vanishes, where $\mathcal{L}$ denotes the Lie differentiation. Furthermore, assuming $h \neq 0$ and $R_{XY}\xi = 0$ for all vector fields X and Y orthogonal to ξ, we prove that the spectrum of h is $\{0,1,-1\}$, with 0 as simple eigenvalue, and M^{2n+1} is locally the product of an $(n+1)$-dimensional manifold of constant curvature -4 and an n-dimensional flat manifold. We provide an example of such a manifold. Comparing with the contact case, one can state the following question: is a locally symmetric almost Kenmotsu manifold either Kenmotsu of constant curvature -1 or locally isometric to the product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$?

As usual, the manifolds involved are assumed to be connected. Furthermore, we denote by $\mathcal{X}(M^{2n+1})$ the space of the C^∞-sections of TM^{2n+1}.

As regards Kenmotsu manifolds, we recall here the basic data related to them. An almost contact metric manifold M^{2n+1}, with structure (φ,ξ,η,g), is said to be a *Kenmotsu manifold* if it is normal, the 1-form η is closed and $d\Phi = 2\eta \wedge \Phi$. It is well known that Kenmotsu manifolds can be characterized by

$$(\nabla_X\varphi)(Y) = g(\varphi X, Y)\xi - \eta(Y)\varphi(X),$$

for any $X, Y, Z \in \mathcal{X}(M^{2n+1})$, which implies that $\nabla_\xi\varphi = 0$. We denote by $\mathcal{D}$ the distribution orthogonal to ξ, that is $\mathcal{D} = Im(\varphi) = Ker(\eta)$. It can be seen that $\nabla_\xi X \in \mathcal{D}$ and $\nabla_X\xi \in \mathcal{D}$ for any vector field $X \in \mathcal{D}$. Moreover, one has $\nabla\xi = -\varphi^2$ and $\nabla\eta = g - \eta\otimes\eta$. Since η is closed, $\mathcal{D}$ is an integrable distribution. It is known that its leaves are $2n$-dimensional totally umbilical Kähler manifolds with mean curvature vector field $H = -\xi$. Kenmotsu manifolds appear for the first time in [9], where they have been locally classified.

Theorem 1. ([9]) *Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be a Kenmotsu manifold. Then, M^{2n+1} is locally a warped product $M' \times_{f^2} N^{2n}$ where N^{2n} is a Kähler manifold, M' is an open interval with coordinate t, and $f^2 = ce^{2t}$ for some positive constant c.*

As proved in [9], a Kenmotsu manifold is locally symmetric if and only if it is a space of constant sectional curvature $K = -1$.

1 Almost Kenmotsu manifolds

An almost contact metric manifold M^{2n+1}, with structure (φ, ξ, η, g), is said to be an *almost Kenmotsu manifold* if the 1-form η is closed and $d\Phi = 2\eta \wedge \Phi$. Obviously, a normal almost Kenmotsu manifold is a Kenmotsu manifold.

Let M^{2n+1} be an almost Kenmotsu manifold with structure (φ, ξ, η, g). Since the 1-form η is closed, we have $\mathcal{L}_\xi \eta = 0$ and $[X, \xi] \in \mathcal{D}$ for any $X \in \mathcal{D}$. The Levi-Civita connection satisfies $\nabla_\xi \xi = 0$ and $\nabla_\xi \varphi = 0$ ([10]), which implies that and $\nabla_\xi X \in \mathcal{D}$ for any $X \in \mathcal{D}$.

Now, we set $A = -\nabla\xi$ and $h = \frac{1}{2}\mathcal{L}_\xi \varphi$. Obviously, $A(\xi) = 0$ and $h(\xi) = 0$. Moreover, the tensor fields A and h are symmetric operators and satisfy the following relations

$$\begin{array}{l} A \circ \varphi + \varphi \circ A = -2\varphi\,, \quad h \circ \varphi + \varphi \circ h = 0 \\ \nabla_X \xi = -\varphi^2 X - \varphi h X, \quad X \in \mathcal{X}(M^{2n+1})\,, \\ \nabla\eta = g - \eta \otimes \eta + g \circ (\varphi \times h), \quad \delta\eta = -2n. \end{array} \tag{1}$$

Hence, M^{2n+1} cannot be compact. We also remark that

$$h = 0 \Leftrightarrow \nabla\xi = -\varphi^2\,. \tag{2}$$

From Lemma 2.2 in [10] we have

$$(\nabla_X \varphi)Y + (\nabla_{\varphi X}\varphi)(\varphi Y) = -\eta(Y)\varphi X - 2g(X, \varphi Y)\xi - \eta(Y)h(X) \tag{3}$$

for any $X, Y \in \mathcal{X}(M^{2n+1})$. The following result is also proved in [10].

Proposition 1. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost Kenmotsu manifold. The integral manifolds of $\mathcal{D}$ are almost Kähler manifolds with mean curvature vector field $H = -\xi$. They are totally umbilical submanifolds of M^{2n+1} if and only if h vanishes.*

Example 1. Let $(N^{2n}, J, \tilde{g})$, $n \geq 2$, be a strictly almost Kähler manifold and consider $\mathbb{R} \times N^{2n}$, with coordinate t on $\mathbb{R}$. We put $\xi = \frac{\partial}{\partial t}$, $\eta = dt$ and define the tensor field φ on $\mathbb{R} \times N^{2n}$ such that $\varphi X = JX$, if X is a vector field on N^{2n}, and $\varphi X = 0$ if X is tangent to $\mathbb{R}$. Furthermore, we consider the metric $g = g_0 + c\, e^{2t}\tilde{g}$, where g_0 denotes the Euclidean metric on $\mathbb{R}$ and $c \in \mathbb{R}^*_+$. Then, the warped product $\mathbb{R} \times_{f^2} N^{2n}$, $f^2 = ce^{2t}$, with the structure (φ, ξ, η, g), is a strictly almost Kenmotsu manifold. Namely, it is easy to verify that the 1-form η is closed and dual of ξ with respect to g, $\varphi^2 = -I + \eta \otimes \xi$ and g is a compatible metric. Computing Φ and $d\Phi$, we get $\Phi = ce^{2t}p_2^*(\tilde{\Omega})$, where p_2 is the projection on N^{2n} and $\tilde{\Omega}$ is the fundamental form of N^{2n}. Then, since $d\tilde{\Omega} = 0$, $d\Phi = 2dt \wedge \Phi = 2\eta \wedge \Phi$. Finally, since the torsion N_J does not vanish, N^{2n} being strictly almost Kähler, we obtain that the structure is not normal.

Remark 1. In [13], Oguro and Sekigawa describe a strictly almost Kähler structure on the Riemannian product $\mathbb{H}^3 \times \mathbb{R}$. Thus, we obtain a 5-dimensional strictly almost Kenmotsu manifold on the warped product $\mathbb{R} \times_{f^2} (\mathbb{H}^3 \times \mathbb{R})$, $f^2 = ce^{2t}$.

Theorem 2. *Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be an almost Kenmotsu manifold and assume that $h=0$. Then, M^{2n+1} is locally a warped product $M'\times_{f^2} N^{2n}$, where N^{2n} is an almost Kähler manifold, M' is an open interval with coordinate t, and $f^2=ce^{2t}$ for some positive constant c.*

Proof. The vector field ξ is geodesic and the orthogonal distribution $\mathcal{D}$ is integrable with totally umbilical almost Kähler leaves. Thus, as a manifold, M^{2n+1} is locally a product $M'\times N^{2n}$ with $TM'=[\xi]$ and $TN^{2n}=\mathcal{D}$. We can choose a neighborhood with coordinates $(t,x^1,\dots,x^{2n})$ such that $\pi_*(\xi)=\frac{\partial}{\partial t}$, π denoting the projection onto M'. Then $\pi: M'\times N^{2n}\to M'$ is a $\mathcal{C}^\infty$-submersion with vertical distribution $\mathcal{V}=TM'$ and horizontal distribution $\mathcal{H}=TN^{2n}$. The splitting $\mathcal{V}\oplus\mathcal{H}$ is orthogonal with respect to g and for any $p\in M^{2n+1}$ we have $g_p(\xi,\xi)=1=g_{\pi(p)}(\pi_*\xi,\pi_*\xi)$; hence, π is a Riemannian submersion. Since the horizontal distribution is integrable, the O'Neill tensor A vanishes. Moreover, the vector field $N=2nH=-2n\xi$ is basic. Now, computing the free-trace part T^0 of the O'Neill tensor T, for any U, V vertical vector fields, we get:

$$T^0_U V = T_U V-\tfrac{1}{2n}g(U,V)N=\alpha(U,V)+g(U,V)\xi=0,$$

$$T^0_U\xi = T_U\xi+\tfrac{1}{2n}g(N,\xi)U=\nabla_U\xi-U=0.$$

Thus $T^0=0$ and M^{2n+1} is locally a warped product of (M',g_0) and $(N^{2n},\tilde g)$ by a positive function f^2 on M', where g_0 is the flat metric and $\tilde g$ is an almost Kähler metric. The vector field $N=-2n\xi$ is π-related to $-\frac{2n}{f}grad_{g_0}f$ ([1], 9.104). It follows that $grad_{g_0}f=f\frac{d}{dt}$, which implies that $f=ke^t$ and $f^2=ce^{2t}$, with c a positive constant. Hence, the warped metric is given by $dt\otimes dt+ce^{2t}\tilde g$.

Proposition 2. *Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be an almost Kenmotsu manifold such that the integral manifolds of $\mathcal{D}$ are Kähler. Then, M^{2n+1} is a Kenmotsu manifold if and only if $\nabla\xi=-\varphi^2$.*

Proof. An easy computation shows that $N(X,\xi)=-2h(\varphi X)$ for any vector field X. Hence, assuming that the structure is normal, then $h(Y)=0$ for any $Y\in\mathcal{D}$. Being $h(\xi)=0$, we get $h=0$ and (2) implies that $\nabla\xi=-\varphi^2$. Vice versa, if $\nabla\xi=-\varphi^2$ then $h=0$ by (2), and thus $N(X,\xi)=0$ for any vector field X. Moreover, for $X,Y\in\mathcal{D}$ we have $N(X,Y)=N_J(X,Y)=0$, the leaves of $\mathcal{D}$ being Kähler manifolds.

Proposition 3. *An almost Kenmotsu manifold M^3 such that $\nabla\xi=-\varphi^2$ is a Kenmotsu manifold.*

Proof. In this case the integral manifolds of the distribution $\mathcal{D}$ are almost Kähler of dimension 2 and thus they are Kähler. The result follows from the previous proposition.

2 Curvature properties and local symmetry

A simple computation gives:

Proposition 4. *Let* $(M^{2n+1},\varphi,\xi,\eta,g)$ *be an almost Kenmotsu manifold. Then, for any* $X,Y\in\mathcal{X}(M^{2n+1})$,

$$R_{XY}\xi=\eta(X)(Y-\varphi hY)-\eta(Y)(X-\varphi hX)+(\nabla_Y\varphi h)X-(\nabla_X\varphi h)Y\,. \quad (4)$$

Proposition 5. *Let* $(M^{2n+1},\varphi,\xi,\eta,g)$ *be an almost Kenmotsu manifold. For any* $X\in\mathcal{X}(M^{2n+1})$ *we have*:

$$R_{\xi X}\xi=\,-\,\varphi^2X-2\varphi hX+h^2X-\varphi(\nabla_\xi h)(X), \quad (5)$$

$$(\nabla_\xi h)X=\,-\,\varphi X-2hX-\varphi h^2X-\varphi(R_{X\xi}\xi), \quad (6)$$

$$\frac{1}{2}(R_{\xi X}\xi-\varphi R_{\xi\varphi X}\xi)=\,-\,\varphi^2X+h^2X. \quad (7)$$

Proof. (5) follows by direct computation, using $\nabla_\xi\varphi=0$ and (1). Applying φ to (5) and remarking that $g((\nabla_\xi h)X,\xi)=0$, we get (6). Finally, we write (5) for φX obtaining

$$R_{\xi\varphi X}\xi=\varphi X+2\varphi^2hX+\varphi h^2X-\varphi(\nabla_\xi h)(\varphi X).$$

Then, we get

$$R_{\xi X}\xi-\varphi R_{\xi\varphi X}\xi=-2\varphi^2X+2h^2X-\varphi(\nabla_\xi h)(X)+\varphi^2(\nabla_\xi h)(\varphi X)$$

which reduces to (7), since $(\nabla_\xi h)\circ\varphi=-\varphi\circ(\nabla_\xi h)$.

Proposition 6. *Let* M^{2n+1} *be a locally symmetric almost Kenmotsu manifold. Then,* $\nabla_\xi h=0$.

Proof. We notice that (7) can be written as

$$\frac{1}{2}(R_{\xi\bullet}\xi-\varphi R_{\xi\varphi\bullet}\xi)=\,-\,\varphi^2+h^2$$

and since the operator $R_{\xi\bullet}\xi$ is parallel with respect to ξ, ξ being a geodesic vector field, we get $\nabla_\xi h^2=0$. Now, writing (6) as $\nabla_\xi h=\,-\,\varphi-2h-\varphi h^2-\varphi(R_{\bullet\xi}\xi)$ and applying ∇_ξ, we obtain $\nabla_\xi(\nabla_\xi h)=-2\nabla_\xi h$. Moreover, $\nabla_\xi h^2=0$ implies $(\nabla_\xi h)\circ h+h\circ\nabla_\xi h=0$, and applying ∇_ξ to this equality, we get $(\nabla_\xi h)^2=0$. Hence, $\nabla_\xi h=0$, since one easily verifies that $\nabla_\xi h$ is a symmetric operator.

Theorem 3. *Let* $(M^{2n+1},\varphi,\xi,\eta,g)$ *be a locally symmetric almost Kenmotsu manifold. Then, the following conditions are equivalent*:

a) M^{2n+1} *is a Kenmotsu manifold*;

b) $h=0$.

Moreover, if any of the above conditions holds, M^{2n+1} *has constant sectional curvature* $K=-1$.

Proof. Assuming that M^{2n+1} is a Kenmotsu manifold, we have $\nabla\xi = -\varphi^2$ and, by (2), $h = 0$. Now, supposing $h = 0$, it follows that $\nabla\xi = -\varphi^2$, $\nabla\eta = g - \eta\otimes\eta$ and, by (4), $R_{XY}\xi = -\eta(Y)X + \eta(X)Y$. Then, we get

$$(\nabla_Z R)(X,Y,\xi) = g(Z,X)Y - g(Z,Y)X - R_{XY}Z.$$

Since $\nabla R = 0$, M^{2n+1} has constant sectional curvature $K = -1$. Now, each integral manifold M' of $\mathcal{D}$ is an almost Kähler, totally umbilical submanifold and then it has constant sectional curvature ([7]). Computing its sectional curvature for orthonormal vectors X, Y we get:

$$k'(X,Y) = k(X,Y) + \|\xi\|^2 = k(X,Y) + 1 = 0$$

and thus M' is Kähler and flat. By Proposition 2, M^{2n+1} is a Kenmotsu manifold. Hence, *a*) and *b*) are equivalent and each of them implies the value $K = -1$ for the curvature.

Theorem 4. *An almost Kenmotsu manifold of constant curvature K is a Kenmotsu manifold and $K = -1$.*

Proof. Clearly, M^{2n+1} is locally symmetric, so $\nabla_\xi h = 0$. Comparing (4) and $R_{XY}\xi = K(\eta(Y)X - \eta(X)Y)$, we obtain

$$(K+1)(\eta(Y)X - \eta(X)Y) - \eta(Y)\varphi hX + \eta(X)\varphi hY - (\nabla_Y\varphi h)X + (\nabla_X\varphi h)Y = 0.$$

Choosing $X = \xi$ and $Y \in \mathcal{D}$, we get $-(K+1)Y + 2\varphi hY - h^2Y = 0$. Now, if Y is an eigenvector of h with eigenvalue λ, then $-(K+1)Y + 2\lambda\varphi Y - \lambda^2 Y = 0$, which implies $\lambda = 0$ and $K = -1$, since Y and φY are linearly independent. Hence $h = 0$, $K = -1$ and we apply the previous theorem.

Now, we consider the rank of the locally symmetric almost Kenmotsu manifold M^{2n+1}. If the rank is equal to one, then M^{2n+1} has constant curvature K, being of odd dimension, it is Kenmotsu, $K = -1$ and $h = 0$. If M^{2n+1} does not have constant curvature then, its rank must be greater than one and $h \neq 0$.

Proposition 7. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a locally symmetric almost Kenmotsu manifold. If M^{2n+1} has rank greater than one, then ± 1 are eigenvalues of h.*

Proof. The hypothesis on the rank implies that there exists a vector X orthogonal to ξ such that $R_{X\xi}\xi = 0$ and by (6) we get $\varphi X + 2hX + \varphi h^2 X = 0$. Let $(\xi, e_1, \dots, e_n, \varphi e_1, \dots, \varphi e_n)$ be a local frame of eigenvectors of h with corresponding eigenvalues $(0, \lambda_1, \dots, \lambda_n, -\lambda_1, \dots, -\lambda_n)$. Writing $X = \sum_{i=1}^n (X^i e_i + \bar{X}^i \varphi e_i)$, we obtain

$$\sum_{i=1}^n \Big((X^i - 2\bar{X}^i\lambda_i + X^i\lambda_i^2)\varphi e_i + (-\bar{X}^i + 2X^i\lambda_i - \bar{X}^i\lambda_i^2)e_i \Big) = 0,$$

which implies

$$\begin{cases} (1+\lambda_i^2)X^i - 2\lambda_i\bar{X}^i = 0 \\ 2\lambda_i X^i - (1+\lambda_i^2)\bar{X}^i = 0 \end{cases}$$

for each $i \in \{1, \dots, n\}$. Since $X \neq 0$, there exists $j \in \{1, \dots, n\}$ such that the corresponding system admits a non trivial solution and this implies $-(1+\lambda_j^2)^2 + 4\lambda_j^2 = 0$ and then $\lambda_j = \pm 1$.

Let us consider the operator $h' = h \circ \varphi$. This operator is symmetric and, if Y is an eigenvector with eigenvalue μ, then φY is an eigenvector with eigenvalue $-\mu$. Moreover, if X is an eigenvector of h with eigenvalue λ, then $X + \varphi X$ is an eigenvector of h' with eigenvalue $-\lambda$, while $X - \varphi X$ is an eigenvector of h' with eigenvalue λ. It follows that h and h' admit the same eigenvalues. Denoting by $[\lambda]$ and $[\lambda]'$ respectively the eigenspaces of h and h' with eigenvalue λ, we have $[\lambda] \oplus [-\lambda] = [\lambda]' \oplus [-\lambda]'$. Furthermore, $\nabla_\xi \varphi = 0$ implies that $\nabla_\xi h' = 0$ if and only if $\nabla_\xi h = 0$.

The operators h and h' are related to the curvature by the following proposition.

Proposition 8. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a locally symmetric almost Kenmotsu manifold. Then,*

1) $k(X, \xi) = -(1 + \lambda^2)$ *for any unit h-eigenvector X with eigenvalue λ,*

2) $k(X, \xi) = -(1 + \mu)^2$ *for any unit h'-eigenvector X with eigenvalue μ.*

Furthermore, $Ric(\xi, \xi) < 0$.

Proof. Since $\nabla_\xi h = 0$, from (5), we have $R_{X\xi}\xi = -X + 2\lambda\varphi X - \lambda^2 X$, and $k(X, \xi) = g(R_{X\xi}\xi, X) = -1 - \lambda^2$ which proves 1).
Analogously, since $\nabla_\xi h' = 0$, applying (5), we have $R_{X\xi}\xi = -X - 2h'(X) - h'^2(X)$ for any $X \in \mathcal{D}$, and $k(X, \xi) = -(1 + \mu)^2$, for any unit eigenvector X of h' with eigenvalue μ.

Proposition 9. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a locally symmetric almost Kenmotsu manifold. Then, for any $X, Y \in \mathcal{X}(M^{2n+1})$, the curvature tensor satisfies*:

$$\begin{aligned} R_{YX}\xi + R_{h'YX}\xi + R_{\xi X}Y + R_{\xi X}h'Y &= -g(X, Y + h'Y)\xi - \eta(X)(Y + h'Y) \\ &\quad +2\eta(Y)(X + 2h'X + h'^2X) \\ &\quad +2(\nabla_Y h')X + (\nabla_Y h'^2)X. \end{aligned} \tag{8}$$

Proof. Since M^{2n+1} is locally symmetric, then $\nabla_\xi h' = 0$. Being $h^2 = h'^2$, from (5), we have

$$R_{\xi X}\xi = X - \eta(X)\xi + 2h'X + h'^2X \tag{9}$$

for any $X \in \mathcal{X}(M^{2n+1})$. Derivating with respect to $Y \in \mathcal{X}(M^{2n+1})$, since $\nabla R = 0$, we get

$$\begin{aligned} R_{\nabla_Y \xi X}\xi + R_{\xi \nabla_Y X}\xi + R_{\xi X}\nabla_Y \xi &= \nabla_Y X - Y(\eta(X))\xi - \eta(X)\nabla_Y \xi \\ &\quad +2\nabla_Y(h'X) + \nabla_Y(h'^2X). \end{aligned} \tag{10}$$

Now, applying (9), $R_{\xi\nabla_Y X}\xi = \nabla_Y X - \eta(\nabla_Y X)\xi + 2h'(\nabla_Y X) + h'^2(\nabla_Y X)$. Moreover, from (1), $\nabla_Y \xi = Y - \eta(Y)\xi + h'Y$ and thus, $Y(\eta(X)) = Y(g(X, \xi)) = g(\nabla_Y X, \xi) + g(X, Y - \eta(Y)\xi + h'Y)$. Substituting in (10), and using again (9), by a simple computation we obtain (8).

In the following, we denote by $[\mu]$ the distribution of the eigenvectors of h' with eigenvalue μ. We remark that the condition $R_{XY}\xi = 0$ for any $X, Y \in \mathcal{X}(M^{2n+1})$, which gives the local decomposition $\mathbb{R}^{n+1} \times S^n(4)$ in the context of locally symmetric contact metric manifolds, in our case has to be relaxed to $X, Y \in \mathcal{D}$, otherwise we get a contradiction with Proposition 8.

Proposition 10. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a locally symmetric almost Kenmotsu manifold and suppose $h' \neq 0$. Then,*

1) $\nabla_Y \xi = 0$ *and* $[\xi, Y] \in [-1]$ *for any* $Y \in [-1]$, *while* $\nabla_Y \xi = 2Y$ *and* $[\xi, Y] \in [+1]$ *for any* $Y \in [+1]$,

2) *the distribution* $[-1]$ *is integrable with totally geodesic leaves or, equivalently, for any* $X, Y \in [-1]$, $R_{XY}\xi = 0$.

Proof. If $Y \in \mathcal{D}$ then we have $\nabla_Y \xi = Y + h'Y$ and this implies that $\nabla_Y \xi = 0$ for any eigenvector Y of h' with eigenvalue -1, $\nabla_Y \xi = 2Y$ for any eigenvector Y with eigenvalue $+1$. Furthermore, $\nabla_\xi h' = 0$ implies $\nabla_\xi [-1] \subset [-1]$, $\nabla_\xi [+1] \subset [+1]$ and 1) holds. From (8) and (4), if X and Y are orthogonal to ξ, we have, respectively,

$$R_{(Y+h'Y)X}\xi + R_{\xi X}(Y + h'Y) = -g(X, Y + h'Y)\xi + 2(\nabla_Y h')X + (\nabla_Y h'^2)X, \quad (11)$$

$$R_{XY}\xi = (\nabla_X h')Y - (\nabla_Y h')X \,. \quad (12)$$

Supposing $X, Y \in [-1]$, (11) gives

$$\nabla_Y X + 2h'(\nabla_Y X) + h'^2(\nabla_Y X) = 0. \quad (13)$$

Let $\{0, +1, -1, \lambda_i, -\lambda_i\}$ be the spectrum of h', where $\lambda_i > 0, \lambda_i \neq +1$. Now, $\nabla_Y X$ decomposes as $\nabla_Y X = A_0 + A_1 + A_{-1} + \sum_i A_{\lambda_i} + \sum_i A_{-\lambda_i}$. Hence,

$$h'(\nabla_Y X) = A_1 - A_{-1} + \textstyle\sum_i \lambda_i A_{\lambda_i} - \sum_i \lambda_i A_{-\lambda_i}$$
$$h'^2(\nabla_Y X) = A_1 + A_{-1} + \textstyle\sum_i \lambda_i^2 A_{\lambda_i} + \sum_i \lambda_i^2 A_{-\lambda_i}.$$

Applying (13), we get $A_0 = A_1 = 0$ and, for any i, $(1+\lambda_i)^2 A_{\lambda_i} = 0$, $(1-\lambda_i)^2 A_{-\lambda_i} = 0$ which imply $A_{\lambda_i} = A_{-\lambda_i} = 0$. Thus $\nabla_Y X \in [-1]$. Being also $\nabla_X Y \in [-1]$, we deduce that $[X, Y] \in [-1]$ and the distribution $[-1]$ is integrable with totally geodesic leaves. From (12), it follows that the integrability of the distribution $[-1]$ is equivalent to $R_{XY}\xi = 0$ for any $X, Y \in [-1]$.

Theorem 5. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a locally symmetric almost Kenmotsu manifold such that $h' \neq 0$ and $R_{XY}\xi = 0$ for any $X, Y \in \mathcal{D}$. Then, the spectrum of h' is $\{0, +1, -1\}$, with 0 as simple eigenvalue. Moreover, choosing $Y \in [-1]$ and $X \in [+1]$ one has $\nabla_Y X \in [+1]$, $\nabla_X Y \in [-1]$ and the distribution $[+1] \oplus [\xi]$ is totally geodesic.*

Proof. We know that $0, +1, -1$ are eigenvalues of h'. First we prove that for any unit eigenvector $X \in [\lambda]$, with $\lambda \neq -1$, and for any unit $Y \in \mathcal{D}$, orthogonal to X, we have

$$k(X, Y) = k(\xi, Y). \quad (14)$$

Namely, since $R_{XY}\xi = 0$, covariantly derivating with respect to X, we get

$$\begin{aligned} 0 &= R_{\nabla_X X\, Y}\xi + R_{X\, \nabla_X Y}\xi + R_{XY}\nabla_X \xi \\ &= g(\nabla_X X, \xi) R_{\xi\, Y}\xi + g(\nabla_X Y, \xi) R_{X\, \xi}\xi + (1+\lambda) R_{XY} X \\ &= -(1+\lambda) R_{\xi Y}\xi + (1+\lambda) R_{XY} X. \end{aligned}$$

Hence $R_{\xi Y}\xi = R_{XY}X$ and, taking the scalar product with Y, we get (14). Now, we suppose that there exists a unit eigenvector $X \in [\lambda]$ with $\lambda \neq \pm 1$ and applying (14) to X and φX we get $k(X, \varphi X) = k(\xi, \varphi X) = -(1-\lambda)^2$. Again, applying (14) to $\varphi X \in [-\lambda]$ and choosing $Y = X$, we have $k(\varphi X, X) = k(\xi, X) = -(1+\lambda)^2$. It follows that $(1-\lambda)^2 = (1+\lambda)^2$ so that $\lambda = 0$ and $Sp(h') = \{0, +1, -1\}$. Finally, let us suppose that $dim\,[0] > 1$ and let X be a unit eigenvector orthogonal to ξ such that $h'(X) = 0$. Applying (14) to X and to a unit $Y \in [+1]$, we get $k(X, Y) = k(\xi, Y) = -4$ and $k(Y, X) = k(\xi, X) = -1$, which is a contradiction.
Now, let be $Y \in [-1]$ and $X \in [+1]$. Since $[-1]$ is totally geodesic, then $\nabla_Y X \in [+1]$. Applying (12) it follows that $0 = R_{XY}\xi = -\nabla_X Y - h'(\nabla_X Y)$ so that $\nabla_X Y \in [-1]$ and $[+1] \oplus [\xi]$ is totally geodesic.

Theorem 6. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a locally symmetric almost Kenmotsu manifold such that $h' \neq 0$ and $R_{XY}\xi = 0$ for any X, $Y \in \mathcal{D}$. Then, M^{2n+1} is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant curvature -4 and a flat n-dimensional manifold.*

Proof. As proved in Proposition 10 and Theorem 5, the distributions $[\xi] \oplus [+1]$, $[-1]$ are integrable and totally geodesic. It follows that M^{2n+1} is locally isometric to the Riemannian product of an integral manifold M^{n+1} of $[\xi] \oplus [+1]$ and an integral manifold M^n of $[-1]$. Therefore, we can choose coordinates $(u^0, \ldots, u^{2n})$ such that $\partial/\partial u^0 \in [\xi]$, $\partial/\partial u^1, \ldots, \partial/\partial u^n \in [+1]$ and $\partial/\partial u^{n+1}, \ldots, \partial/\partial u^{2n} \in [-1]$. Now, we set $X_i = \partial/\partial u^i$ for any $i \in \{1, \ldots, n\}$, so that the distribution $[-1]$ is spanned by the vector fields $\varphi X_1, \ldots, \varphi X_n$. We notice that $[X_i, \varphi X_j] \in [-1]$ for any i, j in $\{1, \ldots, n\}$. Taking the scalar product with any $Z \in [+1]$, since $\nabla_{X_i}\varphi X_j \in [-1]$, we get $g(\nabla_{\varphi X_j} X_i, Z) = 0$ and then $\nabla_{\varphi X_j} X_i = 0$. Applying (3), we have $(\nabla_{X_i}\varphi)X_j - \varphi(\nabla_{\varphi X_i}\varphi X_j) = 0$, which implies

$$\nabla_{\varphi X_i}\varphi X_j = 0, \qquad (\nabla_{X_i}\varphi)X_j = 0,$$

since the two addenda belong to $[-1]$ and $[+1]$, respectively. The first condition implies that M^n is flat. We compute the curvature of M^{n+1}. Applying φ to $(\nabla_{X_i}\varphi)X_j = 0$, we have

$$\nabla_{X_i} X_j + \varphi(\nabla_{X_i}\varphi X_j) = -2g(X_i, X_j)\xi.$$

Derivating with respect to X_k, we obtain:

$$\nabla_{X_k}\nabla_{X_i} X_j + (\nabla_{X_k}\varphi)(\nabla_{X_i}\varphi X_j) + \varphi(\nabla_{X_k}\nabla_{X_i}\varphi X_j) = -2X_k(g(X_i, X_j))\xi - 4g(X_i, X_j)X_k$$

and, by scalar product with X_l,

$$g(\nabla_{X_k}\nabla_{X_i} X_j, X_l) - g(\nabla_{X_k}\nabla_{X_i}\varphi X_j, \varphi X_l) = -4g(X_i, X_j)g(X_k, X_l),$$

since $g((\nabla_{X_k}\varphi)(\nabla_{X_i}\varphi X_j), X_l) = -g(\nabla_{X_i}\varphi X_j, (\nabla_{X_k}\varphi)X_l) = 0$.
Now, we interchange i and k, subtract and, being $[X_i, X_k] = 0$, obtain

$$g(R_{X_kX_i}X_j, X_l) - g(R_{X_kX_i}\varphi X_j, \varphi X_l) = -4g(X_i, X_j)g(X_k, X_l) + 4g(X_k, X_j)g(X_i, X_l).$$

Since $\nabla_{\varphi X_i}X_j = 0 = [\varphi X_i, \varphi X_j]$, then $g(R_{X_kX_i}\varphi X_j, \varphi X_l) = g(R_{\varphi X_j\varphi X_l}X_k, X_i) = 0$, and thus

$$g(R_{X_kX_i}X_j, X_l) = -4(g(X_i, X_j)g(X_k, X_l) - g(X_k, X_j)g(X_i, X_l)).$$

Moreover, we recall that $g(R_{X_iX_j}\xi, X_k) = 0$ and, by (5), $g(R_{X_i\xi}\xi, X_j) = -4g(X_i, X_j)$. We conclude that M^{n+1} is a space of constant curvature -4.

Now, we provide an example of an almost Kenmotsu manifold which is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

Let $\{\xi, X_1, \ldots, X_n\}$ be the standard basis of $\mathbb{R}^{n+1}$ and let us denote by $\mathfrak{h}$ the Lie algebra obtained by defining:

$$[\xi, X_i] = -2X_i, \quad [X_i, \xi] = 2X_i, \quad [X_i, X_j] = 0,$$

for any $i, j \in \{1, \ldots, n\}$. Let $\{Y_1, \ldots, Y_n\}$ be the standard basis of $\mathbb{R}^n$; we consider on $\mathbb{R}^n$ the structure of abelian Lie algebra, denoted by $\mathfrak{k}$. On the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$ define the endomorphism $\varphi : \mathfrak{g} \to \mathfrak{g}$ such that

$$\varphi(\xi) = 0, \quad \varphi(X_i) = Y_i, \quad \varphi(Y_i) = -X_i,$$

for any $i \in \{1, \ldots, n\}$. Let $\eta : \mathfrak{g} \to \mathbb{R}$ be the 1-form defined by

$$\eta(\xi) = 1, \quad \eta(X_i) = \eta(Y_i) = 0,$$

for any $i \in \{1, \ldots, n\}$. We denote by g the inner product on $\mathfrak{g}$ such that the basis $\{\xi, X_i, Y_i\}$ is orthonormal.

Let G, H and K be connected Lie groups with Lie algebras $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{k}$ respectively. Being $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$, we have $G = H \times K$. The vectors ξ, X_i, Y_i determine left-invariant vector fields on G, which we denote in the same manner. Analogously, we denote by φ, η and g the left-invariant tensor fields determined by the corresponding tensors. It can be easily seen that (φ, ξ, η, g) is an almost contact metric structure on G. We prove that it is an almost Kenmotsu structure.

Indeed, for any $X, Y \in \mathfrak{g}$, $\eta(X)$ and $\eta(Y)$ are constant, $[X, Y]$ is orthogonal to ξ and then $d\eta(X, Y) = 0$ follows. It remains to prove that $d\Phi = 2\eta \wedge \Phi$. Since $\Phi(X, Y)$ is constant for any X, $Y \in \mathfrak{g}$, it follows that for any X, Y, $Z \in \mathfrak{g}$,

$$d\Phi(X, Y, Z) = -\frac{1}{3}\left\{\Phi([X, Y], Z) + \Phi([Y, Z], X) + \Phi([Z, X], Y)\right\}. \tag{15}$$

On the other hand,

$$2(\eta \wedge \Phi)(X, Y, Z) = \frac{2}{3}\left\{\eta(X)\Phi(Y, Z) + \eta(Y)\Phi(Z, X) + \eta(Z)\Phi(X, Y)\right\}. \tag{16}$$

Now, if X, Y and Z are orthogonal to ξ, then $\eta(X) = \eta(Y) = \eta(Z) = 0$ and $[X, Y] = [Z, X] = [X, Y] = 0$. Hence, $d\Phi(X, Y, Z) = 2(\eta \wedge \Phi)(X, Y, Z) = 0$. Let us suppose that $X = \xi$ and Y, Z orthogonal to ξ. Using (15) and (16), we have to verify that

$$-\Phi([\xi, Y], Z) - \Phi([Z, \xi], Y) = 2\Phi(Y, Z).$$

If $Y, Z \in \mathfrak{k}$, then $[\xi, Y] = [Z, \xi] = 0$; moreover, $\varphi Z \in \mathfrak{h}$ and thus $\Phi(Y, Z) = g(Y, \varphi Z) = 0$. Let us suppose that $Y, Z \in \mathfrak{h}$. Then, $[\xi, Y] = -2Y$ and $[Z, \xi] = 2Z$ imply $-\Phi([\xi, Y], Z) - \Phi([Z, \xi], Y) = 4\Phi(Y, Z)$ and, since $\varphi Z \in \mathfrak{k}$, we have $\Phi(Y, Z) = g(Y, \varphi Z) = 0$. Finally, we suppose $Y \in \mathfrak{h}$ and $Z \in \mathfrak{k}$. Since $[\xi, Y] = -2Y$ and $[Z, \xi] = 0$, we have $-\Phi([\xi, Y], Z) - \Phi([Z, \xi], Y) = 2\Phi(Y, Z)$.

Furthermore, it can be easily verified that, for any $X, Y \in \mathfrak{h}$, we have $[X, Y] = l(X)Y - l(Y)X$, where $l : \mathfrak{h} \to \mathbb{R}$ is the linear mapping such that $l(\xi) = -2$ and $l(X_i) = 0$ for any $i \in \{1, \dots, n\}$. It follows that H is a space of constant sectional curvature $k = -\|l\|^2 = -4$ (see Example 1.7 in [12]). Hence, H is locally isometric to the hyperbolic space of dimension $n + 1$ and curvature -4, which implies that G is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

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