

# Punishing factors and Chua's conjecture

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## Abstract

Let  $\Omega$  and  $\Pi$  be two simply connected domains in the complex plane  $\mathbb{C}$  which are not equal to the whole plane  $\mathbb{C}$ . We are concerned with the set  $A(\Omega, \Pi)$  of functions  $f : \Omega \rightarrow \Pi$  holomorphic on  $\Omega$  and we prove estimates for  $|f^{(n)}(z)|$ ,  $f \in A(\Omega, \Pi)$ ,  $z \in \Omega$ , of the following type. Let  $\lambda_\Omega(z)$  and  $\lambda_\Pi(w)$  denote the density of the Poincaré metric of  $\Omega$  at  $z$  and of  $\Pi$  at  $w$ , respectively. Then for any pair  $(\Omega, \Pi)$  where  $\Omega$  is convex,  $f \in A(\Omega, \Pi)$ ,  $z \in \Omega$ , and  $n \geq 2$  the inequality

$$\frac{|f^{(n)}(z)|}{n!} \leq (n+1)2^{n-2} \frac{(\lambda_\Omega(z))^n}{\lambda_\Pi(f(z))}$$

is valid.

For functions  $f \in A(\Omega, \Pi)$ , which are injective on  $\Omega$ , the validity of above inequality was conjectured by Chua in 1996.

The most famous result in geometric function theory during the last century was the proof of the Bieberbach conjecture in [6] for the class  $S$  of functions  $f$  holomorphic and injective in the unit disc  $\Delta$  that are normalized by  $f(0) = f'(0) - 1 = 0$ . This proof assures that for  $n \in \mathbb{N}$  and  $f \in S$  the sharp inequality

$$\left| \frac{f^{(n)}(0)}{n!} \right| \leq n$$

holds. Long before this event, Landau proved in 1925 that the inequality

$$\left| \frac{f^{(n)}(z)}{n!} \right| \leq \frac{n + |z|}{(1 - |z|)^{n+2}}, \quad z \in \Delta, \quad (1)$$

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is equivalent with the validity of the Bieberbach conjecture (see [11]). One may consider (1) as the estimate of a local Taylor coefficient of  $f \in \mathcal{S}$ . A result in the same direction was the proof of Jakubowski that the Bieberbach conjecture implies the inequality

$$\left| \frac{f^{(n)}(z)}{f'(z) n!} \right| \leq \frac{(n + |z|)(1 + |z|)^{n-2}}{(1 - |z|^2)^{n-1}}, \quad z \in \Delta, \quad (2)$$

for  $f$  holomorphic and injective in  $\Delta$  and  $n \geq 2$  (see [10]). Possibly, the fact that

$$\lambda_{\Delta}(z) = \frac{1}{1 - |z|^2}$$

represents the density of the Poincaré metric in the unit disc, motivated Yamashita and Chua to consider functions  $f$  holomorphic and injective on hyperbolic domains  $\Omega$  (compare for instance [19], [20], and [7]). In this case the density  $\lambda_{\Omega}$  is defined as follows. Let  $z \in \Omega$  be fixed and  $\Phi_{\Omega,z}$  be the conformal map of  $\Delta$  onto  $\Omega$  normalized by taking  $\Phi_{\Omega,z}(0) = z$  and  $\Phi'_{\Omega,z}(0) > 0$ . Then

$$\lambda_{\Omega}(z) := \frac{1}{\Phi'_{\Omega,z}(0)}.$$

As we are interested here only in the case that  $\Omega$  is convex, we will not cite the numerous results of the above articles in full generality. According to those papers, the following theorem is valid.

**Theorem A.** *Let  $\Omega$  be a convex domain, not equal to the whole plane  $\mathbb{C}$  and let  $f$  be holomorphic and injective on  $\Omega$ . Then for  $z \in \Omega$  and  $n = 2, 3, 4$  the inequalities*

$$\left| \frac{f^{(n)}(z)}{f'(z) n!} \right| \leq (n + 1) 2^{n-2} (\lambda_{\Omega}(z))^{n-1} \quad (3)$$

are valid.

Taking the limit  $z \rightarrow 1, z \in (0, 1)$ , in (2) for the Koebe function shows that the constant  $(n + 1)2^{n-2}$  on the right side of (3) can not be replaced by a smaller one. In [7], Chua published the following conjecture.

**Chua's conjecture.** *Let  $\Omega$  be a convex domain, not equal to the whole plane  $\mathbb{C}$  and let  $f$  be holomorphic and injective on  $\Omega$ . Then for any  $z \in \Omega$  and any  $n \geq 2$  the inequality (3) holds true.*

In the sequel, we will prove a theorem that contains the validity of this conjecture as a special case of punishing factors.

For the explanation of this abbreviation let  $\Omega$  and  $\Pi$  be two simply connected domains in the complex plane  $\mathbb{C}$  which are not equal to the whole plane  $\mathbb{C}$  and

$$A(\Omega, \Pi) = \{f : \Omega \rightarrow \Pi \mid f \text{ holomorphic}\}.$$

Furthermore, let  $\lambda_{\Omega}(z), z \in \Omega$ , and  $\lambda_{\Pi}(w), w \in \Pi$ , denote the density of the Poincaré metric at  $z \in \Omega$  and  $w \in \Pi$ , respectively. In a series of papers (compare in special

[1], [2], and [5]) the authors of the present article considered inequalities of the Schwarz-Pick type

$$\frac{|f^{(n)}(z)|}{n!} \leq C_n(\Omega, \Pi) \frac{(\lambda_\Omega(z))^n}{\lambda_\Pi(f(z))}, \quad z \in \Omega,$$

where  $f \in A(\Omega, \Pi)$  and  $C_n(\Omega, \Pi)$  represents the smallest number possible at that place that is not dependent on  $f$  and  $z \in \Omega$ .

After a colloquium talk of the second author on the results of [1] - [3], Ch. Pommerenke ([14]) suggested looking at this definition in the following way. The quotient  $(\lambda_\Omega(z))^n/\lambda_\Pi(f(z))$  reflects the influence of the positions of the points  $z$  and  $f(z)$  in  $\Omega$  and  $\Pi$  on the  $n$ th derivative  $f^{(n)}(z)$ , whereas the quantities  $C_n(\Omega, \Pi)$  are factors punishing bad behaviour of  $\Omega$  or  $\Pi$  at the boundary.

In [5], we proved that for  $\Omega$  convex,  $\Pi$  linearly accessible, and  $n \geq 2$  the inequality

$$C_n(\Omega, \Pi) \leq (n + 1)2^{n-2} \tag{4}$$

is valid. The equation

$$|f'(z)| = \frac{\lambda_\Omega(z)}{\lambda_\Pi(f(z))} \quad z \in \Omega$$

holds for functions  $f$  injective on  $\Omega$ . Hence, the inequality (4) implies the validity of Chua's conjecture for  $f$  that map  $\Omega$  conformally onto a linearly accessible domain  $\Pi$ . But, in fact more is true.

**Theorem.** *Let  $\Omega$  be convex, not equal to the whole plane  $\mathbb{C}$ , and  $\Pi$  simply connected, not equal to the whole plane  $\mathbb{C}$ . Let further  $n \geq 2$ . Then the inequality (4) is valid.*

**Proof.** Let  $z_0 \in \Omega$ ,  $\Phi_{\Omega, z_0}$  be defined as above and  $\Phi_{\Pi, f(z_0)}$  analogously. To prove our assertion we use a representation for  $f^{(n)}(z_0)$  from [1] and [2]. For  $f \in A(\Omega, \Pi)$ ,  $z_0 \in \Omega$ , we consider the functions

$$s(\zeta) := (\Phi_{\Omega, z_0}(\zeta) - z_0) \lambda_\Omega(z_0), \quad \zeta \in \Delta,$$

and

$$t(\zeta) := (\Phi_{\Pi, f(z_0)}(\zeta) - f(z_0)) \lambda_\Pi(f(z_0)), \quad \zeta \in \Delta.$$

The function  $s$  belongs to the class  $K$  of functions univalent in  $\Delta$  that map  $\Delta$  onto a convex domain and are normalized in the origin as usual, whereas  $t$  belongs to the class  $S$ .

The fact that  $f(\Omega)$  is a subset of  $\Pi$  may be expressed in terms of the function

$$u(\zeta) := (f(\Phi_{\Omega, z_0}(\zeta)) - f(z_0)) \lambda_\Pi(f(z_0)), \quad \zeta \in \Delta.$$

The above inclusion is equivalent to the fact that  $u(\zeta)$  is *subordinate* to  $t(\zeta)$ . This will be denoted by the abbreviation  $u \prec t$  and means that there exists a holomorphic function  $v : \Delta \rightarrow \overline{\Delta}$  such that

$$u(\zeta) = t(\zeta v(\zeta)), \quad \zeta \in \Delta.$$

Using the Taylor expansions

$$u(\zeta) = \sum_{k=1}^{\infty} a_k \lambda_{\Pi}(f(z_0)) \zeta^k$$

and

$$(s^{-1}(w))^k = \sum_{n=k}^{\infty} A_{n,k}(z_0) w^n,$$

where  $s^{-1}(w)$  denotes the function inverse to  $s(\zeta)$ , we get as in [1] and [2]

$$\frac{f^{(n)}(z_0)}{n!} = \sum_{k=1}^n a_k A_{n,k}(z_0) (\lambda_{\Omega}(z_0))^n. \tag{5}$$

Now it is evident that for the proof of (4) via (5) it is sufficient to prove the following proposition, which may deserve some interest of its own.

**Proposition 1.** *Let*

$$g_1(z) = \sum_{n=1}^{\infty} c_n z^n \prec g(z),$$

where  $g \in S$  and let  $F$  be the inverse function to an arbitrary function  $g_2 \in K$ . If the powers  $F^k, k \in \mathbb{N}$ , have the Taylor expansions

$$(F(w))^k = \sum_{n=k}^{\infty} A_{n,k} w^n$$

in a neighbourhood of the origin, then the inequality

$$\left| \sum_{k=1}^n c_k A_{n,k} \right| \leq (n+1)2^{n-2} \tag{6}$$

is valid for any  $n \geq 2$ .

Concerning the inverse function of a convex function, we proceed as in [4] and [5], where we proved inequalities analogous to (6).

For  $m \in \mathbb{N}$ , we consider the Taylor expansions

$$\left( \frac{z}{g_2(z)} \right)^m = \sum_{\nu=0}^{\infty} a_{\nu,m} z^{\nu}.$$

Then the Schur-Jabotinsky theorem (compare for example [9], Theorem 1.9.a), which may be considered as a special case of the Bürmann-Lagrange theorem, implies that for  $1 \leq k \leq n$  the identities

$$A_{n,k} = \frac{k}{n} a_{n-k,n}$$

are valid. Hence, we have to prove that

$$\left| \sum_{l=0}^{n-1} \frac{n-l}{n} c_{n-l} a_{l,n} \right| \leq (n+1)2^{n-2}. \tag{7}$$

To that end we use that for a convex function  $g_2$  one of the well-known Marx-Strohhäcker inequalities (see [12] and [17]), namely

$$\operatorname{Re} \left( \frac{g_2(z)}{z} \right) > \frac{1}{2}, \quad z \in \Delta, \tag{8}$$

holds. The formula (8) is equivalent to the existence of a bounded holomorphic function  $\omega : \Delta \rightarrow \overline{\Delta}$  such that

$$\frac{g_2(z)}{z} = \frac{1}{1 + z\omega(z)}, \quad z \in \Delta.$$

The tool for the proof of (7) is the resulting representation

$$\left( \frac{z}{g_2(z)} \right)^n = (1 + z\omega(z))^n = 1 + \sum_{\sigma=1}^n \binom{n}{\sigma} z^\sigma (\omega(z))^\sigma, \quad z \in \Delta.$$

If we define

$$(\omega(z))^\sigma = \sum_{j=0}^{\infty} d_{j,\sigma} z^j, \quad z \in \Delta,$$

we get the following formula for the sum appearing in (7)

$$\sum_{l=0}^{n-1} \frac{n-l}{n} c_{n-l} a_{l,n} = c_n + \sum_{\sigma=1}^{n-1} \frac{1}{n} \binom{n}{\sigma} \sum_{j=\sigma}^{n-1} (n-j) c_{n-j} d_{j-\sigma,\sigma}. \tag{9}$$

For the proof of (7) via (9) we use that the functions  $\omega^\sigma$  map the disc  $\Delta$  into  $\overline{\Delta}$ , too. Therefore, we may replace the coefficients  $d_{j-\sigma,\sigma}$  by the coefficients  $d_{j-\sigma}$  of a unimodular bounded function when we estimate the modulus of the inner sum in (9). We will prove that

$$\left| \sum_{j=\sigma}^{n-1} (n-j) c_{n-j} d_{j-\sigma,\sigma} \right| \leq (n-\sigma)^2.$$

This is a consequence of the following proposition with  $p = n - \sigma$ . This proposition may be interpreted as a slight generalization of the generalized Bieberbach conjecture or Rogosinski conjecture (see for instance [13]). The validity of this conjecture was proved by de Branges, too. It assures that for functions  $g_1$  subordinated to a function  $g \in S$  and  $n \geq 2$  the assertion

$$\left| \frac{g_1^{(n)}(0)}{n!} \right| \leq n$$

holds true.

**Proposition 2.** *Let*

$$\tilde{\omega}(z) = \sum_{\tau=0}^{\infty} d_\tau z^\tau$$

be holomorphic in the unit disc and such that  $\tilde{\omega}(\Delta) \subset \overline{\Delta}$  and let  $g_1$  be as in Proposition 1. Then for  $p \in \mathbb{N}$  the inequality

$$\left| \sum_{\tau=0}^{p-1} (p - \tau) c_{p-\tau} d_{\tau} \right| \leq p^2 \quad (10)$$

is valid.

Before we go into the proof of this proposition, we remark that the Rogosinski conjecture is the special case  $\tilde{\omega}(z) \equiv 1$  of (10). Hence, it seems not to be surprising that the proposition 2 is a consequence of a very deep theorem concerning schlicht functions due to Sheil-Small (see [16]). When this was published in 1973, it was not known whether the Robertson conjecture (see [15]) is true. Therefore, in the original version of this theorem, Sheil-Small could show the assertion only for those natural numbers for which the Robertson conjecture was proved. Since we owe the proof of this conjecture to de Branges, too, we may use the following theorem. It follows immediately from Lemma 4.2 in [16], the considerations (4.10) in [16], and the validity of the Robertson conjecture.

**Theorem B.** *Let the expansions*

$$F(z) = \sum_{\nu=0}^{\infty} B_{\nu} z^{\nu} \quad \text{and} \quad G(z) = \sum_{\nu=0}^{\infty} D_{\nu} z^{\nu}$$

be valid in a neighbourhood of the origin. Then the Hadamard product of  $F$  and  $G$  is defined by

$$(F * G)(z) = \sum_{\nu=0}^{\infty} B_{\nu} D_{\nu} z^{\nu}.$$

If  $g_1$  is subordinated to a function  $g \in S$  and  $P$  is a polynomial of degree  $\leq p$ , then for  $z \in \overline{\Delta}$  the inequality

$$|(P * g_1)(z)| \leq p \max\{|P(z)|, |z| = 1\} \quad (11)$$

is valid.

To prove (10) with the help of (11) we consider the polynomial

$$P(z) = \sum_{\tau=0}^{p-1} d_{\tau} (p - \tau) z^{p-\tau}.$$

Because of the identity

$$(P * g_1)(1) = \sum_{\tau=0}^{p-1} (p - \tau) c_{p-\tau} d_{\tau}$$

it is sufficient for the proof of (10) to show that

$$\left| P(e^{-i\theta}) \right| \leq p, \quad \theta \in [0, 2\pi].$$

Since the family of functions  $\tilde{\omega}$ , where  $\tilde{\omega}$  is chosen as in the proposition 2, is invariant against rotations of the unit disc, it remains to prove the inequality

$$\left| \sum_{\tau=0}^{p-1} (p - \tau) d_{\tau} \right| \leq p.$$

This inequality is known since long as Fejér's inequality. Proofs can be found in [8] and [18]. This concludes the proof of Proposition 2.

After these preparations it is not difficult to prove (7). Using the triangle inequality together with (9) and Proposition 2, we get

$$\left| \sum_{l=0}^{n-1} \frac{n-l}{n} c_{n-l} a_{l,n} \right| \leq |c_n| + \sum_{\sigma=1}^{n-1} \binom{n}{\sigma} \frac{(n-\sigma)^2}{n} \leq \sum_{\sigma=1}^n \binom{n}{\sigma} \frac{\sigma^2}{n} = (n+1)2^{n-2}.$$

This proves Proposition 1 and therefore, according to the above, the proof of our theorem follows immediately.

#### Remarks.

(1) In [5], we have presented some examples showing that equality in (4) is attained for special choices of  $\Omega$  and  $\Pi$ .

(2) Further, we are indebted to the referee for the following comment:

Chua's conjecture follows immediately from our Proposition 1 and formula (12) in [7], p.68.

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