Parallel surfaces in the motion groups $E(1,1)$ and $E(2)$

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Abstract

We give a classification of parallel surfaces in the groups of rigid motions of Minkowski plane and Euclidean plane, equipped with a general left-invariant metric. Our result completes the classification of parallel surfaces in the eight three-dimensional model geometries of Thurston and in three-dimensional unimodular Lie groups with maximal isometry group.

1 Introduction

In recent years, there has been done a lot of research concerning curves and surfaces in 3-dimensional homogeneous spaces. Initial work was done in [4] and [2]. One of the reasons for this success is that the classification of these spaces is well-understood and that they form a natural generalization of 3-dimensional real space forms. Indeed, let M^3 be a simply connected homogeneous manifold with isometry group $I(M^3)$, i.e. $I(M^3)$ acts transitively on M^3. Then $\dim I(M^3) \in \{3,4,6\}$ and moreover:

(i) if $\dim I(M^3) = 6$, then M^3 is a real space form, i.e. Euclidean space $\mathbb{E}^3$, hyperbolic space $\mathbb{H}^3(c)$ or a three-sphere $\mathbb{S}^3(c)$,

(ii) if $\dim I(M^3) = 4$, then M^3 is either a Riemannian product $\mathbb{H}^2(c)\times\mathbb{R}$ or $\mathbb{S}^2(c)\times\mathbb{R}$, or one of following Lie groups with left-invariant metric: the special unitary group SU_2, the universal covering of the special linear group $\widetilde{\mathrm{SL}_2\mathbb{R}}$ or the Heisenberg group Nil_3,

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(iii) if $\dim I(M^3) = 3$, then M^3 is a general 3-dimensional Lie group with left-invariant metric, for example the Lie group Sol_3.

Another reason is that the above classification contains the eight model geometries of Thurston ([7]), namely $\mathbb{E}^3$, $\mathbb{H}^3$, $\mathbb{S}^3$, $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{R}$, $\widetilde{\mathrm{SL}_2\mathbb{R}}$, Nil_3 and Sol_3. The famous geometrization conjecture of Thurston states that these eight spaces are the 'building blocks' to construct *any* 3-dimensional manifold.

Parallel submanifolds are an important class to study. Their second fundamental form is covariantly constant and hence their extrinsic invariants do not change from point to point. Parallel and even higher order parallel surfaces have been classified in the 3-dimensional homogeneous spaces M^3 with $\dim I(M^3) \in \{4, 6\}$, see [1], [3], [9]. In the present paper we investigate parallel surfaces in $E(1,1)$, the group of rigid motions of Minkowski plane, and in the universal covering of $E(2)$, the group of rigid motions of Euclidean plane, both equipped with general left-invariant metrics. The first family includes the remaining model geometry Sol_3.

2 Parallel hypersurfaces

Let $f : M^n \to \widetilde{M}^{n+1}$ be an isometric immersion of Riemannian manifolds. Denote by N a unit normal vector field on the hypersurface and by ∇ and $\widetilde{\nabla}$ the Levi Civita connections of M^n and $\widetilde{M}^{n+1}$ respectively. We define for all $X, Y \in T_pM^n$, $p \in M^n$, the shape-operator S by $SX = -\widetilde{\nabla}_X N$ and the second fundamental form h by $h(X,Y) = \langle SX, Y\rangle = \langle X, SY\rangle$. The hypersurface is said to be *totally umbilical* if S is a scalar multiple of the identity transformation at every point, and *totally geodesic* if $S = 0$. The formula of Gauss relates the Levi Civita connections and the second fundamental form: if X and Y are vector fields tangent to M^n, then

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y)N. \tag{1}$$

Now let R be the curvature tensor of M^n and $\widetilde{R}$ that of $\widetilde{M}^{n+1}$, then the equations of Gauss and Codazzi state respectively

$$\langle \widetilde{R}(X,Y)Z, W\rangle = \langle R(X,Y)Z, W\rangle + h(X,Z)h(Y,W) - h(Y,Z)h(X,W), \tag{2}$$
$$\langle \widetilde{R}(X,Y)Z, N\rangle = (\nabla h)(X,Y,Z) - (\nabla h)(Y,X,Z), \tag{3}$$

for $p \in M^n$ and $X, Y, Z, W \in T_pM^n$. Here, the covariant derivative of h is defined by

$$(\nabla h)(X,Y,Z) = X[h(Y,Z)] - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

Remark that for $n = 2$, the equation of Gauss can be reformulated as

$$K(p) = \langle \widetilde{R}(E_1, E_2)E_2, E_1\rangle + \det S, \tag{4}$$

where K denotes the Gaussian curvature of M^2 and $\{E_1, E_2\}$ is an orthonormal basis of T_pM^2.

We say that M^n is *parallel* in $\widetilde{M}^{n+1}$ if $\nabla h = 0$ and that M^n is *semi-parallel* in $\widetilde{M}^{n+1}$ if $R \cdot h = 0$, where

$$(R \cdot h)(X,Y,Z,W) = -h(R(X,Y)Z, W) - h(Z, R(X,Y)W)$$

for every $p \in M^n$ and $X, Y, Z, W \in T_pM^n$. It is easy to see that parallellism implies semi-parallellism and for $n = 2$, the following result is easily obtained:

Lemma 1. *Let M^2 be a surface immersed in a three-dimensional Riemannian manifold. Then M^2 is semi-parallel if and only if it is flat or totally umbilical.*

3 The motion groups $E(1,1)$ and $E(2)$

3.1 Unimodular Lie groups

A Lie group G is said to be *unimodular* if its left-invariant Haar measure is right-invariant. Milnor gave an infinitesimal reformulation of unimodularity for 3-dimensional Lie groups. We recall it briefly here.

Let $\mathfrak{g}$ be a 3-dimensional oriented Lie algebra with an inner product $\langle \cdot, \cdot \rangle$. We define the vector product operation $\times : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ as the skew-symmetric bilinear map which is uniquely determined by the following conditions:

(i) $\langle X, X \times Y \rangle = \langle Y, X \times Y \rangle = 0$,

(ii) $|X \times Y|^2 = \langle X, X \rangle \langle Y, Y \rangle - \langle X, Y \rangle^2$,

(iii) if X and Y are linearly independent, then $\det(X, Y, X \times Y) > 0$,

for all $X, Y \in \mathfrak{g}$. On the other hand, the Lie-bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a skew-symmetric bilinear map. Comparing these two operations, we get a linear endomorphism $L_\mathfrak{g}$ which is uniquely determined by the formula

$$[X, Y] = L_\mathfrak{g}(X \times Y), \quad X, Y \in \mathfrak{g}.$$

Now let G be an oriented 3-dimensional Lie group equipped with a left-invariant Riemannian metric. Then the metric induces an inner product on the Lie algebra $\mathfrak{g}$. With respect to the orientation on $\mathfrak{g}$ induced from G, the endomorphism field $L_\mathfrak{g}$ is uniquely determined. The unimodularity of G is characterized as follows.

Proposition 1. *[5] Let G be an oriented 3-dimensional Lie group with a left-invariant Riemannian metric. Then G is unimodular if and only if the endomorphism $L_\mathfrak{g}$ is self-adjoint with respect to the metric.*

Let G be a 3-dimensional unimodular Lie group with a left-invariant metric. Then there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of the Lie algebra $\mathfrak{g}$ such that

$$[e_1, e_2] = c_3 e_3, \quad [e_2, e_3] = c_1 e_1, \quad [e_3, e_1] = c_2 e_2, \qquad c_i \in \mathbb{R}.$$

Three-dimensional unimodular Lie groups are classified by Milnor as follows:

Signature of (c_1, c_2, c_3)	Simply connected Lie group	Property
$(+,+,+)$	SU_2	compact and simple
$(+,+,-)$	$\widetilde{\mathrm{SL}_2\mathbb{R}}$	non-compact and simple
$(+,+,0)$	$\widetilde{E(2)}$	solvable
$(+,-,0)$	$E(1,1)$	solvable
$(+,0,0)$	Heisenberg group Nil_3	nilpotent
$(0,0,0)$	$(\mathbb{R}^3,+)$	Abelian

Remark that parallel surfaces are classified in the first two Lie groups and in the last two Lie groups, in the case that they are equipped with a left-invariant metric giving rise to a maximal isometry group. In this paper, we study parallel surfaces in $E(1,1)$ and $\widetilde{E(2)}$.

3.2 The group of rigid motions of Minkowski plane

Let $E(1,1)$ be the motion group of the Minkowski plane:

$$E(1,1) = \left\{ \begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x, y, z \in \mathbb{R} \right\}.$$

The Lie algebra $\mathfrak{e}(1,1)$ is given explicitly by

$$\mathfrak{e}(1,1) = \left\{ \begin{pmatrix} w & 0 & u \\ 0 & -w & v \\ 0 & 0 & 0 \end{pmatrix} \,\middle|\, u, v, w \in \mathbb{R} \right\}.$$

Consider the basis

$$F_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

of $\mathfrak{e}(1,1)$. Then the left-translated vector fields of $\{F_1, F_2, F_3\}$ are given by

$$f_1 = e^z \frac{\partial}{\partial x}, \quad f_2 = e^{-z} \frac{\partial}{\partial y}, \quad f_3 = \frac{\partial}{\partial z}.$$

The dual coframe field is

$$\omega^1 = e^{-z} dx, \quad \omega^2 = e^z dy, \quad \omega^3 = dz.$$

Now we take the following left-invariant vector fields u_1, u_2, u_3:

$$u_1 = \frac{1}{\sqrt{2}}(-f_1 + f_2), \quad u_2 = \frac{1}{\sqrt{2}}(f_1 + f_2), \quad u_3 = f_3.$$

This left-invariant frame field satisfies the commutation relations

$$[u_1, u_2] = 0, \quad [u_2, u_3] = u_1, \quad [u_3, u_1] = -u_2.$$

We equip $E(1,1)$ with a left-invariant Riemannian metric such that $\{e_1, e_2, e_3\}$, with $e_i = \frac{u_i}{\lambda_i}$, is orthonormal, where $\lambda_1, \lambda_2, \lambda_3$ are positive constants. The resulting Riemannian metric is

$$g_{(\lambda_1,\lambda_2,\lambda_3)} = \frac{\lambda_1^2}{2}(-\omega^1 + \omega^2)^2 + \frac{\lambda_2^2}{2}(\omega^1 + \omega^2)^2 + \lambda_3^2(\omega^3)^2.$$

Moreover we have the following:

Theorem 1. ([6, Proposition 2.4]) *Any left-invariant metric on $E(1,1)$ is isometric to one of the metrics $g_{(\lambda_1,\lambda_2,\lambda_3)}$ with $\lambda_1 \geq \lambda_2 > 0$ and $\lambda_3 = \frac{1}{\lambda_1\lambda_2}$.*

For simplicity of notation, we put $g(\lambda_1,\lambda_2) = g_{(\lambda_1,\lambda_2,\frac{1}{\lambda_1\lambda_2})}$.

The Riemannian homogeneous manifold $\mathrm{Sol}_3 = (E(1,1), g(1,1))$ is the model space of solve-geometry in the sense of Thurston. Thus, we have obtained the fact that Sol_3 has a natural 2-parametric deformation family

$$\{(E(1,1), g(\lambda_1,\lambda_2)) \mid \lambda_1 \geq \lambda_2 > 0\}.$$

Note that this deformation preserves the unimodularity property, because all these spaces have common underlying Lie group $E(1,1)$.

3.3 The group of rigid motions of Euclidean plane

The Euclidean motion group $E(2)$ is given explictly by the following matrix group:

$$E(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x, y \in \mathbb{R},\ \theta \in \mathbb{S}^1 \right\}.$$

Let $\widetilde{E(2)}$ denote the universal covering group of $E(2)$. Then $\widetilde{E(2)}$ is $\mathbb{R}^3$ with multiplication

$$(x,y,z)\cdot(x',y',z') = (x + x'\cos z - y'\sin z\ , y + x'\sin z + y'\cos z\ , z + z').$$

Take positive constants λ_1, λ_2 and λ_3 and a left-invariant frame

$$e_1 = \frac{1}{\lambda_2}\left(-\sin z\frac{\partial}{\partial x} + \cos z\frac{\partial}{\partial y}\right), \quad e_2 = \frac{1}{\lambda_3}\frac{\partial}{\partial z}, \quad e_3 = \frac{1}{\lambda_1}\left(\cos z\frac{\partial}{\partial x} + \sin z\frac{\partial}{\partial y}\right).$$

Then this frame satisfies the following commutation relations:

$$[e_1,e_2] = c_1 e_3, \quad [e_2,e_3] = c_2 e_1, \quad [e_3,e_1] = 0,$$

with $c_1 = \frac{\lambda_1}{\lambda_2\lambda_3}$ and $c_2 = \frac{\lambda_2}{\lambda_1\lambda_3}$. The left-invariant Riemannian metric determined by the condition that $\{e_1,e_2,e_3\}$ is orthonormal, is given by

$$g_{(\lambda_1,\lambda_2,\lambda_3)} = \lambda_1^2(\cos z\,dx + \sin z\,dy)^2 + \lambda_2^2(-\sin z\,dx + \cos z\,dy)^2 + \lambda_3^2\,dz^2.$$

Also in this case, we have:

Theorem 2. ([6, Proposition 2.3]) *Any left-invariant metric on $\widetilde{E(2)}$ is isometric to one of the metrics $g_{(\lambda_1,\lambda_2,\lambda_3)}$ with $\lambda_1 > \lambda_2 > 0$ and $\lambda_3 = \frac{1}{\lambda_1\lambda_2}$, or $\lambda_1 = \lambda_2 = \lambda_3 = 1$. In particular, $\widetilde{E(2)}$ with metric $g_{(1,1,1)}$ is isometric to Euclidean 3-space $\mathbb{E}^3$.*

For simplicity of notation, we put $g(\lambda_1,\lambda_2) = g_{(\lambda_1,\lambda_2,\frac{1}{\lambda_1\lambda_2})}$.

4 Parallel surfaces in $E(1,1)$

We start with the general left-invariant metric on $E(1,1)$, given in Theorem 1:

$$g(\lambda_1,\lambda_2)=\frac{\lambda_1^2}{2}(-e^{-z}\,dx+e^z\,dy)^2+\frac{\lambda_2^2}{2}(e^{-z}\,dx+e^z\,dy)^2+\frac{1}{\lambda_1^2\lambda_2^2}\,dz^2$$

with $\lambda_1\geq\lambda_2>0$. The orthonormal frame field

$$e_1=\frac{1}{\lambda_1\sqrt{2}}\left(-e^z\frac{\partial}{\partial x}+e^{-z}\frac{\partial}{\partial y}\right),\quad e_2=\frac{1}{\lambda_2\sqrt{2}}\left(e^z\frac{\partial}{\partial x}+e^{-z}\frac{\partial}{\partial y}\right),\quad e_3=\lambda_1\lambda_2\frac{\partial}{\partial z}$$

satisfies the following commutation relations:

$$[e_1,e_2]=0,\quad [e_2,e_3]=\lambda_1^2e_1,\quad [e_3,e_1]=-\lambda_2^2e_2.$$

Using Koszul's formula, one can then compute the Levi Civita connection $\widetilde{\nabla}$. This is completely determined by

$$\begin{array}{ccc}\widetilde{\nabla}_{e_1}e_1=0, & \widetilde{\nabla}_{e_1}e_2=-\frac{\lambda_1^2+\lambda_2^2}{2}e_3, & \widetilde{\nabla}_{e_1}e_3=\frac{\lambda_1^2+\lambda_2^2}{2}e_2,\\ \widetilde{\nabla}_{e_2}e_1=-\frac{\lambda_1^2+\lambda_2^2}{2}e_3, & \widetilde{\nabla}_{e_2}e_2=0, & \widetilde{\nabla}_{e_2}e_3=\frac{\lambda_1^2+\lambda_2^2}{2}e_1,\\ \widetilde{\nabla}_{e_3}e_1=\frac{\lambda_1^2-\lambda_2^2}{2}e_2, & \widetilde{\nabla}_{e_3}e_2=-\frac{\lambda_1^2-\lambda_2^2}{2}e_1, & \widetilde{\nabla}_{e_3}e_3=0.\end{array}\tag{5}$$

The Riemann Christoffel curvature tensor $\widetilde{R}$ is then determined by the following sectional curvatures:

$$\begin{aligned}\langle\widetilde{R}(e_1,e_2)e_2,e_1\rangle &= \frac{(\lambda_1^2+\lambda_2^2)^2}{4},\\ \langle\widetilde{R}(e_2,e_3)e_3,e_2\rangle &= -\frac{(\lambda_1^2+\lambda_2^2)(3\lambda_1^2-\lambda_2^2)}{4},\\ \langle\widetilde{R}(e_1,e_3)e_3,e_1\rangle &= \frac{(\lambda_1^2+\lambda_2^2)(\lambda_1^2-3\lambda_2^2)}{4}.\end{aligned}\tag{6}$$

Remark that for $\lambda_1=\lambda_2$ (for example in Sol_3, where $\lambda_1=\lambda_2=1$), the curvature tensor can be expressed as

$$\begin{aligned}\langle\widetilde{R}(X,Y)Z,W\rangle = {} & \lambda_1^4[\langle X,W\rangle\langle Y,Z\rangle-\langle X,Z\rangle\langle Y,W\rangle\\ & +2\langle X,Z\rangle\langle Y,e_3\rangle\langle W,e_3\rangle+2\langle Y,W\rangle\langle X,e_3\rangle\langle Z,e_3\rangle\\ & -2\langle X,W\rangle\langle Y,e_3\rangle\langle Z,e_3\rangle-2\langle Y,Z\rangle\langle X,e_3\rangle\langle W,e_3\rangle].\end{aligned}$$

Now consider a surface M^2 in $E(1,1)$. Let $N=\alpha e_1+\beta e_2+\gamma e_3$ denote a unit normal vector field, then $V_1=\gamma e_1-\alpha e_3$ and $V_2=\gamma e_2-\beta e_3$ span the tangent plane to M^2 at every point. If M^2 is parallel, the normal component of $\widetilde{R}(X,Y)Z$ should vanish for every $p\in M^2$ and $X,Y,Z\in T_pM^2$, due to Codazzi's equation (3). Therefore, we prove the following:

Lemma 2. *With the notations from above, we have:*

(i) $\mathrm{Nor}(\widetilde{R}(V_1,V_2)V_1)=(\lambda_1^2+\lambda_2^2)\beta\gamma(\lambda_1^2\alpha^2-\lambda_2^2(\alpha^2+\gamma^2))N,$

(ii) $\mathrm{Nor}(\widetilde{R}(V_1,V_2)V_2) = (\lambda_1^2+\lambda_2^2)\alpha\gamma(\lambda_1^2(\beta^2+\gamma^2)-\lambda_2^2\beta^2)N$.

Proof. We will only prove the first equality. The second one can be proven analoguously. Using (6), we find

$$\begin{aligned}
\widetilde{R}(V_1,V_2)V_1 &= \widetilde{R}(\gamma e_1-\alpha e_3,\gamma e_2-\beta e_3)(\gamma e_1-\alpha e_3)\\
&= \gamma^3\widetilde{R}(e_1,e_2)e_1-\alpha\gamma^2\widetilde{R}(e_1,e_2)e_3-\beta\gamma^2\widetilde{R}(e_1,e_3)e_1\\
&\quad +\alpha\beta\gamma\widetilde{R}(e_1,e_3)e_3-\alpha\gamma^2\widetilde{R}(e_3,e_2)e_1+\alpha^2\gamma\widetilde{R}(e_3,e_2)e_3\\
&= \frac{\lambda_1^2+\lambda_2^2}{4}[-\gamma^3(\lambda_1^2+\lambda_2^2)e_2+\beta\gamma^2(\lambda_1^2-3\lambda_2^2)e_3\\
&\quad +\alpha\beta\gamma(\lambda_1^2-3\lambda_2^2)e_1+\alpha^2\gamma(3\lambda_1^2-\lambda_2^2)e_2]. \qquad (7)
\end{aligned}$$

If we develop $\frac{4}{\lambda_1^2+\lambda_2^2}\widetilde{R}(V_1,V_2)V_1$ in the basis $\{V_1,V_2,N\}$, we get

$$\begin{aligned}
\frac{4}{\lambda_1^2+\lambda_2^2}\widetilde{R}(V_1,V_2)V_1 &= pV_1+qV_2+rN\\
&= p(\gamma e_1-\alpha e_3)+q(\gamma e_2-\beta e_3)+r(\alpha e_1+\beta e_2+\gamma e_3)\\
&= (\gamma p+\alpha r)e_1+(\gamma q+\beta r)e_2+(-\alpha p-\beta q+\gamma r)e_3. \qquad (8)
\end{aligned}$$

Comparing (7) and (8), we get

$$\begin{cases}
\gamma p+\alpha r = \alpha\beta\gamma(\lambda_1^2-3\lambda_2^2)\\
\gamma q+\beta r = -\gamma^3(\lambda_1^2+\lambda_2^2)+\alpha^2\gamma(3\lambda_1^2-\lambda_2^2)\\
-\alpha p-\beta q+\gamma r = \beta\gamma^2(\lambda_1^2-3\lambda_2^2),
\end{cases}$$

from which $r = 4\beta\gamma(\lambda_1^2\alpha^2-\lambda_2^2(\alpha^2+\gamma^2))$. This proves the first formula. □

We can now formulate the main theorem of this section.

Theorem 3. *Let M^2 be a parallel surface in $(E(1,1),g(\lambda_1,\lambda_2))$. Then M^2 is one of the following:*

(i) an integral surface of the distribution spanned by $\left\{\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right\}$,

(ii) an integral surface of the distribution spanned by $\left\{\frac{\partial}{\partial x},\frac{\partial}{\partial z}\right\}$ or $\left\{\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right\}$,

the latter case only occuring if $\lambda_1=\lambda_2$. Moreover the surfaces described in (i) *are flat and minimal, but not totally geodesic and the surfaces in* (ii) *are totally geodesic and have constant Gaussian curvature $-\lambda_1^4$.*

Proof. Since M^2 is parallel, Codazzi's equation (3) implies $\mathrm{Nor}(\widetilde{R}(V_1,V_2)V_1) = \mathrm{Nor}(\widetilde{R}(V_1,V_2)V_2) = 0$. From Lemma 2, it follows that there are two cases to consider: $N=\pm e_3$ and $N=\alpha e_1+\beta e_2$, with $\alpha^2+\beta^2=1$.

Case 1: $N=\pm e_3$

We assume $N=e_3$ (if $N=-e_3$, the argument is analoguous). Remark that $TM^2=\mathrm{span}\{e_1,e_2\}$ and that the shape-operator S is given by $Se_1=-\widetilde{\nabla}_{e_1}e_3 = -\frac{\lambda_1^2+\lambda_2^2}{2}e_2$, $Se_2=-\widetilde{\nabla}_{e_2}e_3=-\frac{\lambda_1^2+\lambda_2^2}{2}e_1$, i.e.

$$S=-\frac{\lambda_1^2+\lambda_2^2}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}.$$

From Gauss' equation (4) we have that the Gaussian curvature of M^2 is

$$\begin{aligned} K &= \langle \widetilde{R}(e_1,e_2)e_2,e_1\rangle + \det S \\ &= \frac{(\lambda_1^2+\lambda_2^2)^2}{4} - \left(-\frac{\lambda_1^2+\lambda_2^2}{2}\right)^2 \\ &= 0. \end{aligned}$$

Hence M^2 is flat and minimal but not totally geodesic, even not totally umbilical. Gauss' formula (1) yields that the Levi Civita connection ∇ of M^2 satisfies $\nabla_{e_1}e_1 = \nabla_{e_1}e_2 = \nabla_{e_2}e_1 = \nabla_{e_2}e_2 = 0$, which implies that $(\nabla h)(e_i,e_j,e_k) = 0$ for $i,j,k \in \{1,2\}$. Hence M^2 is indeed parallel.

Case 2: $N = \alpha e_1 + \beta e_2$, with $\alpha^2+\beta^2 = 1$

Now the tangent plane to M^2 is spanned by the orthonormal basis $\{E_1,E_2\}$, with $E_1 = \beta e_1 - \alpha e_2$, $E_2 = e_3$. A necessary and sufficient condition for this distribution to be integrable, is given by Frobenius' theorem, namely $[E_1,E_2] \in \mathrm{span}\{E_1,E_2\}$. In this case, we have

$$\begin{aligned} [E_1,E_2] &= \widetilde{\nabla}_{\beta e_1-\alpha e_2}e_3 - \widetilde{\nabla}_{e_3}(\beta e_1 - \alpha e_2) \\ &= (-\alpha\lambda_1^2 - E_2[\beta])e_1 + (\beta\lambda_2^2 + E_2[\alpha])e_2, \end{aligned}$$

which is in the distribution $\mathrm{span}\{E_1,E_2\}$ if and only if it lies in the direction of E_1, i.e. if and only if

$$\alpha E_2[\beta] - \beta E_2[\alpha] = \lambda_2^2\beta^2 - \lambda_1^2\alpha^2. \tag{9}$$

To determine the shape operator of M^2, we compute

$$\begin{aligned} \widetilde{\nabla}_{E_1}N &= \widetilde{\nabla}_{\beta e_1-\alpha e_2}(\alpha e_1+\beta e_2) = (\beta E_1[\alpha] - \alpha E_1[\beta])E_1 + \frac{\lambda_1^2+\lambda_2^2}{2}(\alpha^2-\beta^2)E_2, \\ \widetilde{\nabla}_{E_2}N &= \widetilde{\nabla}_{e_3}(\alpha e_1+\beta e_2) = (\beta E_2[\alpha] - \alpha E_2[\beta] - \frac{\lambda_1^2-\lambda_2^2}{2})E_1, \end{aligned}$$

from which

$$S = \begin{pmatrix} \alpha E_1[\beta] - \beta E_1[\alpha] & \alpha E_2[\beta] - \beta E_2[\alpha] + \frac{\lambda_1^2-\lambda_2^2}{2} \\ \frac{\lambda_1^2+\lambda_2^2}{2}(\beta^2-\alpha^2) & 0 \end{pmatrix}.$$

The symmetry condition for S is the same as (9) and we get

$$S = \begin{pmatrix} \alpha E_1[\beta] - \beta E_1[\alpha] & \frac{\lambda_1^2+\lambda_2^2}{2}(\beta^2-\alpha^2) \\ \frac{\lambda_1^2+\lambda_2^2}{2}(\beta^2-\alpha^2) & 0 \end{pmatrix}.$$

Recall that a parallel surface is automatically semi-parallel and from Lemma 1, we get that M^2 is flat or totally umbilical.

M^2 is flat if and only if

$$\langle \widetilde{R}(\beta e_1 - \alpha e_2, e_3)e_3, \beta e_1 - \alpha e_2\rangle + \det S = 0,$$

or equivalently

$$(\lambda_1^2+\lambda_2^2)\beta^4 - 2\lambda_1^2\beta^2 + \lambda_1^2 = 0,$$

which has obviously no solutions for β.

On the other hand, the expression for S shows that M^2 is totally umbilical if and only if it is totally geodesic, implying $\alpha^2 = \beta^2$ and $\alpha E_1[\beta] = \beta E_1[\alpha]$, or equivalently, $\alpha = \pm\beta = \frac{1}{\sqrt{2}}$ (since $\alpha^2 + \beta^2 = 1$). This is compatible with (9) if and only if $\lambda_1 = \lambda_2$. For $\alpha = \beta = \frac{1}{\sqrt{2}}$, we have $E_1 = -\frac{1}{\lambda_1} e^z \frac{\partial}{\partial x}$, $E_2 = \lambda_1^2 \frac{\partial}{\partial z}$ and for $\alpha = -\beta = \frac{1}{\sqrt{2}}$ we have $E_1 = -\frac{1}{\lambda_1} e^{-z} \frac{\partial}{\partial y}$, $E_2 = \lambda_1^2 \frac{\partial}{\partial z}$. The Gaussian curvature of both families of surfaces can be computed, using Gauss' equation, as $K = -\lambda_1^4$. □

Remark 1. The complete surface through the identity, which belongs to the first family described in the theorem, is given by the parametrization $\varphi : \mathbb{R}^2 \to E(1,1) : (u,v) \mapsto \varphi(u,v)$, with

$$\begin{aligned}
\varphi(u,v) &= \exp\left(u\frac{\partial}{\partial x}\right)\exp\left(v\frac{\partial}{\partial y}\right) \\
&= \exp\begin{pmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\exp\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & u \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

This is exactly the group of all translations of Minkowski plane. Analogously, the surfaces through the identity of the families given in the second part of the theorem, are generated by isometries preserving the origin and translations in the x- resp. the y-direction.

Corollary 1. *A parallel surface in* Sol_3 *is an integral surface of one of the following distributions:* $\operatorname{span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$, $\operatorname{span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right\}$ *or* $\operatorname{span}\left\{\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\}$. *Surfaces of the first type are flat and minimal, but not totally geodesic, whereas surfaces of the second and the third type are totally geodesic and have constant Gaussian curvature* -1.

5 Parallel surfaces in $\widetilde{E(2)}$

We start with the general left-invariant metric on $\widetilde{E(2)}$, given in Theorem 2:

$$g(\lambda_1,\lambda_2) = \lambda_1^2(\cos z\,dx + \sin z\,dy)^2 + \lambda_2^2(-\sin z\,dx + \cos z\,dy)^2 + \frac{1}{\lambda_1^2\lambda_2^2}\,dz^2$$

with $\lambda_1 > \lambda_2 > 0$ or $\lambda_1 = \lambda_2 = 1$. In the last case, we are dealing with Euclidean space and hence we exclude this case. The orthonormal frame field

$$e_1 = \frac{1}{\lambda_2}\left(-\sin z\frac{\partial}{\partial x} + \cos z\frac{\partial}{\partial y}\right), \quad e_2 = \lambda_1\lambda_2\frac{\partial}{\partial z}, \quad e_3 = \frac{1}{\lambda_1}\left(\cos z\frac{\partial}{\partial x} + \sin z\frac{\partial}{\partial y}\right),$$

satisfies the following commutation relations:

$$[e_1, e_2] = \lambda_1^2 e_3, \quad [e_2, e_3] = \lambda_2^2 e_1, \quad [e_3, e_1] = 0.$$

The Levi Civita connection is given by

$$\begin{array}{lll} \widetilde{\nabla}_{e_1} e_1 = 0, & \widetilde{\nabla}_{e_1} e_2 = \frac{\lambda_1^2-\lambda_2^2}{2} e_3, & \widetilde{\nabla}_{e_1} e_3 = -\frac{\lambda_1^2-\lambda_2^2}{2} e_2, \\ \widetilde{\nabla}_{e_2} e_1 = -\frac{\lambda_1^2+\lambda_2^2}{2} e_3, & \widetilde{\nabla}_{e_2} e_2 = 0, & \widetilde{\nabla}_{e_2} e_3 = \frac{\lambda_1^2+\lambda_2^2}{2} e_1, \\ \widetilde{\nabla}_{e_3} e_1 = -\frac{\lambda_1^2-\lambda_2^2}{2} e_2, & \widetilde{\nabla}_{e_3} e_2 = \frac{\lambda_1^2-\lambda_2^2}{2} e_1, & \widetilde{\nabla}_{e_3} e_3 = 0. \end{array} \tag{10}$$

The curvature of the space is completely determined by

$$\begin{aligned} \langle \widetilde{R}(e_1, e_2)e_2, e_1 \rangle &= -\frac{(\lambda_1^2 - \lambda_2^2)(3\lambda_1^2 + \lambda_2^2)}{4}, \\ \langle \widetilde{R}(e_2, e_3)e_3, e_2 \rangle &= \frac{(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 + 3\lambda_2^2)}{4}, \\ \langle \widetilde{R}(e_1, e_3)e_3, e_1 \rangle &= \frac{(\lambda_1^2 - \lambda_2^2)^2}{4}. \end{aligned} \tag{11}$$

Now consider a surface M^2 in $\widetilde{E(2)}$. We will use the same notations as in the previous section, namely $N = \alpha e_1 + \beta e_2 + \gamma e_3$ is a unit normal vector and $V_1 = \gamma e_1 - \alpha e_3$ and $V_2 = \gamma e_2 - \beta e_3$ span the tangent plane to M^2 at every point. The proof of the following lemma is completely analoguous to that of Lemma 2.

Lemma 3. *With the notations from above, we have:*

(i) $\mathrm{Nor}(\widetilde{R}(V_1, V_2)V_1) = (\lambda_1^2 - \lambda_2^2)\beta\gamma(\lambda_1^2\gamma^2 - \lambda_2^2\alpha^2)N,$

(ii) $\mathrm{Nor}(\widetilde{R}(V_1, V_2)V_2) = -(\lambda_1^2 - \lambda_2^2)\alpha\gamma(\lambda_1^2\gamma^2 - \lambda_2^2(\beta^2 + \gamma^2))N.$

This enables us to prove the classification of parallel surfaces in $\widetilde{E(2)}$.

Theorem 4. *The only parallel surfaces in $\widetilde{E(2)}$ are integral surfaces of the distribution spanned by $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$. These surfaces are flat and minimal, but not totally geodesic.*

Proof. Since the proof is similar to that of Theorem 3, we will not go into much details.

Codazzi's equation (3) and Lemma 3 imply that there are two cases to consider, namely $N = \pm e_3$ and $N = \alpha e_1 + \beta e_2$, with $\alpha^2 + \beta^2 = 1$. In the first case, we have that the tangent plane to M^2 is spanned by $\{e_1, e_2\}$. But this is impossible due to Frobenius' theorem, since $[e_1, e_2] = \lambda_1^2 e_3$.

So the only possibility is $N = \alpha e_1 + \beta e_2$, with $\alpha^2 + \beta^2 = 1$, in which case $E_1 = \beta e_1 - \alpha e_2$ and $E_2 = e_3$ form an orthonormal basis for the tangent plane. The integrability condition is

$$\alpha E_2[\beta] - \beta E_2[\alpha] + \lambda_2^2 \alpha^2 = 0 \tag{12}$$

and we have

$$\begin{aligned}\widetilde{\nabla}_{E_1}N &= (\beta E_1[\alpha]-\alpha E_1[\beta])E_1+(\frac{\lambda_1^2+\lambda_2^2}{2}-\lambda_2^2\beta^2)E_2,\\ \widetilde{\nabla}_{E_2}N &= (\beta E_2[\alpha]-\alpha E_2[\beta]+\frac{\lambda_1^2-\lambda_2^2}{2})E_1.\end{aligned}$$

The symmetry condition for the shape-operator S coincides with (12), and we obtain

$$S=\begin{pmatrix}\alpha E_1[\beta]-\beta E_1[\alpha] & \lambda_2^2\beta^2-\frac{\lambda_1^2+\lambda_2^2}{2}\\ \lambda_2^2\beta^2-\frac{\lambda_1^2+\lambda_2^2}{2} & 0\end{pmatrix}.$$

From Lemma 1, we know that M^2 should be totally umbilical or flat.

It is clear that M^2 is totally umbilical if and only if it is totally geodesic. This implies $\beta^2=\frac{\lambda_1^2+\lambda_2^2}{2\lambda_2^2}$ and thus $\alpha^2=\frac{\lambda_2^2-\lambda_1^2}{2\lambda_2^2}$, which is impossible since $\lambda_1>\lambda_2>0$.

Finally, M^2 is flat if and only if

$$\langle\widetilde{R}(\beta e_1-\alpha e_2,e_3)e_3,\beta e_1-\alpha e_2\rangle-\left(\lambda_2^2\beta^2-\frac{\lambda_1^2+\lambda_2^2}{2}\right)^2=0,$$

which is equivalent to $\beta^2=1$. This means that e_1 and e_3 span the tangent plane to M^2, proving the theorem. □

From the proof, the following is clear:

Corollary 2. ([8]) $\widetilde{E(2)}$ *does not admit any totally geodesic surfaces.*

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