Short proof of a metrization theorem*

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Abstract

In this note we provide a new insight into trans-separable spaces, *i.e.* those which are separable by seminorms. This approach enables getting an easy proof of the fact that in a wide class of uniform spaces, containing (DF) and (LM)-spaces, precompact subsets are metrizable.

1 Introduction

A uniform space X is *trans-separable* [3] if every uniform cover [10, p. 210] of X admits a countable subcover. This general notion had already been used by Garnir, de Wilde and Schmets [5] as *separable seminormed space* and had proven to be useful while studying concrete uniform spaces problems including locally convex spaces (lcs), and more recently has been used by Robertson [8] in order to study the metrizability of (pre)compact sets in uniform spaces. His goal was an attempt to clarify the first of the following two fundamental questions which arise in a natural way concerning compactness in any lcs E:

Question 1. When are the compact sets in E metrizable?

Question 2. When is E weakly angelic?

Partial answers were known since Pfister [6] proved that every precompact set in a (DF)-space is metrizable by using the fact that a lcs E is *trans-separable* if and only if each equicontinuous subset of the dual space E' of E is $\sigma(E', E)$-metrizable.

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Pfister's result was extended by Valdivia [9, p. 68 (28)] in dual metric spaces by using *quasi-Souslin* spaces.

Later on Cascales and Orihuela [2] studied the aforementioned two problems in a large class of lcs, called $\mathfrak{G}$ which will be conveniently recalled hereafter, and include (LF)- and (DF)-spaces. Their approach provided a general result [2, Theorem 1] stating that all precompact sets in a lcs in class $\mathfrak{G}$ are metrizable and every space in class $\mathfrak{G}$ is weakly angelic. They also showed that if E is a lcs in class $\mathfrak{G}$ and K is a weakly compact set in E contained in a separable subspace of E, then K is metrizable [2, 1.14] by using K-analytic structures connected with ordered families of compact sets in uniform spaces. Cascales, Kąkol and Saxon [1, Theorem 3.1] provided some alternative proofs for (LM)-spaces and dual metric spaces.

A different form of Cascales-Orihuela's theorem was presented by Robertson [8] who proved that a uniform space is trans-separable whenever it is covered by an adequately ordered family of precompact sets, applying this result to show that precompact sets in a lcs E are metrizable if its topological dual E', endowed with the topology $\mathfrak{T}_{pr}$ of uniform convergence on precompact subsets of E, is covered by an ordered family of $\mathfrak{T}_{pr}$ precompact subsets.

The aim of this note is to apply Robertson's result in order to provide a new and simple proof of the following result of Cascales and Orihuela [2] whose original proof was based on some results of K-analytic spaces, angelic spaces and lower semi-continuous maps.

Theorem 1. Let $(X,\mathcal{U})$ be a uniform space such that the uniformity $\mathcal{U}$ has got a base $B = \left\{N_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\right\}$ verifying: *(*)* for every $\alpha, \beta \in \mathbb{N}^\mathbb{N}$, with $\alpha \leq \beta$ (that is with $\alpha(n) \leq \beta(n)$ for each n) it holds $N_\beta \subset N_\alpha$. Then the precompact subsets of $(X,\mathcal{U})$ are metrizable in the induced uniformity.

Let us recall that a lcs belongs to class $\mathfrak{G}$ if there is a family $\mathfrak{A} = \left\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\right\}$ of subsets of E', called its $\mathfrak{G}$-representation, such that:

1. $\mathfrak{A}$ covers E';

2. the A_α satisfy condition (*) above;

3. sequences in each A_α are equicontinuous.

Class $\mathfrak{G}$ enjoys good stability properties and includes many important lcs classes such as dual metric spaces (hence (DF)-spaces), (LM)-spaces (hence (LF)-spaces), the space of distributions $D'(\Omega)$ and real analytic $A(\Omega)$-spaces for open $\Omega \subset \mathbb{R}^\mathbb{N}$, [1, 2]. Condition (c) implies that each A_α is countably $\sigma(E', E)$-relatively compact, hence $\mathfrak{T}_{pr}$-precompact.

Finally note that trans-separability for a lcs E means exactly that E is separable with respect to every continuous seminorm on E, or equivalently, for every neighbourhood of the origin U in E there exists a countable subset N in E such that $E = N + U$, [6]. Pfister also noted the easy and useful following observation.

Lemma 2. Let E be a lcs. Then the space E is trans-separable iff every equicontinuous set in E' is metrizable in the weak topology $\sigma(E', E)$.

2 Proof of Theorem

Let $C_{pr}^{buc}(X)$ be the lcs of bounded real-valued uniformly continuous functions defined on X with the topology of uniform convergence on precompact subsets of X, and for each $\alpha = (m, \alpha_1, \alpha_2, ..., \alpha_n, ...) \in \mathbb{N} \times \left(\mathbb{N}^{\mathbb{N}}\right)^{\mathbb{N}}$ let us denote by A_α the set formed by those $f \in C_p^{buc}(X)$ such that

$$\|f\|_\infty \leq m, |f(s) - f(t)| \leq 1/n \text{ for } (s,t) \in (X \times X) \cap N_{\alpha_n}, n \in \mathbb{N}.$$

Clearly $A_\alpha \subset A_\beta$ whenever $\alpha \leq \beta$ and $C_{pr}^{buc}(X) = \bigcup\{A_\alpha : \alpha \in \mathbb{N} \times \left(\mathbb{N}^{\mathbb{N}}\right)^{\mathbb{N}}\}$.

Note that each A_α is pointwise compact and, by equicontinuity, the pointwise topology and the induced topology by $C_{pr}^{buc}(X)$ coincide on each A_α, therefore each A_α is a compact subset of $C_{pr}^{buc}(X)$. By the aforementioned Robertson's result [8] it follows that $C_{pr}^{buc}(X)$ is trans-separable. Hence by Lemma 2 each equicontinuous subset of the dual $C_{pr}^{buc}(X)'$ is $\sigma\left(C_{pr}^{buc}(X)', C_{pr}^{buc}(X)\right)$ $-$metrizable.

Finally let K be a precompact subset of X and let us consider

$$W := \{f \in C_{pr}^{buc}(X) : \sup_{x \in K} |f(x)| \leq 1\}.$$

Let φ be the evaluation map from X into $C_{pr}^{buc}(X)'$, which is an isomorphism onto $\varphi(X)$ provided with the topology induced by the weak*-topology by the construction given in [4, Section 6.8]. From the equality $\varphi(K) = W^\circ \cap \varphi(X)$, it follows that K is metrizable.

Remark 1. Cascales and Orihuela approach to Question 1 [2, Theorem 2] uses Theorem 1 applied to the topology τ' of uniform convergence on the sets A_α. It turns out that for quasi-barrelled spaces in class $\mathfrak{G}$ the argument is even easier since then τ' is coarser than τ and, both topologies coinciding on any τ-precompact subset, the proven theorem may be applied.

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