Lagrangian submanifolds attaining equality in the improved Chen's inequality *

J. Bolton L. Vrancken

Abstract

In [7] Oprea gave an improved version of Chen's inequality for Lagrangian submanifolds of $\mathbb{C}P^n(4)$. For minimal submanifolds this inequality coincides with a previous version proved in [5]. We consider here those non-minimal 3-dimensional Lagrangian submanifolds in $\mathbb{C}P^3(4)$ attaining at all points equality in the improved Chen inequality. We show how all such submanifolds may be obtained starting from a minimal Lagrangian surface in $\mathbb{C}P^2(4)$.

1 Introduction

In the early nineties Chen [4] introduced a new invariant, called δ_M, for a Riemannian manifold M. Specifically, $\delta_M : M \to \mathbb{R}$ is given by:

$$\delta_M(p) = \tau(p) - (\inf K)(p),$$

where $(\inf K)(p) = \inf \left\{ K(\pi) \,|\, \pi \text{ is a 2-dimensional subspace of } T_pM \right\}$, with $K(\pi)$ being the sectional curvature of π, and $\tau(p) = \sum_{i<j} K(e_i \wedge e_j)$ denotes the scalar curvature defined in terms of an orthonormal basis $\{e_1, \ldots, e_n\}$ of the tangent space T_pM of M at p. In the same paper, he discovered, for submanifolds of real space forms, an inequality relating this invariant with the length of the mean curvature vector H. A similar inequality was proved in [5] and [6] for n-dimensional Lagrangian

*This work was done during a research visit of the second author at Durham University. Both authors are grateful for the support of the universities of Durham and Valenciennes

Received by the editors April 2006.

Communicated by L. Vanhecke.

1991 *Mathematics Subject Classification* : 53B25, 53B20.

Key words and phrases : Lagrangian submanifold, complex projective space, Chen inequality.

submanifolds of a complex space form $\tilde{M}^n(4c)$ of constant holomorphic sectional curvature $4c$. Indeed, it was shown that

$$\delta_M \leq \tfrac{(n-2)(n+1)}{2}\tfrac{c}{4} + \tfrac{n^2}{2}\tfrac{n-2}{n-1}\|H\|^2 . \tag{1}$$

Note that, for $n = 2$, both sides of the above inequality are zero.

Let $\mathbb{C}P^n(4)$ denote complex projective n-space of constant holomorphic sectional curvature 4. For $n \geq 3$, Lagrangian submanifolds of $\mathbb{C}P^n(4)$ attaining at every point equality in (1) were studied in, amongst others, [5], [6], [1] and [2]. In particular, in [5] and [6], it was shown that such submanifolds are minimal, and in [1] and [2] a complete classification was obtained of 3-dimensional Lagrangian submanifolds of $\mathbb{C}P^3(4)$ attaining at each point equality in (1). Such submanifolds are obtained starting from minimal surfaces with ellipse of curvature a circle in the unit 5-sphere.

However, Oprea [7] has recently shown that the inequality (1) is not optimal, and, for $n \geq 3$, can be improved to

$$\delta_M \leq \tfrac{(n-2)(n+1)}{2}\tfrac{c}{4} + \tfrac{n^2}{2}\tfrac{2n-3}{2n+3}\|H\|^2 . \tag{2}$$

This explains why a Lagrangian submanifold of $\mathbb{C}P^n(4)$ attaining at every point equality in (1) must be minimal, since both inequalities coincide in this case.

2 Classification

Let M be a Lagrangian submanifold of $\mathbb{C}P^n(4)$. A careful analysis of Oprea's arguments shows that equality in (2) is obtained at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of the tangent space T_pM such that the symmetric cubic form C on M constructed using the second fundamental form h, the complex structure J and the Riemannian metric $< , >$ on $\mathbb{C}P^n(4)$ given by

$$C(X, Y, Z) =< h(X, Y), JZ >,$$

has the following form,

$$< h(e_2, e_2), Je_2 > = - < h(e_3, e_3), Je_2 > \tag{3}$$

$$4 < h(e_2, e_2), Je_1 >= 4 < h(e_3, e_3), Je_1 > =< h(e_1, e_1), Je_1 >= 3 < h(e_j, e_j), Je_1 >, \tag{4}$$

where $j \in \{4, \ldots, n\}$, and all other components of C are zero unless they can be obtained from the above using the symmetric nature of C.

In this paper we show that the inequality (2) is optimal, and we show how to construct all non-minimal Lagrangian submanifolds of $\mathbb{C}P^3(4)$ which attain everywhere equality in (2) (the classification in the minimal case having been found in [5] and [6]).

We now assume that M is a non-minimal Lagrangian submanifold of $\mathbb{C}P^3(4)$ attaining at all points equality in the improved Chen inequality (2). Then C satisfies (3) and (4) at all points. Thus, using the notation and terminology of [9], M is a non-minimal submanifold of Type 2 with the additional condition that $\lambda_1 = 4\lambda_2 \neq 0$, where $\lambda_1 =< h(e_1, e_1), Je_1 >$ and $\lambda_2 =< h(e_2, e_2), Je_1 >$. We have chosen the above orthonormal basis $\{e_1, e_2, e_3\}$ so that the notation agrees with [9], and, in particular, the plane for which the minimal sectional curvature is attained is that spanned by e_2 and e_3.

Since M is Lagrangian, there is a horizontal lift $E_0 : M \to S^7(1) \subset \mathbb{R}^8 = \mathbb{C}^4$ to the unit 7-sphere [8], and if dE_0 denotes the derivative of E_0, we put $E_j = dE_0(e_j)$, for $j = 1, 2, 3$. We will often identify a point of M with its image under E_0.

It follows from [9] that, for some suitable function b_1,

$$D_{E_1}E_1 = 4\lambda_2 iE_1 - E_0, \tag{5}$$
$$D_{E_j}E_1 = (b_1 + i\lambda_2)E_j, \qquad j = 2, 3, \tag{6}$$

where D denotes the standard flat covariant derivative on $\mathbb{C}^4$. We also get from (41), (42), (50) and (51) of [9] that the functions λ_2 and b_1 have zero derivative with respect to E_2 and E_3, and from (40) and (46) of [9] that their derivatives in the E_1 direction are given by

$$E_1(\lambda_2) = 2\lambda_2 b_1, \tag{7}$$
$$E_1(b_1) = -(1 + b_1^2 + 3\lambda_2^2). \tag{8}$$

The following lemma is immediate from (5) and (6).

Lemma 1 *The brackets $[E_1, E_2]$, $[E_1, E_3]$, $[E_2, E_3]$ are linear combinations of E_2 and E_3.*

In [9], submanifolds of the type we are considering are divided into 3 further subcases depending on the relative values of $a =< h(e_2, e_2), Je_2 >$ and λ_2. One of these cases is easy to deal with, namely that in which $a \neq 0$ but $a^2 - 2\lambda_2^2 = 0$. In this case, it follows from equations (33) - (45) of [9] that $b_1 = 0$ which contradicts (8). Hence this case cannot occur.

We now consider the other two cases, namely those where $a = 0$, or both a and $a^2 - 2\lambda_2^2$ are non-zero. We introduce a function θ defined locally on M having zero derivative with respect to E_2 and E_3 and satisfying $E_1(\theta) = -\lambda_2$. It follows from Lemma 1 that the integrability conditions of the this system for θ are satisfied, and hence such a function θ exists.

We now consider the maps into $S^7(1)$ given by

$$V = (-(b_1 + i\lambda_2)E_0 + E_1)/\sqrt{1 + b_1^2 + \lambda_2^2}, \tag{9}$$
$$W = e^{i\theta}(E_0 - (-b_1 + i\lambda_2)E_1)/\sqrt{1 + b_1^2 + \lambda_2^2}. \tag{10}$$

It follows easily that $D_{E_2}V = D_{E_3}V = 0$ and $D_{E_1}V = 3\lambda_2 iV$. This implies that V is contained in the unit circle of a complex plane $\mathbb{C}$, and, taking t as the standard parameter along this circle, we also have that $E_1(t) = 3\lambda_2$. Hence, after applying a translation if necessary, we may assume that $\theta = -t/3$.

Lemma 2 *The map W describes a minimal horizontal surface in the unit sphere $S^5(1)$ of the orthogonal complement in $\mathbb{C}^4$ of the complex plane containing V.*

Proof: It is clear that W is orthogonal to V and iV, so the image of W is contained in the indicated $S^5(1)$. We now use arguments similar to those employed for V above to complete the proof. In fact,

$$\begin{aligned}
&D_{E_1}W = 0,\\
&D_{E_j}W = \sqrt{1+b_1^2+\lambda_2^2}e^{i\theta}E_j, \quad j=2,3,\\
&D_{E_2}(D_{E_2}W) = b_3D_{E_3}W + iaD_{E_2}W - (1+b_1^2+\lambda_2^2)W,\\
&D_{E_2}(D_{E_3}W) = -b_3D_{E_2}W - iaD_{E_3}W,\\
&D_{E_3}(D_{E_2}W) = c_2D_{E_3}W - iaD_{E_3}W,\\
&D_{E_3}(D_{E_3}W) = -c_2D_{E_2}W - iaD_{E_2}W - (1+b_1^2+\lambda_2^2)W,
\end{aligned}$$

from which the proof of the lemma quickly follows. qed

We can now state and prove our classification theorem.

Theorem 1 *Let M be a non-minimal Lagrangian submanifold of $\mathbb{C}P^3(4)$ which attains equality at every point in Oprea's improvement* (2) *of Chen's inequality. Then there is a minimal Lagrangian surface $\tilde{W}(z,\bar{z})$ in $\mathbb{C}P^2(4)$ such that M can be locally written as $[E_0]$ where*

$$E_0(t,z,\bar{z}) = \frac{e^{it/3}}{\sqrt{1+b_1^2+\lambda_2^2}}(0,W(z,\bar{z})) + \frac{(-b_1+i\lambda_2)}{\sqrt{1+b_1^2+\lambda_2^2}}(e^{it},0,0,0),$$

where b_1 and λ_2 are solutions of the following system of ordinary differential equations:

$$\frac{db_1}{dt} = -\frac{1+3\lambda_2^2+b_1^2}{3\lambda_2}, \qquad \frac{d\lambda_2}{dt} = \frac{2}{3}b_1, \tag{11}$$

and W is a horizontal lift to $S^5(1)$ of $\tilde{W}$. Conversely any 3 dimensional Lagrangian submanifold obtained in this way attains equality at each point in (2).

Proof: By [8], minimal horizontal surfaces in $S^5(1)$ correspond to minimal Lagrangian surfaces in $\mathbb{C}P^2(4)$. Solving (9) and (10) for E_0, we find that, after applying a suitable element of $SU(4)$, the original immersion is the projection onto $\mathbb{C}P^3(4)$ of the map E_0 given above, where, from (7) and (8), b_1 and λ_2 are solutions of the system (11). Conversely, it is clear that any submanifold obtained in this way has an orthonormal basis of the tangent space at each point satisfying (3) and (4). Hence equality is attained in (2) at each point. qed

Remarks (i) It is clear that $\lambda_2(1+\lambda_2^2+b_1^2)$ is a first integrand of the system (11).
(ii) An alternative method of proof would be to apply immediately Theorem 7 or Theorem 9 of [9]. However the result in that case would have been less explicit.
(iii) Lagrangian immersions into $\mathbb{C}P^n(4)$ constructed from a curve in $S^3(1)$ and a lower dimensional Lagrangian immersion have been studied in [3].

References

[1] J. Bolton, C. Scharlach, L. Vrancken and L. M. Woodward. From certain minimal Lagrangian submanifolds of the 3-dimensional complex projective space to minimal surfaces in the 5-sphere. *Proceedings of the Fifth Pacific Rim Geometry Conference, Tohoku University.* Tohoku Mathematical Publication No. **20**, 23–31, 2001.

[2] J. Bolton, C. Scharlach and L. Vrancken. From surfaces in the 5-sphere to 3-manifolds in complex projective 3-space. *Bull. Austral. Math. Soc.* **66**, 465–475, 2002.

[3] I. Castro, H. Li and F. Urbano. Hamiltonian-minimal Lagrangian submanifolds in complex space forms. *Pacific Journal Math.* **227**, 43-63, 2006.

[4] B.-Y. Chen Some pinching and classification theorems for minimal submanifolds. *Archiv Math.* **60**, 568–578, 1993.

[5] B.-Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken. Totally real submanifolds of $\mathbb{C}P^n$ satisfying a basic equality. *Arch. Math.* **63**, 553–564, 1994.

[6] B.-Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken. An exotic totally real minimal immersion of S^3 into $\mathbb{C}P^3$ and its characterization. *Proc. Royal Soc. Edinburgh* **126** 153–165, 1996.

[7] T. Oprea. Chen's equality in Lagrangian case. Preprint arXiv:math.DG/0511087, 2005.

[8] H. Reckziegel. Horizontal lifts of isometric immersions into the bundle space of a pseudo-Riemannian submersion. In *Global Differential Geometry and Global Analysis (1984)*, Lecture Notes in Mathematics 1156, Springer Verlag (1985), 264-279.

[9] C. Rodriguez Montealegre and L. Vrancken. Lagrangian submanifolds of the three dimensional complex projective space. *J. Math. Soc. Japan* **53**, 603–631, 2001.

J. Bolton,
Dept of Mathematical Sciences, University of Durham,
Durham DH1 3LE, UK.
E-mail: john.bolton@dur.ac.uk

L. Vrancken,
LAMATH, ISTV2, Université de Valenciennes,
Campus du Mont Houy, 59313 Valenciennes Cedex 9, France.
E-mail: luc.vrancken@univ-valenciennes.fr