On the angular distribution of mass by Besov functions

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Abstract

Let $\mathbb{D}$ be the open unit disk in the complex plane. For $\varepsilon > 0$ we consider the sector $\Sigma_\varepsilon = \{z \ : \ |\arg z| < \varepsilon\}$. We will prove that for certain classes of functions f in the Besov's space $B_p(\mathbb{D})$ such that $f(0) = 0$, the B_p norm is obtained by integration over $f^{-1}(\Sigma_\varepsilon)$.

1 Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane. For $p > 1$, we denote by $B_p = B_p(\mathbb{D})$ the Besov space of the holomorphic functions on $\mathbb{D}$ such that

$$\|f\|_{B_p}^p = \int_{\mathbb{D}} \left(1 - |z|^2\right)^{p-2} |f'(z)|^p dA(z) < \infty,$$

where $z = x + iy = re^{i\theta}$ and $dA(z) = \dfrac{1}{\pi} dxdy = \dfrac{1}{\pi} rdrd\theta$ is the two-dimensional Lebesgue measure on $\mathbb{D}$. Each $f \in B_p$ induces a Borel measure μ_f on the plane defined by

$$\mu_f(E) = \int_E \left(1 - |z|^2\right)^{p-2} |f'(z)|^p \, dA(z).$$

Our problem concerns the angular distribution given by such a measure. We are interested in knowing if given $\varepsilon > 0$, we can find a constant $\delta > 0$, depending only on p and ε, such that

$$\int_{f^{-1}(\Sigma_\varepsilon)} \left(1 - |z|^2\right)^{p-2} |f'(z)|^p dA(z) > \delta \|f\|_{B_p}^p, \tag{1.1}$$

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for all functions $f \in B_p$ satisfying $f(0) = 0$, where

$$\Sigma_\varepsilon = \{w \in \mathbb{C} : |arg\,(w)| < \varepsilon\}.$$

This question was suggested by an article of D. Marshall and W. Smith [1] where they analyze a problem of this type for functions in the classic Bergman's space (without weight) A^p. The principal result of [1, Theorem 1.1] ensures that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\int_{f^{-1}(\Sigma_\varepsilon)} |f(z)|\, dA(z) > \delta \int_{\mathbb{D}} |f(z)|\, dA(z),$$

for any univalent function in A^1 fixing the origin. It is an open problem to show whether their result still holds if the hypothesis that f is univalent were omitted.

Pérez-González and Ramos [2, 4] have extended the results of Marshall and Smith to the widest class of weighted Bergman space A^p_α. They proved the following theorem:

Theorem 1. If $\alpha > -1$ and $p \geq 1$ satisfy $\alpha > 2p-1$, then for all $\varepsilon > 0$, there exists a constant $\delta > 0$, depending only on p, α and ε, such that

$$\int_{f^{-1}(\Sigma_\varepsilon)} |f(z)|^p \left(1-|z|^2\right)^\alpha dA(z) > \delta \int_{\mathbb{D}} |f(z)|^p \left(1-|z|^2\right)^\alpha dA(z), \tag{1.2}$$

for any univalent function $f \in A^p_\alpha$ with $f(0) = 0$.

Also, they gave an example where the above theorem does not hold for all $\varepsilon > 0$ when $\alpha < 2p-2$. If $p = 1$ and $\alpha \geq 0$, then Theorem 1 is true (see [3]). The case when $2p-2 \leq \alpha \leq 2p-1$ needs to be explored. On the other hand, if we omit the condition that the function is univalent, does the result continue to be true?

In this article, we consider a similar problem, but in the spaces of Besov B_p with $p > 1$. We will prove the following result:

Main Theorem. Assume $p > 1$ and $\varepsilon > 0$. Then there exists a constant $K > 0$ depending only on p such that

$$\int_{f^{-1}(\Sigma_\varepsilon)} \left(1-|z|^2\right)^{p-2} |f'(z)|^p\, dA(z) \geq K(p)\varepsilon \frac{|f'(0)|^{p+4}}{\|f\|^4_{B_p}}, \tag{1.3}$$

for any nonnull function $f \in B_p$ with $f(0) = 0$.

The above theorem implies that the inequality (1.1) is true for certain class of functions in B_p. For instance, if we consider functions in the Besov's space that fix the origin and such that $|f'(0)| \geq k\|f\|_{B_p}$ for some constant $k > 0$, we obtain the following result:

Corollary 1. Suppose $\varepsilon > 0$, $p > 1$ and k a positive constant. Then there exists a constant $\beta > 0$, depending on p and k, such that

$$\int_{f^{-1}(\Sigma_\varepsilon)} \left(1-|z|^2\right)^{p-2} |f'(z)|^p\, dA(z) \geq \beta(p,k)\varepsilon\, \|f\|^p_{B_p}, \tag{1.4}$$

for any function $f \in B_p$ such that $f(0) = 0$ and $|f'(0)| \geq k\|f\|_{B_p}$.

The third section of this paper is devoted to constructing an example using conformal mapping, that shows that the inequality (1.4) is not true, for all $\varepsilon > 0$, if we omit the hypothesis $|f'(0)| \geq k\|f\|_{B_p}$. However, in the last section of the paper we will prove the following result for conformal mapping:

Theorem 2. Suppose $\varepsilon > 0$ and let h be a conformal map on B_p, $p > 1$, with $h(0) = 0$. Then there exists a constant $K > 0$, depending on p such that

$$\int_{h^{-1}(\Sigma_\varepsilon)} \left(1 - |z|^2\right)^{p-2} |h'(z)|^p \, dA(z) \geq K(p) \, |h'(0)|^p \, \varepsilon. \tag{1.5}$$

2 Proof of the Main Theorem

The following well known result will play an important role in the proof of the main theorem. We will denote by $D(a, r)$ the Euclidean disk with center a and radius r.

Theorem 3. [1/4-Koebe] If $g(0) = 0$ and $g'(0) = 1$, then $D(0, \frac{1}{4}) \subset \Omega$, where g is a conformal map from the unit disk $\mathbb{D}$ into a domain Ω.

We now give the proof of our main theorem. We can observe that inequality (1.3) is true if $f'(0) = 0$ therefore we can suppose that $f'(0) \neq 0$. First, suppose that $\|f\|_{B_p} = 1$. Since f is an analytic function on the disk $D\left(0, \frac{1}{2}\right)$, we can apply the Cauchy integral formula to obtain

$$|f'(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{\left|f'\left(re^{i\theta}\right)\right|}{|re^{i\theta} - z|} d\theta,$$

where $|z| < \frac{1}{2}$. Multiplying both sides of the above inequality by r and integrating from $r = \frac{3}{4}$ to $r = \frac{7}{8}$ we can see that there exists a constant $K_1 > 0$ such that

$$|f'(z)| \leq K_1 \int_{\{\frac{3}{4} < |s| < \frac{7}{8}\}} |f'(s)| \, dA(s),$$

with $|z| < \frac{1}{2}$. By Hölder's inequality we have

$$|f'(z)| \leq K_2 \, \|f\|_{B_p} = K_2,$$

for all z such that $|z| < \frac{1}{2}$. Here we have used the fact that if $s \in \{\frac{3}{4} < |s| < \frac{7}{8}\}$, then $(1 - |s|^2)^{\frac{p-2}{p}} \geq \frac{15}{64}$. Next, we define $h(w) = \frac{1}{2k_2}\{f'\left(\frac{w}{2}\right) - f'(0)\}$, for $w \in \mathbb{D}$. It is not hard to see that:

(i) $h \in H(\mathbb{D})$,

(ii) $h(0) = 0$,

(iii) $|h(w)| \leq 1$.

Invoking Schwarz's lemma, we obtain

$$|h(w)| \leq |w|, \tag{2.1}$$

for all $w \in \mathbb{D}$. Taking $w = 2z$ from (2.1) we have

$$|f'(z) - f'(0)| \leq 4K_2|z|,$$

for $|z| < \frac{1}{2}$.

Now, if we take

$$R = \frac{1}{8K_2} |f'(0)|,$$

then

$$|f'(z) - f'(0)| < \frac{1}{2} |f'(0)|,$$

with $|z| < R$. In particular, for $z_1, z_2 \in D(0,R)$ with $z_1 \neq z_2$, we have

$$|f(z_2) - f(z_1) - f'(0)(z_2 - z_1)| \leq \int_{z_1}^{z_2} |f'(z) - f'(0)| \, dz < \frac{1}{2} |z_2 - z_1| \, |f'(0)| .$$

Thus, $f(z_1) \neq f(z_2)$ and the function f is one to one on the disk $D(0,R)$. Thus, if we define the function

$$g(z) = \frac{1}{Rf'(0)} f(Rz), \quad z \in \mathbb{D},$$

we can see that $g(0) = 0$ and $g'(0) = 1$. Then by the 1/4-Koebe theorem, we have $D\left(0, \frac{1}{4}\right) \subset g(\mathbb{D})$. This implies that $D(0,\sigma) \subset f(D(0,R))$, where

$$\sigma = \frac{|f'(0)|^2}{32K_2}.$$

Therefore

$$\begin{aligned}
&\int_{f^{-1}(\Sigma_\varepsilon)} \left(1 - |z|^2\right)^{p-2} |f'(z)|^p \, dA(z) \\
&\qquad \geq \int_{f^{-1}(\Sigma_\varepsilon \cap D(0,\sigma)) \cap D(0,R)} \left(1 - |z|^2\right)^{p-2} |f'(z)|^p \, dA(z) \\
&\qquad \geq K_3(p) \, |f'(0)|^p \int_{f^{-1}(\Sigma_\varepsilon \cap D(0,\sigma)) \cap D(0,R)} dA(z) \\
&\qquad \geq \frac{K_3(p)}{K_2^2} \, |f'(0)|^p \int_{f^{-1}(\Sigma_\varepsilon \cap D(0,\sigma)) \cap D(0,R)} |f'(z)|^2 \, dA(z) \\
&\qquad = K_4(p)\varepsilon \, |f'(0)|^{p+4},
\end{aligned}$$

where in the second inequality we have used $|z| < R < \frac{1}{8}$, $|f'(z)| > \frac{1}{2} |f'(0)|$, and also, the fact that if f is 1-1 on the set E then

$$\int_E |f'(z)|^2 \, dA(z) = area(f(E)).$$

This concludes the proof of the Theorem. ∎

3 An example

In this section we give an example where we show that the inequality (1.1) is not true, for all $\varepsilon > 0$, if we omitted the condition $|f'(0)| \geq k\|f\|_{B_p}$.

For each $n \in \mathbb{N}$ with $n \geq 2$, consider the Riemann map $f_n : \mathbb{D} \to D\,(1-n, n)$ given by

$$f_n(z) = \frac{1-2n}{(1-n)\,z+n} z.$$

It is not hard to see that $f_n(0) = 0$ and $f_n'(0) < 0$. Furthermore, since f_n is analytic in a neighborhood of the closed disk $\overline{\mathbb{D}}$ is clear that $f_n \in B_p$ for all $n \geq 2$. Also, there exist positive constants C_1 and C_2 such that

$$\frac{C_1}{\left(1-\left|\frac{n-1}{n}\right|^2\right)^p} \leq \int_{\mathbb{D}} \frac{(1-|z|^2)^{p-2}}{\left|1-\frac{n-1}{n}z\right|^{2p}} dA(z) \leq \frac{C_2}{\left(1-\left|\frac{n-1}{n}\right|^2\right)^p}$$

for n large enough (see [6], Lemma 4.2.2). Hence $lim_{n\to\infty} \|f_n\|_{B_p} = \infty$.

In this example we observe that

$$|f_n'(0)| = 2 - \frac{1}{n} \leq 2$$

therefore, there is no constant $k > 0$ such that

$$|f_n'(0)| > k\,\|f_n\|_{B_p}, \quad \text{for all } n \in \mathbb{N}.$$

On the other hand, if we fix $\varepsilon < \frac{\pi}{2}$, then by elementary calculations from trigonometry we have $|f_n(z)| < K(\varepsilon) = \sec(\varepsilon)$ for all $n > 2$ and for any $z \in f_n^{-1}(T)$, where $T = \Sigma_\varepsilon \cap D\,(1-n, n)$. Also, since f_n is a linear fractional transformation, we can see that

$$|f_n'(z)| \leq \left|\frac{2n-1}{n-1} - f_n(z)\right|^2.$$

Thus, there exists a constant $K_1(\varepsilon) > 0$ such that

$$|f_n'(z)| \leq K_1(\varepsilon), \quad z \in f_n^{-1}(T).$$

Therefore, for $n \in \mathbb{N}$ large enough we have

$$\begin{aligned}\int_{f_n^{-1}(T)} \left(1-|z|^2\right)^{p-2} |f_n'(z)|^p \, dA(z) \;&\leq\; K_1^p(\varepsilon) \int_{f_n^{-1}(T)} \left(1-|z|^2\right)^{p-2} dA(z)\\ &\leq\; \frac{K_1^p(\varepsilon)}{p-1}.\end{aligned}$$

Hence, there is no a constant $\delta > 0$ such that

$$\int_{f_n^{-1}(\Sigma_\varepsilon)} \left(1-|z|^2\right)^{p-2} |f_n'(z)|^p \, dA(z) > \delta\,\|f_n\|^p_{B_p},$$

for all $n \in \mathbb{N}$ and for any $\varepsilon < \frac{\pi}{2}$.

Remark 1. The referee has pointed out to us the failure of (1.1) is due to the Möbius invariance of the Besov norm. To see this we can note that

$$\|f\|_{B_p} = \|f_a\|_{B_p},$$

where $f_a = f \circ \varphi_a - f(a)$ and $\varphi_a(z) = (a-z)/(1-\overline{a}z)$ is the Möbius map of the disk $\mathbb{D}$ that interchanges 0 and $a \in \mathbb{D}$. So $f_a(0) = 0$ and a change of variables shows

$$\int_{f_a^{-1}(\Sigma_\varepsilon)} \left(1-|z|^2\right)^{p-2} |f_a'(z)|^p \, dA(z) = \int_{f^{-1}(\Sigma_\varepsilon + f(a))} \left(1-|z|^2\right)^{p-2} |f'(z)|^p \, dA(z).$$

So if (1.1) held it would follow that

$$\delta \|f\|_{B_p} = \delta \|f_a\|_{B_p} < \int_{f^{-1}(\Sigma_\varepsilon + f(a))} \left(1-|z|^2\right)^{p-2} |f'(z)|^p \, dA(z),$$

for all $a \in \mathbb{D}$ and $f \in B_p$, which is not possible.

4 The angular distribution of mass by conformal mappings

In this section, we show that an estimate like (1.3) is true when we consider conformal maps that fix the origin on B_p, with $p > 1$. Before the proof of theorem 2 we gather some results about conformal maps that we will need for our goal (see [5]).

We consider $z \in \mathbb{D}$, g a conformal map from the unit disk $\mathbb{D}$ into a domain Ω. We denote by $\delta_\Omega(g(z))$ the Euclidean distance from $g(z)$ to $\partial\Omega$ where $\partial\Omega$ denotes the boundary of Ω.

Theorem 4. [Distortion] If $g(0) = 0$, then

(i) $$|g'(0)| \frac{|z|}{(1+|z|)^2} \le |g(z)| \le |g'(0)| \frac{|z|}{(1-|z|)^2},$$

(ii) $$|g'(0)| \frac{1-|z|}{(1+|z|)^3} \le |g'(z)| \le |g'(0)| \frac{1+|z|}{(1-|z|)^3},$$

(iii) $$\frac{1}{4}\left(1-|z|^2\right)|g'(z)| \le \delta_\Omega(f(z)) \le \left(1-|z|^2\right)|g'(z)|,$$

for any $z \in \mathbb{D}$. The key to our results is the following lemma:

Lemma 1. Suppose $\varepsilon > 0$ and let h be a conformal map in B_p, $p > 1$, from the open unit disk $\mathbb{D}$ into a domain Ω_h with $h(0) = 0$. Then there exist a Euclidean ball $B_r \subset \Omega_h$ and a constant $K(p) > 0$ such that

$$\int_{h^{-1}(\Sigma_\varepsilon \cap B_r)} \left(1-|z|^2\right)^{p-2} |h'(z)|^p \, dA(z) \ge K(p) \, |h'(0)|^p \, \varepsilon. \tag{4.1}$$

Proof. We consider $r = \frac{1}{5}\delta_{\Omega_h}(0)$ and let $B_r = D(0, r)$ be the ball with center at the origin with radius r. By the distortion Theorem, if $w = h(z) \in B_r$ then

$$|z| < \frac{1}{|h'(0)|}(1+|z|)^2 |h(z)| \le \frac{4}{5}. \tag{4.2}$$

Thus, there are positive constants $K_0(p)$ and $K_1(p)$ such that

$$K_0(p) \leq \left(1-|z|^2\right)^{p-2} \leq K_1(p), \quad z \in h^{-1}\left(B_r\right).$$

On the other hand, applying the distortion theorem again and the estimate (4.2), we can see that there is a universal constant $K_2 > 0$ such that

$$|h'(z)| \geq |h'(0)| \frac{1-|z|}{(1+|z|)^3} \geq K_2 \left|h'(0)\right|, \quad z \in h^{-1}\left(B_r\right).$$

Similarly, there is a universal constant $K_3 > 0$ such that

$$|h'(z)| \leq |h'(0)| \frac{1+|z|}{(1-|z|)^3} \leq K_3 \left|h'(0)\right|, \quad z \in h^{-1}\left(B_r\right).$$

Therefore,

$$\begin{aligned}
&\int_{h^{-1}(\Sigma_\varepsilon \cap B_r)} \left(1-|z|^2\right)^{p-2} |h'(z)|^p \, dA(z) \\
&\qquad \geq K_0(p) K_2^p \left|h'(0)\right|^p \int_{h^{-1}(\Sigma_\varepsilon \cap B_r)} dA(z) \\
&\qquad \geq \frac{K_0(p) K_2^p}{K_3^2} \left|h'(0)\right|^{p-2} \int_{h^{-1}(\Sigma_\varepsilon \cap B_r)} \left|h'(z)\right|^2 dA(z) \\
&\qquad \geq K_4(p) \left|h'(0)\right|^p \varepsilon.
\end{aligned}$$

The proof of Lemma 1 is now complete. Now we can proceed to the proof of Theorem 2 ■

Proof (of Theorem 2).
Theorem 2 follows from Lemma 1, because $\Sigma_\varepsilon \cap B_r \subset \Sigma_\varepsilon$, which completes the proof. ■

An application of the Theorem 2 is the following result on angular distribution of a conformal mapping in B_p fixing the origin.

Corollary 2. Suppose $\varepsilon > 0$ and $p > 1$. There exists a constant $K(p) > 0$ such that

$$\int_{h^{-1}(\Sigma_\varepsilon)} \left(1-|z|^2\right)^{p-2} |h'(z)|^p \, dA(z) \geq K(p) \left|h'(0)\right|^{2p} \varepsilon \|h\|_{B_p}^p, \tag{4.3}$$

for any univalent function $h \in B_p$, with $h(0) = 0$ and $|h'(0)| \leq 1/\|h\|_{B_p}$.

Proof. The corollary follows from Theorem 2 because $1 \geq |h'(0)|^p \, \|h\|_{B_p}^p$. ■

Remark. Since the inequality (1.1) is trivially true for all $\varepsilon \geq \pi$, we can ask if there exists an angle $\varepsilon_0 < \pi$ such that if $\varepsilon \geq \varepsilon_0$ then there is a constant $\delta > 0$ satisfying

$$\int_{f^{-1}(\Sigma_\varepsilon)} \left(1-|z|^2\right)^{p-2} |f'(z)|^p dA_\alpha(z) > \delta \|f\|_{B_p}^p, \tag{4.4}$$

for all functions $f \in B_p$ with $f(0) = 0$?

A similar estimation is indeed true if we consider the Bergman spaces A_α^1 in [4] Pérez-González and Ramos proved the following theorem:

Theorem 5. [4] Given $\alpha > -1$, there exist an angle $\varepsilon_0 \in (0, \frac{\pi}{2})$ and $\delta > 0$, depending only on α, such that

$$\int_{\mathbb{D}} |f(z)| \left(1 - |z|^2\right)^{\alpha} dA(z) \leq \delta \int_{f^{-1}\left(\Sigma_{\varepsilon_0}\right)} |f(z)| \left(1 - |z|^2\right)^{\alpha} dA(z)$$

for any $f \in A^1_\alpha$ satisfying $f(0) = 0$.

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