Spectral Method for a Class of Systems of Generalized Zakharov Equations

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Abstract

In this paper, a Fourier spectral method for an initial boundary value problem for a class of systems of generalized Zakharov equations is proposed. Semi-discrete and fully discrete Fourier spectral schemes are given. In fully discrete case we have established a two level scheme which is convenient and saves time in real computation. An energy estimation method is used to obtain error estimates for approximate solutions.

1 Introduction

Zakharov [1] has proved that propagation of Langmuir waves in Plasma Physics, which describe a system called Zakharov equations nowadays. In this paper, we consider the following initial boundary value problem for a class of systems of generalized

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Zakharov equations:

$$i\partial_t u + \partial_x^2 u - \alpha n u + \beta(|u|^2)u = 0, \qquad x \in R, t \geq 0, \tag{1.1}$$

$$\partial_t v + \partial_x f(v) - \delta\partial_t\partial_x^2 v + \partial_x n + \mu\partial_x(|u|^2) = \phi(v), \qquad x \in R, t \geq 0, \tag{1.2}$$

$$\partial_t n + \partial_x v = 0, \qquad x \in R, t \geq 0, \tag{1.3}$$

$$u(x,0) = u_0, v(x,0) = v_0, n(x,0) = n_0, \qquad x \in R, \tag{1.4}$$

$$u(x+2\pi,t) = u(x,t), v(x+2\pi,t) = v(x,t), n(x+2\pi,t) = n(x,t), \qquad t \geq 0, \tag{1.5}$$

where the complex function $u(x,t)$ represents the envelop of the electric field, $n(x,t)$ is the deviation of the ion density from its equilibrium value, $v(x,t)$ is speed of soliton. The parameters $\alpha, \beta, \delta, \mu$ are real constants, $f(v)$ and $\phi(v)$ are nonlinear functions. The existence and uniqueness of the global solution of the generalized Zakharov equations in one dimension is proved in [2].

A lot of work have been done on the numerical solution of Zakharov type equations. For example, in [6] Guangye used the pseudospectral method for (1.1)-(1.5) and proved the error estimation of semi-discrete and fully discrete pseudospectral schemes. Ma Shuqing and Chang Qianshun [8] has studied the dissipative Zakharov equations, in which they apply pseudospectral method and proved the convergence by priori estimates. Payne et al. [5] designed a spectral method for one dimensional Zakharov system (ZS). They used a truncated Fourier expansion in their schemes to estimate the aliasing error. Shi Jin et al. [9] studied the time splitting spectral methods for the generalized ZS. They analyzed the behavior of numerical solution in the subsonic regime and studied the collision behavior of two solitons in the transonic region. Glassey [3] presented an energy preserving finite difference schemes for the ZS in one dimension and proved its convergence in [4]. In [10, 11], Chang et al. presented a conservative difference scheme for the generalized ZS. This scheme can be implicit or semi explicit depending on the choice of parameter. They also proved the convergence of their method. More recently Bao et al. [12] proposed a time splitting spectral scheme to solve the generalized ZS. Their method was also extended to the vector ZS for multi-component plasmas [13].

The aim of this paper is to investigate the first order finite difference approximation in time, combined with spectral approximation in space, for solving (1.1)-(1.5). Both the semi-discrete and the fully discrete schemes are analyzed and error estimation for both are found. The rate of convergence of the resulting schemes are $O(N^{-S})$ and $O(k^2 + N^{-S})$ where N is the number of spatial Fourier modes, k is the discrete mesh spacing of the time variable t and where S is depending only on the smoothness of an exact solution.

This paper is organized as follows: we introduce some notations and lemmas in section 2; the semi discrete and fully discrete spectral schemes are studied in section 3 and section 4 respectively; finally the conclusion is given in section 5.

2 Notations

Let $\Omega = [0, 2\pi]$, the inner product and the norm are defined by $(u, v) = \int_\Omega u(x)v(x)dx$ and $\|u\|^2 = (u, u)$ respectively. Let $\|u\|_\infty = ess \sup_{x\in\Omega} |u(x)|$ and the periodic Sobolev spaces H_p^S is defined by

$$H_p^S(\Omega) = \{u \in H^S(R) : u(x + 2\pi) = u(x)\}.$$

For any positive integer S, the Sobolev norm and semi-norm are defined by

$$\|u\|_S = \left(\sum_{j=0}^{S} \left\|\frac{\partial^j u}{\partial x^j}\right\|^2\right)^{1/2}, \quad |u|_j = \left\|\frac{\partial^j u}{\partial x^j}\right\|^2.$$

We define

$$L^\infty(0, T; H_p^S(\Omega)) = \{u(\cdot, t) \in H_p^S(\Omega) : \sup_{0\le t<T} \|u(\cdot, t)\|_S < \infty\}.$$

For any even integer N, set $S_N = \text{Span}\left\{\varphi_k = \frac{1}{\sqrt{2\pi}} e^{ikx} : |k| \le N\right\}$. P_N denotes the L^2 orthogonal projection operator of $H_P^S(\Omega)$ upon S_N.

Lemma 2.1. *[14] For any periodic discrete function u^m, there are*

$$(u^m, u_t^m) = \frac{1}{2}\|u^m\|_t^2 - \frac{k}{2}\|u_t^m\|^2, \quad \textit{where} \quad \|u^m\|_t^2 = \frac{1}{k}(\|u^{m+1}\|^2 - \|u^m\|^2).$$

Lemma 2.2. *[7] Assume that $u \in H_p^S(\Omega)$, for any $0 \le \mu \le S$, there exists C independent of u and N*

$$\|u - P_N u\|_\mu \le CN^{\mu-S}|u|_S.$$

Lemma 2.3. *[7]* ***(Inverse Property)*** *Assume that $u \in S_N$, for any $0 \le \mu \le \sigma$, there exists C independent of u and N*

$$\|u\|_\sigma \le CN^{\sigma-\mu}\|u\|_\mu.$$

3 The Semi-Discrete Spectral Method and Error Estimation

The semi-discrete spectral approximation of problem (1.1)–(1.5) consists in finding $u_N, v_N, n_N \in S_N$, satisfying, for any $\psi \in S_N$, such that

$$(i\partial_t u_N + \partial_x^2 u_N - \alpha n_N u_N + \beta(|u_N|^2)u_N, \psi) = 0, \tag{3.1}$$
$$(\partial_t v_N + \partial_x f(v_N) - \delta\partial_t\partial_x^2 v_N + \partial_x n_N + \mu\partial_x|u_N|^2 - \phi(v_N), \psi) = 0, \tag{3.2}$$
$$(\partial_t n_N + \partial_x v_N, \psi) = 0, \tag{3.3}$$
$$u_N(x, 0) = P_N u_0, v_N(x, 0) = P_N v_0, n_N(x, 0) = P_N n_0. \tag{3.4}$$

Suppose that (u, v, n) are the solutions of (1.1)-(1.5) and (u_N, v_N, n_N) are the solutions of (3.1)-(3.4). Setting

$$\begin{aligned} e_1 &= u - u_N = (u - P_N u) + (P_N u - u_N) = \xi_1 + \eta_1, \\ e_2 &= v - v_N = (v - P_N v) + (P_N v - v_N) = \xi_2 + \eta_2, \\ e_3 &= n - n_N = (n - P_N n) + (P_N n - n_N) = \xi_3 + \eta_3. \end{aligned}$$

By Lemma 2.2 and using $(\xi_\ell, \psi) = 0, \ell = 1, 2, 3, \forall \psi \in S_N$, implies that

$$\|e_\ell\| \leq \|\xi_\ell\| + \|\eta_\ell\| \leq CN^{-S} + \|\eta_\ell\|, \quad \ell = 1, 2, 3. \tag{3.5}$$

Taking the inner product of (1.1) with $\psi \in S_N$, implies that

$$(i\partial_t u + \partial_x^2 u - \alpha n u + \beta(|u|^2)u, \psi) = 0. \tag{3.6}$$

Subtracting (3.1) from (3.6) yields

$$(i\partial_t e_1 + \partial_x^2 e_1 - \alpha(nu - n_N u_N) + \beta(|u|^2 u - |u_N|^2 u_N), \psi) = 0. \tag{3.7}$$

Note that

$$\begin{aligned} (\partial_t e_\ell, \psi) &= (\partial_t \xi_\ell, \psi) + (\partial_t \eta_\ell, \psi) = (\partial_t \eta_\ell, \psi), \ell = 1, 2, 3 \\ (\partial_x^2 e_\ell, \psi) &= -(\partial_x e_\ell, \partial_x \psi) = -(\partial_x \eta_\ell, \partial_x \psi), \ell = 1, 2. \end{aligned}$$

Setting $\psi = \eta_1$, and taking the imaginary part, (3.7) follows

$$\frac{1}{2}\frac{d}{dt}\|\eta_1\|^2 = \alpha I_m(nu - n_N u_N, \eta_1) - \beta I_m(|u|^2 u - |u_N|^2 u_N, \eta_1). \tag{3.8}$$

Throughout this paper, we shall use C to denote a general positive constant independent of k and N. It can be of different values in different cases.

$$\begin{aligned} |(nu - n_N u_N, \eta_1)| &\leq \|n\|_\infty \|\eta_1\|^2 + \|u_N\|_\infty \|e_2\| \|\eta_1\| \\ &\leq C(N^{-2S} + \|\eta_2\|^2 + \|\eta_1\|^2), \end{aligned}$$

$$\begin{aligned} (|u|^2 u - |u_N|^2 u_N, \eta_1) &= (((|u|^2 - |u_N|^2)u + |u_N|^2 e_1, \eta_1) \\ &\leq (((\|u\|_\infty + \|u_N\|_\infty)\|u\|_\infty + \|u_N\|^2)\|e_1\| \|\eta_1\| \\ &\leq C(N^{-S} + \|\eta_1\|)\|\eta_1\| \\ &\leq C(N^{-2S} + \|\eta_1\|^2). \end{aligned}$$

Then (3.8) gives

$$\frac{1}{2}\frac{d}{dt}\|\eta_1\|^2 \leq C(N^{-2S} + \|\eta_1\|^2 + \|\eta_2\|^2). \tag{3.9}$$

Next by replacing u, n with u_N, n_N in (1.1), it follows

$$i\partial_t u_N + \partial_x^2 u_N - \alpha n_N u_N + \beta(|u_N|^2)u_N = 0. \tag{3.10}$$

Differentiate (1.1) and (3.10) with respect to x respectively, subtracting one equation from another, taking imaginary parts, and setting $\psi = \partial_x \eta_1$, we have

$$\frac{1}{2}\frac{d}{dt}\|\partial_x \eta_1\|^2 = \alpha I_m \partial_x((nu - n_N u_N), \partial_x \eta_1) + \beta I_m(\partial_x(|u|^2 u - |u_N|^2 u_N), \partial_x \eta_1). \tag{3.11}$$

Similar to (3.8), we get

$$|(nu - n_N u_N, \partial_x^2 \eta_1)| \le C(N^{-2S} + \|\eta_1\|^2 + \|\eta_2\|^2 + \|\partial_x^2 \eta_1\|^2),$$
$$(|u|^2 u - |u_N|^2 u_N, \partial_x^2 \eta_1) \le C(N^{-2S} + \|\eta_1\|^2 + \|\partial_x^2 \eta_1\|^2).$$

Then (3.11) gives

$$\frac{1}{2}\frac{d}{dt}\|\eta_1\|^2 \le C(N^{-2S} + \|\eta_1\|^2 + \|\eta_2\|^2 + \|\partial_x^2 \eta_1\|^2). \tag{3.12}$$

On other hand, taking the inner product of (1.2) with $\psi \in S_N$, implies that

$$(\partial_t v + \partial_x f(v) - \delta \partial_t \partial_x^2 v + \partial_x n + \mu \partial_x |u|^2 - \phi(v), \psi) = 0. \tag{3.13}$$

Subtracting (3.13) from (3.2), implies that

$$(\partial_t e_2 + \partial_x (f(v) - f(v_N)) - \delta \partial_t \partial_x^2 e_2 + \partial_x e_3 + \mu \partial_x (|u|^2 - |u_N|^2) - (\phi(v) - \phi(v_N), \psi) = 0. \tag{3.14}$$

Set $\psi = \eta_2$, (3.14) follows

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\eta_2\|^2 + \frac{\delta}{2}\frac{d}{dt}\|\partial_x \eta_2\|^2 + (\partial_x (f(v) - f(v_N)), \eta_2) + (\partial_x e_3, \eta_2) \\ + (\mu \partial_x (|u|^2 - |u_N|^2), \eta_2) - (\phi(v) - \phi(v_N), \eta_2) = 0. \end{aligned} \tag{3.15}$$

Similar to above derivation, we have

$$\begin{aligned} |(\partial_x (|u|^2 - |u_N|^2), \eta_2)| &= |(|u|^2 - |u_N|^2, \partial_x \eta_2)| \\ &\le (\|u\|_\infty - \|u_N\|_\infty)\|e_1\|\|\partial_x \eta_2\| \\ &\le C(N^{-2S} + \|\eta_1\|^2 + \|\partial_x \eta_2\|^2), \end{aligned}$$

$$\begin{aligned} |(\partial_x (f(v) - f(v_N)), \eta_2)| &= |(f(v) - f(v_N), \partial_x \eta_2)| \\ &\le \|\frac{\partial f}{\partial v}\|_\infty \|e_2\|\|\partial_x \eta_2\| \\ &\le C(N^{-2S} + \|\eta_2\|^2 + \|\partial_x \eta_2\|^2), \end{aligned}$$

$$\begin{aligned} |(\partial_x (\phi(v) - \phi(v_N)), \eta_2)| &\le \|\frac{\partial \phi}{\partial v}\|_\infty \|e_2\|\|\eta_2\| \\ &\le C(N^{-2S} + \|\eta_2\|^2), \end{aligned}$$

$$|\partial_x e_3, \eta_2)| \le C(N^{-2S} + \|\eta_3\|^2 + \|\partial_x \eta_2\|^2).$$

Then (3.15) gives

$$\frac{1}{2}\frac{d}{dt}(\|\eta_2\|^2 + \delta\|\partial_x \eta_2\|^2) \le C(N^{-2S} + \|\eta_1\|^2 + \|\eta_2\|^2 + \|\partial_x \eta_2\|^2 + \|\eta_3\|^2). \tag{3.16}$$

Finally, taking the inner product of (1.3) with $\psi \in S_N$, implies that

$$(\partial_t n + \partial_x v, \psi) = 0. \tag{3.17}$$

Subtracting (3.3) from (3.17), we get

$$(\partial_t e_3 + \partial_x e_2, \psi) = 0. \tag{3.18}$$

Set $\psi = \eta_3$, (3.18) follows

$$\frac{1}{2}\frac{d}{dt}\|\eta_3\|^2 \le C(N^{-2S} + \|\eta_2\|^2 + \|\partial_x \eta_3\|^2). \tag{3.19}$$

Combining (3.9), (3.12), (3.16), and (3.19), we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\eta_1\|^2 + \|\eta_2\|^2 + \|\eta_3\|^2 + \|\eta_1\|^2_{H^1} + \|\eta_2\|^2_{H^1}) \le C(N^{-2S} + \|\eta_1\|^2 \\ + \|\eta_2\|^2 + \|\eta_3\|^2 + \|\partial_x^2 \eta_1\|^2 + \|\partial_x \eta_2\|^2 + \|\partial_x \eta_3\|^2).\end{aligned} \tag{3.20}$$

Note that $\|\eta_\ell(0)\| = 0, \ell = 1, 2, 3$ and by applying Gronwall's inequality, we obtain

$$\|\eta_1\|^2_{H^1} + \|\eta_2\|^2_{H^1} + \|\eta_3\|^2 \le CN^{-2s}$$

Theorem 1. *Suppose that $f(v), \phi(v) \in C^1(\Omega)$, $u_0(x)$, $v_0(x)$ and $n_0(x) \in H^S(\Omega)$, $(S \ge 1)$. Assume that u, v and n are solution of equations (1.1)-(1.5). Then there exists a unique solution u_N, v_N and n_N of the problem (3.1)-(3.3). Moreover there exists a positive constant C, the following error estimate holds*

$$\|u - u_N\|_{H^1} + \|v - v_N\|_{H^1} + \|n(t) - n_N(t)\| \le CN^{-S}.$$

4 The Fully Discrete Spectral Method and Error Estimation

Let J be positive integer, $k = \frac{T}{J}$ be time-length step. The approximation $Z_N^m = (u_N^m, v_N^m, n_N^m)^T$ to $Z_N = (u_N, v_N, n_N)^T$ at $t = mk$ given by the spectral method is defined by $Z_N^0 = P_N Z_0$, and for $m > 0$, $\forall \psi \in S_N$, find $u_N^m, v_N^m, n_N^m \in S_N$, such that

$$(iu_{Nt}^m + \partial_x^2 \widehat{u}_N^m - \alpha n_N^m \widehat{u}_N^m + \beta(|\widehat{u}_N^m|^2)\widehat{u}_N^m, \psi) = 0, \tag{4.1}$$

$$(v_{Nt}^m + \partial_x f(v_N^m) - \delta \partial_x^2 v_{Nt}^m + \partial_x \widehat{n}_N^m + \mu \partial_x(|u_N^m|^2) - \phi(v_N^m), \psi) = 0, \tag{4.2}$$

$$(n_{Nt}^m + \partial_x \widehat{v}_N^m, \psi) = 0, \tag{4.3}$$

$$u_N^m(x,0) = P_N u_0^m, v_N^m(x,0) = P_N v_0^m, n_N^m(x,0) = P_N n_0^m, \tag{4.4}$$

where

$$u_{Nt}^m = \frac{1}{k}[u_N^{m+1} - u_N^m], \quad v_{Nt}^m = \frac{1}{k}[v_N^{m+1} - v_N^m],$$
$$n_{Nt}^m = \frac{1}{k}[n_N^{m+1} - n_N^m], \quad \widehat{u}_N^m = \frac{1}{2}[u_N^{m+1} + u_N^m].$$

Setting

$$\begin{aligned}
e_1^m &= u^m - u_N^m = (u^m - P_N u^m) + (P_N u^m - u_N^m) = \xi_1^m + \eta_1^m,\\
e_2^m &= v^m - v_N^m = (v^m - P_N v^m) + (P_N v^m - v_N^m) = \xi_2^m + \eta_2^m,\\
e_3^m &= n^m - n_N^m = (n^m - P_N n^m) + (P_N n^m - n_N^m) = \xi_3^m + \eta_3^m.
\end{aligned}$$

By Lemma 2.2 and using $(\xi_\ell^m, \psi) = 0, \ell = 1,2,3, \forall \psi \in S_N$, implies that

$$\|e_\ell^m\| \le \|\xi_\ell^m\| + \|\eta_\ell^m\| \le CN^{-S} + \|\eta_\ell^m\|, \quad \ell = 1,2,3. \tag{4.5}$$

Substituting the solution $u(t_m)$, $v(t_m)$, $n(t_m)$, into (1.1)- (1.3), and subtracting (4.1) from (1.1), (4.2) from (1.2) and (4.3) from (1.3) respectively, we have

$$(ie_{1t}^m + \partial_x^2 \widehat{e}_1^m - \alpha(n^m u^m - n_N^m \widehat{u}_N^m) + \beta(|u^m|^2 u^m - |\widehat{u}_N^m|^2 \widehat{u}_N^m), \psi) = (\tau_1^m, \psi), \tag{4.6}$$

$$\begin{aligned}
&(e_{2t}^m + \partial_x(f(v^m) - f(v_N^m)) - \delta\partial_x^2 e_{2t}^m + \partial_x \widehat{e}_3^m + \mu\partial_x(|u^m|^2 - |u_N^m|^2)\\
&\quad - (\phi(v^m) - \phi(v_N^m)), \psi) = (\tau_2^m, \psi),
\end{aligned} \tag{4.7}$$

$$(e_{3t}^m + \partial_x \widehat{e}_2^m, \psi) = (\tau_3^m, \psi). \tag{4.8}$$

where $\tau_1^m = O(k^2), \quad \tau_2^m = O(k^2), \quad \tau_3^m = O(k^2)$ are truncation errors. Setting $\psi = \eta_1^m$ in (4.6) and taking the imaginary parts, we have

$$\frac{1}{2}\|\eta_1^m\|_t^2 - \frac{k}{2}\|\eta_{1t}^m\|^2 + F_1^m + F_2^m = (\tau_1^m, \eta_1^m), \tag{4.9}$$

where

$$\begin{aligned}
F_1^m &= -\alpha I_m(n^m u^m - n_N^m \widehat{u}_N^m, \eta_1^m),\\
F_2^m &= \beta I_m(|u^m|^2 u^m - |\widehat{u}_N^m|^2 \widehat{u}_N^m, \eta_1^m).
\end{aligned}$$

By applying Lemma 2.2 and Cauchy Schwartz inequality, we have

$$\begin{aligned}
|F_1^m| &\le C(\|(n^m u^m - n_N^m \widehat{u}_N^m\|^2 + \|\eta_1^m\|^2)\\
&\le C(N^{-2s} + \|\widehat{\eta}_1^m\|^2 + \|\eta_1^m\|^2 + \|\eta_2^m\|^2,\\
|F_2^m| &\le C((\||u^m|^2 u^m - |\widehat{u}_N^m|^2 \widehat{u}_N^m\|^2) + \|\eta_1^m\|^2)\\
&\le C([(\|u^m\|_\infty + \|\widehat{u}^m\|_\infty)\|u^m\|_\infty + \|\widehat{u}_N^m\|_\infty](\|\widehat{e}_1^m\| + \|e_1^m\|))\\
&\le C(N^{-2s} + \|\widehat{\eta}_1^m\|^2 + \|\eta_1^m\|^2),\\
|(\tau_1^m, \eta_1^m)| &\le C(\|\eta_1^m\|^2 + \|\tau_1^m\|^2).
\end{aligned}$$

Substituting the above estimates into (4.9), we have

$$\frac{1}{2}\|\eta_1^m\|_t^2 - \frac{k}{2}\|\eta_{1t}^m\|^2 \le C(N^{-2S} + \|\widehat{\eta}_1^m\|^2 + \|\eta_1^m\|^2 + \|\eta_2^m\|^2 + \|\tau_1^m\|^2). \tag{4.10}$$

Differentiate (1.1) and (4.6) with respect to x, subtracting one equation from another, setting $\psi = \partial_x \eta_1^m$ and taking the imaginary parts, we get

$$\frac{1}{2}\|\partial_x \eta_1^m\|_t^2 - \frac{k}{2}\|\partial_x \eta_{1t}^m\|^2 + F_3^m + F_4^m = (\tau_1^m, \partial_x \eta_1^m), \tag{4.11}$$

where

$$F_3^m = \alpha I_m(\partial_x(n^m u^m - n_N^m \widehat{u}_N^m), \partial_x \eta_1^m),$$
$$F_4^m = \beta I_m(\partial_x(|u^m|^2 u^m - |\widehat{u}_N^m|^2 \widehat{u}_N^m), \partial_x \eta_1^m).$$

We can estimate $|F_3^m|$, $|F_4^m|$, similar to the above method and putting in (4.11), we have

$$\frac{1}{2}\|\partial_x \eta_1^m\|_t^2 - \frac{k}{2}\|\partial_x \eta_{1t}^m\|^2 \le C(N^{-2S} + \|\widehat{\eta}_1^m\|^2 + \|\widehat{\eta}_2^m\|^2 + \|\partial_x^2 \eta_1^m\|^2 + \|\eta_2^m\|^2 + \|\tau_1^m\|^2). \tag{4.12}$$

Setting $\psi = \eta_2^m$ in (4.7), we get

$$\frac{1}{2}(\|\eta_2^m\|_t^2 + \|\partial_x \eta_2^m\|_t^2) - \frac{k}{2}(\|\eta_{2t}^m\|^2 + \|\partial_x \eta_{2t}^m\|^2) + F_5^m + F_6^m + F_7^m + F_8^m = (\tau_2^m, \partial_x \eta_2^m), \tag{4.13}$$

where

$$F_5^m = (f(v^m) - f(v_N^m), \partial_x \eta_2^m),$$
$$F_6^m = (\widehat{e}_3^m, \partial_x \eta_2^m),$$
$$F_7^m = \mu(|u^m|^2 - |u_N^m|^2, \partial_x \eta_2^m),$$
$$F_5^m = (\phi(v^m) - \phi(v_N^m), \partial_x \eta_2^m),$$

$$\begin{aligned}
|F_5^m| &\le C(\|f(v^m) - f(v_N^m)\|^2 + \|\partial_x \eta_2^m\|^2) \le C(\left\|\frac{\partial f}{\partial v^m}\right\|_\infty \|e_2^m\|^2 + \|\partial_x \eta_2^m\|^2) \\
&\le C(N^{-2S} + \|\eta_2^m\|^2 + \|\partial_x \eta_2^m\|^2), \\
|F_6^m| &\le C(\|\widehat{e}_3^m\|^2 + \|\partial_x \eta_2^m\|^2) \le C(N^{-2S} + \|\widehat{\eta}_3^m\|^2 + \|\partial_x \eta_2^m\|^2), \\
|F_7^m| &\le C(\||u^m|^2 - |u_N^m|^2\|^2 + \|\partial_x \eta_2^m\|^2) \le C((\|u^m\|_\infty^2 + \|u_N^m\|_\infty^2)\|e_1^m\|^2 + \|\partial_x \eta_2^m\|^2) \\
&\le C(N^{-2S} + \|\eta_1^m\|^2 + \|\partial_x \eta_2^m\|^2), \\
|F_8^m| &\le C(\|\phi(v^m) - \phi(v_N^m)\|^2 + \|\partial_x \eta_2^m\|^2) \le C(\left\|\frac{\partial \phi}{\partial v^m}\right\|_\infty \|e_2^m\|^2 + \|\partial_x \eta_2^m\|^2), \\
&\le C(N^{-2S} + \|\eta_2^m\|^2 + \|\partial_x \eta_2^m\|^2).
\end{aligned}$$

Substituting the above estimate into (4.13), we get

$$\begin{aligned}
\frac{1}{2}(\|\eta_2^m\|_t^2 + \|\partial_x \eta_2^m\|_t^2) - \frac{k}{2}(\|\eta_{2t}^m\|^2 + \|\partial_x \eta_{2t}^m\|^2) \le C(N^{-2S} + \|\partial_x \eta_2^m\|^2 \\
+ \|\widehat{\eta}_3^m\|^2 + \|\eta_1^m\|^2 + \|\eta_2^m\|^2 + \|\tau_2^m\|^2).
\end{aligned} \tag{4.14}$$

Setting $\psi = \eta_3^m$ in (4.8), we get

$$\frac{1}{2}\|\eta_3^m\|_t^2 - \frac{k}{2}\|\eta_{3t}^m\|^2 \le C(N^{-2S} + \|\widehat{\eta}_2^m\|^2 + \|\partial_x \eta_2^m\|^2 + \|\eta_3^m\|^2 + \|\tau_3^m\|^2). \tag{4.15}$$

Let $\psi = \eta_{1t}^m$ in (4.6) and taking the imaginary parts, we get

$$\|\eta_{1t}^m\|^2 = \alpha I_m(n^m u^m - n_N^m \widehat{u}_N^m, \eta_{1t}^m) - \beta I_m(|u^m|^2 u^m - |\widehat{u}_N^m|^2 \widehat{u}_N^m, \eta_{1t}^m) + (\tau_1^m, \eta_{1t}^m).$$

Using Cauchy-Schwartz inequality and algebraic inequality, $ab \le \varepsilon a^2 + \frac{1}{4\varepsilon}b^2$, we get

$$\|\eta_{1t}^m\|^2 \le \frac{3}{4}\|\eta_{1t}^m\|^2 + |\alpha|^2(\|n\|_\infty\|\eta_1^m\| + \|\widehat{u}_N^m\|_\infty\|e_2^m\|)^2 \\ + |\beta|^2[(\|u^m\|_\infty + \|\widehat{u}_N^m\|_\infty)\|u^m\|_\infty + \|\widehat{u}_N^m\|_\infty]^2(\|\widehat{e}_1^m\|^2 + \|e_1^m\|^2) + \|\tau_1^m\|^2,$$

which implies that

$$\|\eta_{1t}^m\|^2 \le C(N^{-2S} + \|\widehat{\eta}_1^m\|^2 + \|\eta_1^m\|^2 + \|\eta_2^m\|^2 + \|\tau_1^m\|^2). \tag{4.16}$$

Let $\psi = \eta_{2t}^m$ in (4.10), we obtain

$$\|\eta_{2t}^m\|^2 + \|\eta_{2xt}^m\|^2 \le \frac{2}{3}\|\eta_{2t}^m\|^2 + \frac{3}{4}\|\partial_x\eta_{2t}^m\|^2 + \|\widehat{\eta}_3^m\|^2 + \|\frac{\partial f}{\partial v^m}\|_\infty\|e_2^m\| \\ + |\alpha|[\|u^m\|_\infty + \|u_N^m\|_\infty]^2\|e_1^m\|^2 + \|\frac{\partial \phi}{\partial v^m}\|_\infty\|e_2^m\|^2 + \|\tau_2^m\|^2.$$

Thus we have

$$\|\eta_{2t}^m\|^2 + \|\partial_x\eta_{2t}^m\|^2 \le C(N^{-2S} + \|\eta_1^m\|^2 + \|\eta_2^m\|^2 + \|\widehat{\eta}_3^m\|^2 + \|\tau_2^m\|^2). \tag{4.17}$$

Let $\psi = \eta_{3t}^m$ in (4.8), we get

$$\|\eta_{3t}^m\|^2 \le C(N^{-2S} + \|\partial_x\eta_{3t}^m\|^2 + \|\eta_2^m\|^2 + \|\tau_3^m\|^2). \tag{4.18}$$

In fact

$$\|\widehat{u}^m\|^2 \le \frac{1}{2}\left(\|u^{m+1}\|^2 + \|u^m\|^2\right).$$

Combining (4.10), (4.12) and (4.14)- (4.18), using Lemma 2.3, Lemma 2.1, we get

$$E^n = \left\|\eta_1^{n+1}\right\|^2 + \left\|\eta_2^{n+1}\right\|^2 + \left\|\eta_1^{n+1}\right\|_{H^1}^2 + \left\|\eta_2^{n+1}\right\|_{H^1}^2 + \left\|\eta_3^{n+1}\right\|^2 \\ \le C\left(k^4 + N^{-2s} + \left\|\eta_1^0\right\|^2 + \left\|\eta_2^0\right\|^2 + \left\|\eta_1^0\right\|_{H^1}^2 + \left\|\eta_2^0\right\|_{H^1}^2 + \left\|\eta_3^0\right\|^2\right) \\ + Ck\sum_{m=0}^{n}\left(\|\eta_1^{m+1}\|^2 + \|\eta_2^{m+1}\|^2 + \|\eta_3^{m+1}\|^2 + \|\eta_1^m\|_{H^1}^2 + \|\eta_2^m\|_{H^1}^2\right),$$

and hence

$$E^n \le C\left(k^4 + N^{-2s} + \left\|\eta_1^0\right\|^2 + \left\|\eta_2^0\right\|^2 + \left\|\eta_1^0\right\|_{H^1}^2 + \left\|\eta_2^0\right\|_{H^1}^2 + \left\|\eta_3^0\right\|^2\right) + Ck\sum_{m=0}^{n} E^{m-1}. \tag{4.19}$$

Note that $\|\eta_\ell^0\| = 0, \ell = 1, 2, 3$, equation (4.19) can be written as

$$E^n(t) \le C\left(k^4 + N^{-2s}\right) + Ck\sum_{m=0}^{n} E^{m-1}. \tag{4.20}$$

By applying Gronwall's inequality, we obtain

$$C\left(k^4 + N^{-2s}\right) \le Me^{-cT},$$

and so the estimate for E^n in (4.20) takes the form

$$E^n(t) \le C\left(k^4 + N^{-2s}\right)e^{c(n+1)k}, \qquad \forall (n+1)k \le T$$

Thus we have proved:

Theorem 2. *Assume that u^m, v^m and n^m are solutions of equations (1.1)-(1.5) and u_N^m, v_N^m and n_N^m are solutions of the problem (3.1)-(3.3) respectively, then there exists a positive constant M, independent of k and N. The following error estimate holds*

$$\sup_{1\le m\le[\frac{T}{k}]}(\|u^m-u_N^m\|_{H^1}+\|v^m-v_N^m\|_{H^1}+\|n^m-n_N^m\|)\le M(k^2+N^{-s}).$$

5 Conclusion

A Fourier spectral method was applied to the initial boundary value problem for a class of generalized Zakharov equations. An energy estimation method was used to find error estimates for semi-discrete and fully discrete of the spectral schemes. A Crank-Nicolson implicit scheme was used for nonlinear Schrödinger equation in Zakharov system. The rate of convergence of the semi-discrete and fully discrete were obtained $O(N^{-S})$ and $O(k^2+N^{-S})$ respectively. This method is also very useful to other similar nonlinear partial differential equations.

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