

\mathcal{D} -homothetic transformations for a generalization of contact metric manifolds*

Beniamino Cappelletti Montano

Luigia Di Terlizzi

Abstract

Curvature properties of some generalizations of contact metric manifolds are studied, with special attention to (κ, μ) -nullity conditions in the framework of \mathcal{S} -manifolds.

1 Basic definitions

An extensive research about contact geometry is done in recent years. In the present paper we are concerned with a certain generalization of contact metric manifolds in the context of f -manifolds. We recall the precise definitions. Let M be a $(2n + s)$ -dimensional manifold. We say that M is equipped with an f -structure with a parallelizable kernel, more briefly $f.pk$ -structure, if there are given on M an f -structure φ , s global vector fields ξ_1, \dots, ξ_s and 1-forms η_1, \dots, η_s on M satisfying the following conditions

$$\varphi(\xi_i) = 0, \quad \eta_i \circ \varphi = 0, \quad \varphi^2 = -\text{Id} + \sum_{j=1}^s \eta_j \otimes \xi_j, \quad \eta_i(\xi_j) = \delta_j^i \quad (1.1)$$

for all $i, j \in \{1, \dots, s\}$; we denote by \mathcal{D} the bundle $\text{Im}(\varphi)$, and we set $\bar{\xi} := \xi_1 + \dots + \xi_s$, $\bar{\eta} := \eta_1 + \dots + \eta_s$. The structure (φ, ξ_i, η_j) on M is said to be *normal* if and only if $\mathcal{N}_\varphi = 0$, where \mathcal{N}_φ is the $(2, 1)$ -tensor on M given by $\mathcal{N}_\varphi := [\varphi, \varphi] + 2 \sum_{i=1}^s d\eta_i \otimes \xi_i$.

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On a manifold equipped with an $f.pk$ -structure there always exists a *compatible* Riemannian metric g in the sense that for each $X, Y \in \Gamma(TM)$

$$g(X, Y) = g(\varphi(X), \varphi(Y)) + \sum_{j=1}^s \eta_j(X)\eta_j(Y). \tag{1.2}$$

However such a metric on M is not unique: we fix one of them; then the structure obtained is called a *metric $f.pk$ -structure*. Let F be the Sasaki form of φ defined by $F(X, Y) := g(X, \varphi Y)$ for $X, Y \in \Gamma(TM)$. It may be observed that \mathcal{D} is the orthogonal complement of the bundle $\ker(\varphi) = \langle \xi_1, \dots, \xi_s \rangle$.

The metric $f.pk$ -manifold $(M, \varphi, \xi_i, \eta_j, g)$ is said to be an *almost \mathcal{S} -manifold* if and only if $d\eta_1 = \dots = d\eta_s = F$. Almost \mathcal{S} -manifolds which are normal are called \mathcal{S} -manifolds.

The study of f -manifolds was started by D.E. Blair, S.I. Goldberg, K. Yano, J. Vanzura, cf. [1, 11, 14]. Almost \mathcal{S} -structures were studied, without being precisely named, by J.L. Cabrerizo, L.M. Fernández and M. Fernández, cf. [4]. Then K. Duggal, A.M. Pastore and S. Ianus, cf. [10], also studied such manifolds and gave them the name "almost \mathcal{S} -manifolds". \mathcal{S} -manifolds were introduced by D.E. Blair (cf. [1]), who proved that the space of a principal toroidal bundle over a Kähler manifold is an \mathcal{S} -manifold. \mathcal{S} -structures are a natural generalization of Sasakian structures, but unlike Sasakian manifolds, no \mathcal{S} -structure can be realized on a simply connected, compact manifold (cf. [6]). In [8] there is an example of an even dimensional principal toroidal bundle over a Kähler manifold which does not carry any Sasakian structure. On the other hand, there is constructed an \mathcal{S} -structure on the even dimensional manifold $U(2)$. It is well known that $U(2)$ does not admit a Kähler structure. We conclude that there exist manifolds such that the best structure we can hope to obtain on them is an \mathcal{S} -structure.

On an almost \mathcal{S} -manifold $(M, \varphi, \xi_i, \eta_j, g)$ there are defined the (1,1)-tensor fields $h_i := (1/2)L_{\xi_i}\varphi$ for $i = 1, \dots, s$, cf. [4, (2.5)]. We use extensively the properties of these tensor fields in the present paper. In particular these operators are self adjoint, traceless, anticommute with φ and for each $i, j \in \{1, \dots, s\}$

$$h_i \xi_j = 0, \quad \eta_i \circ h_j = 0, \tag{1.3}$$

cf. [4]. Moreover the following identities hold, cf. [10],

$$\nabla_X \xi_i = -\varphi X - \varphi h_i X, \quad \nabla_{\xi_i} \varphi = 0, \quad \nabla_{\xi_i} \xi_j = 0 \tag{1.4}$$

where ∇ is the Levi Civita connection of g , $X \in \Gamma(TM)$ and $i, j \in \{1, \dots, s\}$. We shall sometimes use the following curvature identity related to ∇

$$R_{\xi_i X} \xi_j - \varphi(R_{\xi_i \varphi X} \xi_j) = 2 \left((h_i \circ h_j) X + \varphi^2 X \right) \tag{1.5}$$

which can be immediately obtained combining the first equation on [4, pag. 158] and (1.4).

In 1995 D. Blair, T. Koufogiogios and B.J. Papantoniou, cf. [2], studied contact metric manifolds such that the characteristic vector field belongs to the (κ, μ) -nullity distribution. We generalize this concept for almost \mathcal{S} -manifolds as follows.

Definition 1.1. Let M be an almost \mathcal{S} -manifold, κ, μ real constants. We say that M verifies the (κ, μ) -nullity condition if and only if for each $i \in \{1, \dots, s\}$, $X, Y \in \Gamma(TM)$ the following identity holds

$$R_{XY}\xi_i = \kappa (\bar{\eta}(X)\varphi^2Y - \bar{\eta}(Y)\varphi^2X) + \mu (\bar{\eta}(Y)h_iX - \bar{\eta}(X)h_iY). \tag{1.6}$$

Lemma 1.1. Let M be an almost \mathcal{S} -manifold verifying the (κ, μ) -nullity condition. Then we have

- (i). $h_i \circ h_j = h_j \circ h_i$, for each $i, j \in \{1, \dots, s\}$
- (ii). $\kappa \leq 1$
- (iii). if $\kappa < 1$ then, for each $i \in \{1, \dots, s\}$, h_i has eigenvalues $0, \pm\sqrt{1 - \kappa}$.

Proof. From (1.6) it follows that for each $X \in \Gamma(TM)$, $i, j \in \{1, \dots, s\}$ $R_{\xi_j X}\xi_i - \varphi R_{\xi_j \varphi X}\xi_i = 2\kappa\varphi^2X$. Using (1.5) we obtain

$$(h_i \circ h_j)X = (\kappa - 1)\varphi^2X = (h_j \circ h_i)X \tag{1.7}$$

and then (i) is verified. Next, from (1.7) we get

$$h_i^2 = (\kappa - 1)\varphi^2 \tag{1.8}$$

$$h_i^2X = (1 - \kappa)X, \quad X \in \Gamma(\mathcal{D}). \tag{1.9}$$

Then, using (1.3), (1.9) we obtain that the eigenvalues of h_i^2 are 0 and $1 - \kappa$. Moreover, since h_i is symmetric, $\|h_iX\|^2 = (1 - \kappa)\|X\|^2$. Hence $\kappa \leq 1$. Finally, let t be a real eigenvalue of h_i and X be an eigenvector corresponding to t . Then $t^2\|X\|^2 = \|h_iX\|^2 = (1 - \kappa)\|X\|^2$ and $t = \pm\sqrt{1 - \kappa}$. Taking (1.3) into account we get (iii). ■

Proposition 1.1. Let M be an almost \mathcal{S} -manifold verifying the (κ, μ) -nullity condition. Then

$$h_1 = \dots = h_s. \tag{1.10}$$

Proof. If $\kappa = 1$ then from (1.8) and the symmetry of each h_i we have $h_1 = \dots = h_s = 0$. Let now $\kappa < 1$. We fix $x \in M$ and $i \in \{1, \dots, s\}$. Since h_i is symmetric then we have $\mathcal{D}_x = (\mathcal{D}_+)_x \oplus (\mathcal{D}_-)_x$, where $(\mathcal{D}_+)_x$ is the eigenspace of h_i corresponding to the eigenvalue $\lambda = \sqrt{1 - \kappa}$ and $(\mathcal{D}_-)_x$ is the eigenspace of h_i corresponding to the eigenvalue $-\lambda$. If $X \in \mathcal{D}_x$ then we can write $X = X_+ + X_-$, where $X_+ \in (\mathcal{D}_+)_x$, $X_- \in (\mathcal{D}_-)_x$, so that $h_iX = \lambda(X_+ - X_-)$. We fix $j \in \{1, \dots, s\}$, $j \neq i$. Then from (1.7) we get $h_jX = h_j(X_+ + X_-) = h_j\left(\frac{1}{\lambda}h_iX_+ - \frac{1}{\lambda}h_iX_-\right) = \frac{1}{\lambda}(h_j \circ h_i)(X_+ - X_-) = \lambda(X_+ - X_-) = h_iX$. Taking (1.3) into account we obtain (1.10). ■

Remark 1.1. Throughout all this paper whenever (1.6) holds we put $h := h_1 = \dots = h_s$. Then (1.6) becomes

$$R_{XY}\xi_i = \kappa (\bar{\eta}(X)\varphi^2Y - \bar{\eta}(Y)\varphi^2X) + \mu (\bar{\eta}(Y)hX - \bar{\eta}(X)hY). \tag{1.11}$$

Furthermore, using (1.11), the symmetry properties of the curvature tensor and the symmetry of φ^2 and h , we get

$$R_{\xi_i X}Y = \kappa (\bar{\eta}(Y)\varphi^2X - g(X, \varphi^2Y)\bar{\xi}) + \mu (g(X, hY)\bar{\xi} - \bar{\eta}(Y)hX). \tag{1.12}$$

Remark 1.2. Let M be an almost \mathcal{S} -manifold verifying the (κ, μ) -nullity condition, with $\kappa \neq 1$. We denote by \mathcal{D}_+ and \mathcal{D}_- the n -dimensional distributions of the eigenspaces of $\lambda = \sqrt{1-\kappa}$ and $-\lambda$, respectively. We have that \mathcal{D}_+ and \mathcal{D}_- are mutually orthogonal. Moreover, since φ anticommutes with h , we have $\varphi(\mathcal{D}_+) = \mathcal{D}_-$ and $\varphi(\mathcal{D}_-) = \mathcal{D}_+$. In other words, \mathcal{D}_+ is a Legendrian distribution and \mathcal{D}_- is the conjugate Legendrian distribution of \mathcal{D}_+ (cf. [5]).

Proposition 1.2. *Let M be an almost \mathcal{S} -manifold verifying the (κ, μ) -nullity condition. Then M is an \mathcal{S} -manifold if and only if $\kappa = 1$.*

Proof. We observed in the proof of Proposition 1.1 that if $\kappa = 1$ then $h = 0$. It follows that (1.6) reduces to $R_{XY}\xi_i = \bar{\eta}(X)\varphi^2Y - \bar{\eta}(Y)\varphi^2X$. From [4, Proposition 3.4 and Theorem 4.3] we get the claim. ■

Remark 1.3. Let M be an almost \mathcal{S} -manifold verifying the (κ, μ) -nullity condition. If there exists $i \in \{1, \dots, s\}$ such that ξ_i is a Killing vector field then from [4, Theorem 2.6] we have $h = h_i = 0$. From (1.9) we get $\kappa = 1$ and using Proposition 1.2 we have that M is an \mathcal{S} -manifold.

The notion of \mathcal{D} -homothetic transformation for contact metric manifolds has been deeply studied (cf. for example [13]). Now we generalize this concept for a metric $f.pk$ -manifold (in particular for an almost \mathcal{S} -manifold).

Definition 1.2. Let $(\varphi, \xi_i, \eta_j, g)$ be an $f.pk$ -structure on a manifold M^{2n+s} and a be a real positive constant. By a \mathcal{D} -homothetic transformation of constant a we mean a change of the structure tensors in the following way:

$$\tilde{\varphi} = \varphi \quad \tilde{\eta}_i = a\eta_i \quad \tilde{\xi}_i = \frac{1}{a}\xi_i \quad \tilde{g} = ag + a(a-1) \sum_{j=1}^s \eta_j \otimes \eta_j \tag{1.13}$$

for each $i \in \{1, \dots, s\}$.

It is straightforward to prove that if $(\tilde{\varphi}, \tilde{\xi}_i, \tilde{\eta}_j, \tilde{g})$, $i, j \in \{1, \dots, s\}$, is a structure on the manifold M obtained by a \mathcal{D} -homothetic transformation from the $f.pk$ -structure $(\varphi, \xi_i, \eta_j, g)$, then $(\tilde{\varphi}, \tilde{\xi}_i, \tilde{\eta}_j, \tilde{g})$ is an (almost) \mathcal{S} -structure if and only if $(\varphi, \xi_i, \eta_j, g)$ is an (almost) \mathcal{S} -structure.

Lemma 1.2. *Let M^{2n+s} be a manifold and $(\tilde{\varphi}, \tilde{\xi}_i, \tilde{\eta}_j, \tilde{g})$, $i, j \in \{1, \dots, s\}$, be an almost \mathcal{S} -structure on M obtained from the almost \mathcal{S} -structure $(\varphi, \xi_i, \eta_j, g)$ by a \mathcal{D} -homothetic transformation. Then for each $i \in \{1, \dots, s\}$, $X, Y \in \Gamma(TM)$ the following identities hold*

$$a\tilde{h}_i = h_i \tag{1.14}$$

$$a\tilde{\nabla}_X\tilde{\xi}_i = \nabla_X\xi_i + (1-a)\varphi X \tag{1.15}$$

$$\eta_i(\tilde{\nabla}_X Y) = X(\eta_i(Y)) - g(Y, \varphi X + \varphi\tilde{h}_i X) \tag{1.16}$$

$$a\tilde{\nabla}_X Y = a\nabla_X Y + (1-a) \left(\sum_{l=1}^s g(\varphi h_l X, Y)\xi_l + a(\bar{\eta}(Y)\varphi X + \bar{\eta}(X)\varphi Y) \right) \tag{1.17}$$

where $\tilde{\nabla}$ and ∇ denote the Levi Civita connections of \tilde{g} and g , respectively, and $\tilde{h}_i = \frac{1}{2}L_{\tilde{\xi}_i}\tilde{\varphi}$.

Proof. Identity (1.14) is an immediate consequence of the definitions of h_i, \tilde{h}_i . By an easy direct computation, from (1.4), using (1.14), we get (1.15). Since $\eta_i(X) = g(X, \xi_i)$, for each $i \in \{1, \dots, s\}$, $X \in \Gamma(TM)$, using (1.14) we have (1.16). Next, applying the Koszul formulas, cf. [12, pag.160], for $\tilde{\nabla}, \nabla$ and $d\eta_1 = \dots = d\eta_s = F$ we get

$$2\tilde{g}(\tilde{\nabla}_X Y, Z) = 2ag(\nabla_X Y, Z) + a(a-1) \sum_{i=1}^s [2g(X, \varphi Z)\eta_i(Y) + 2g(Y, \varphi Z)\eta_i(X) + g((X(\eta_i(Y)) + Y(\eta_i(X)) + \eta_i([X, Y]))\xi_i, Z)].$$

Here we substitute the expression of \tilde{g} in (1.13) and then using (1.16) and $d\eta_1 = \dots = d\eta_s = F$ we obtain

$$\tilde{\nabla}_X Y = \nabla_X Y + (1-a) \sum_{i=1}^s [\eta_i(Y)\varphi X + \eta_i(X)\varphi Y - \frac{1}{2}(X(\eta_i(Y)) + Y(\eta_i(X)) + \eta_i([X, Y]))\xi_i + (X(\eta_i(Y)) + g(\varphi X, Y) + g(\varphi \tilde{h}_i X, Y))\xi_i].$$

Taking (1.14) into account we get (1.17). ■

Remark 1.4. Under the same hypotheses of Proposition 1.2, from (1.14) and [4, Theorem 2.6] it follows that ξ_i is a Killing vector field if and only if $\tilde{\xi}_i$ is a Killing vector field, $i \in \{1, \dots, s\}$.

Proposition 1.3. *Let M^{2n+s} be a manifold and $(\tilde{\varphi}, \tilde{\xi}_i, \tilde{\eta}_j, \tilde{g})$, $i, j \in \{1, \dots, s\}$, be an almost \mathcal{S} -structure on M obtained from the almost \mathcal{S} -structure $(\varphi, \xi_i, \eta_j, g)$ by a \mathcal{D} -homothetic transformation of constant a . Then for each $i \in \{1, \dots, s\}$, $X, Y \in \Gamma(TM)$ the following identity holds*

$$a\tilde{R}_{XY}\tilde{\xi}_i = R_{XY}\xi_i + \frac{1-a}{a} \sum_{l=1}^s (g(h_l Y, h_l X) - g(h_l X, h_l Y))\xi_l + (1-a) [\bar{\eta}(X)(h_i Y - \varphi^2 Y) - \bar{\eta}(Y)(h_i X - \varphi^2 X) + (\nabla_X \varphi)Y - (\nabla_Y \varphi)X] + (1-a)^2 (\bar{\eta}(X)\varphi^2 Y - \bar{\eta}(Y)\varphi^2 X). \tag{1.18}$$

Proof. Using (1.15), (1.17), (1.2), (1.4), (1.3) and the symmetry of each h_i we can straightforwardly obtain (1.18). ■

2 Properties of the curvature

Let $(M^{2n+s}, \varphi, \xi_i, \eta_j, g)$, $i, j \in \{1, \dots, s\}$, be an almost \mathcal{S} -manifold. We consider the $(1, 1)$ -tensor fields defined by

$$l_{ij}(X) = R_{X\xi_i}\xi_j$$

for each $i, j \in \{1, \dots, s\}$, $X \in \Gamma(TM)$ and put $l_i = l_{ii}$.

Lemma 2.1. For each $i, j, k \in \{1, \dots, s\}$ the following identities hold

$$\varphi \circ l_{ij} \circ \varphi - l_{ij} = 2(h_j \circ h_i + \varphi^2) \tag{2.1}$$

$$\eta_k \circ l_{ij} = 0 \tag{2.2}$$

$$l_{ij}(\xi_k) = 0 \tag{2.3}$$

$$\nabla_{\xi_i} h_j = \varphi - \varphi \circ l_{ij} - \varphi \circ h_j \circ h_i + \varphi \circ (h_j - h_i) \tag{2.4}$$

$$\nabla_{\xi_i} h_i = \varphi - \varphi \circ l_i - \varphi \circ h_i^2. \tag{2.5}$$

Proof. Identity (2.1) is a rewriting of [9, (3.4)]; (2.2) and (2.3) are an immediate consequence of (2.1). Next from (1.4) and $\eta_l \circ (\nabla_{\xi_i} h_k) = 0$ we get $(\varphi - \varphi \circ l_{ij} - \varphi \circ h_j \circ h_i)(X) = -\varphi^2((\nabla_{\xi_i} h_j)X) - (\varphi \circ h_i - \varphi \circ h_j)(X) = (\nabla_{\xi_i} h_j)(X) + \varphi((h_j - h_i)(X))$, for each $X \in \Gamma(TM)$, from which it follows (2.4). Finally, identity (2.5) is (2.4) when $i = j$. ■

Remark 2.1. In the case when ξ_i is Killing for each $i \in \{1, \dots, s\}$, from [4, Theorem 2.6] we get that (2.4) reduces to $\varphi \circ l_{ij} = \varphi$. Then from (2.2) we have $l_{ij} = -\varphi^2$ so that all the l_{ij} 's coincide.

Remark 2.2. Let M be an almost \mathcal{S} -manifold verifying the (κ, μ) -nullity condition. Then for each $i, j \in \{1, \dots, s\}$ we have

$$l_{ij} = -\kappa\varphi^2 + \mu h. \tag{2.6}$$

It follows that all the l_{ij} 's coincide. We put $l = l_{ij}$

Lemma 2.2. Let M be an almost \mathcal{S} -manifold verifying the (κ, μ) -nullity condition. Then for each $X, Y \in \Gamma(TM)$, $i \in \{1, \dots, s\}$, the following identities hold

$$\nabla_{\xi_i} h = \mu h \circ \varphi \tag{2.7}$$

$$l \circ \varphi - \varphi \circ l = 2\mu h \circ \varphi \tag{2.8}$$

$$l \circ \varphi + \varphi \circ l = 2\kappa \varphi \tag{2.9}$$

$$Q\xi_i = 2n\kappa \bar{\xi}. \tag{2.10}$$

Proof. From (2.5), using (2.6), we obtain (2.7). Identities (2.8) and (2.9) follow directly from (2.6) using $h \circ \varphi = -\varphi \circ h$. For the proof of (2.10) we fix $x \in M$ and $\{E_1, \dots, E_{2n+s}\}$ a local φ -basis around x with $E_{2n+1} = \xi_1, \dots, E_{2n+s} = \xi_s$. Then using (1.12) and $\text{trace}(h) = 0$ we get $Q\xi_i = \sum_{\alpha=1}^{2n} R_{\xi_i E_\alpha} E_\alpha = \sum_{\alpha=1}^{2n} \kappa g(\varphi^2 E_\alpha, E_\alpha) \bar{\xi} = \kappa \sum_{\alpha=1}^{2n} \delta_{\alpha\alpha} \bar{\xi} = 2n\kappa \bar{\xi}$. ■

Remark 2.3. Let M be an almost \mathcal{S} -manifold. Then from [7, (2.2)] using $(\nabla_{h_i X} F)(Y, Z) = -g((\nabla_{h_i X} \varphi)Y, Z)$, for each $X, Y, Z \in \Gamma(TM)$, we get

$$\begin{aligned} (\nabla_{h_i X} \varphi)Y &= \frac{1}{2}(\varphi R_{\xi_i \varphi X} Y - R_{\xi_i \varphi X} \varphi Y - \varphi R_{\xi_i X} \varphi Y - R_{\xi_i X} Y) \\ &\quad -g(\varphi^2 X - h_i X, Y) \bar{\xi} + \bar{\eta}(Y)(\varphi^2 X - h_i X). \end{aligned} \tag{2.11}$$

Lemma 2.3. *Let M be an almost \mathcal{S} -manifold verifying the (κ, μ) -nullity condition. Then the following identities hold*

$$(\nabla_X \varphi)Y = g(Y, hX - \varphi^2 X)\bar{\xi} - \bar{\eta}(Y)(hX - \varphi^2 X) \tag{2.12}$$

$$\begin{aligned} (\nabla_X h)Y - (\nabla_Y h)X &= (1 - \kappa) \left(2g(X, \varphi Y)\bar{\xi} + \bar{\eta}(X)\varphi Y - \bar{\eta}(Y)\varphi X \right) \\ &\quad + (1 - \mu) (\bar{\eta}(X)\varphi hY - \bar{\eta}(Y)\varphi hX). \end{aligned} \tag{2.13}$$

Proof. From (2.11) we obtain $(\nabla_{hX} \varphi)Y = \kappa \left(g(X, \varphi^2 Y)\bar{\xi} - \bar{\eta}(Y)\varphi^2 X \right) - g(\varphi^2 X - hX, Y)\bar{\xi} + \bar{\eta}(Y)(\varphi^2 X - hX)$. Here we replace X with hX and by a direct computation, taking (1.4), (1.8) into account, we get (2.12). From (2.12), since h and φ^2 are self-adjoint, we have $(\nabla_X(\varphi \circ h))Y - (\nabla_Y(\varphi \circ h))X = \varphi \left((\nabla_X h)Y - (\nabla_Y h)X \right)$. It follows that for each $Z \in \Gamma(TM)$

$$\begin{aligned} g(R_{XY}\xi_i, Z) &= g\left(g(X, hZ - \varphi^2 Z)\bar{\xi}, Y\right) - g\left(\bar{\eta}(X)(hZ - \varphi^2 Z), Y\right) \\ &\quad + g\left(\varphi((\nabla_Y h)X - (\nabla_X h)Y), Z\right), \end{aligned} \tag{2.14}$$

where we use (2.1) of [7] and (2.12). From (2.14) and the symmetry of h and φ^2 it follows that $\varphi((\nabla_Y h)X - (\nabla_X h)Y) = R_{XY}\xi_i - \bar{\eta}(Y)(hX - \varphi^2 X) + \bar{\eta}(X)(hY - \varphi^2 Y)$. Then, applying φ to both the sides of the last identity, using (1.11) and $\eta_l((\nabla_Y h)X - (\nabla_X h)Y) = 2(k - 1)g(X, \varphi Y)$, $l \in \{1, \dots, s\}$, we get (2.13). ■

Theorem 2.1. *Let $\mathcal{Z} = (M^{2n+s}, \varphi, \xi_i, \eta_j, g)$ be an almost \mathcal{S} -manifold and $(\tilde{\varphi}, \tilde{\xi}_i, \tilde{\eta}_j, \tilde{g})$ be an almost \mathcal{S} -structure on M obtained by a \mathcal{D} -homothetic transformation of constant a . If \mathcal{Z} verifies the (κ, μ) -nullity condition for certain real constants (κ, μ) then $(M, \tilde{\varphi}, \tilde{\xi}_i, \tilde{\eta}_j, \tilde{g})$ verifies the $(\tilde{\kappa}, \tilde{\mu})$ -nullity condition, where*

$$\tilde{\kappa} = \frac{\kappa + a^2 - 1}{a^2}, \quad \tilde{\mu} = \frac{\mu + 2(a - 1)}{a}.$$

Proof. From (1.14) and Proposition 1.1 it follows that $\tilde{h}_1 = \dots = \tilde{h}_s$. Then, using (1.18) and (2.12), by a direct calculation we get the claim. ■

Remark 2.4. In [7] there are studied almost \mathcal{S} -manifolds such that $R_{XY}\xi_i = 0$ for all $X, Y \in \Gamma(TM)$, $i \in \{1, \dots, s\}$. This is the case when (1.6) is verified for $\kappa = \mu = 0$. If we consider $a > 0$ and a \mathcal{D} -homothetic transformation of constant a on such a manifold, then from Theorem 2.1 we obtain an almost \mathcal{S} -manifold verifying the $(\tilde{\kappa}, \tilde{\mu})$ -nullity condition where $\tilde{\kappa} = \frac{a^2-1}{a^2}$ and $\tilde{\mu} = \frac{2(a-1)}{a}$. This result can be applied for the examples of flat \mathcal{S} -manifolds of dimension $2 + s$, $s \geq 2$ given in [9] so that we easily obtain examples of \mathcal{S} -manifolds of dimension $2 + s$ verifying the (κ, μ) -nullity condition with $(\kappa, \mu) \neq (0, 0)$ and $(\kappa, \mu) \neq (1, 0)$.

Lemma 2.4. *Let M be an almost \mathcal{S} -manifold verifying the (κ, μ) -nullity condition. Then*

$$X, Y \in \Gamma(\mathcal{D}_+) \Rightarrow \nabla_X Y \in \Gamma(\mathcal{D}_+) \tag{2.15}$$

$$X, Y \in \Gamma(\mathcal{D}_-) \Rightarrow \nabla_X Y \in \Gamma(\mathcal{D}_-) \tag{2.16}$$

$$X \in \Gamma(\mathcal{D}_+), Y \in \Gamma(\mathcal{D}_-) \Rightarrow \nabla_X Y \in \Gamma(\mathcal{D}_- \oplus \ker(\varphi)) \tag{2.17}$$

$$X \in \Gamma(\mathcal{D}_-), Y \in \Gamma(\mathcal{D}_+) \Rightarrow \nabla_X Y \in \Gamma(\mathcal{D}_+ \oplus \ker(\varphi)) \tag{2.18}$$

Proof. From (2.13) we get $g((\nabla_X h)\varphi Z - (\nabla_{\varphi Z} h)X, Y) = 0$, for each $X, Y, Z \in \Gamma(\mathcal{D}_+)$. On the other hand, since h is symmetric, from Remark 1.2 we have $g((\nabla_X h)\varphi Z - (\nabla_{\varphi Z} h)X, Y) = -2\lambda g(\nabla_X(\varphi Z), Y)$.

Then $g(\varphi Z, \nabla_X Y) = -g(\nabla_X(\varphi Z), Y) = 0$, that is $\nabla_X Y$ is normal to \mathcal{D}_- . Moreover from (1.4) and Remark 1.2 it follows that, for each $i \in \{1, \dots, s\}$, $g(\nabla_X Y, \xi_i) = -g(Y, \nabla_X \xi_i) = 0$. Then we have (2.15). The proof of (2.16) is analogous. If $X \in \Gamma(\mathcal{D}_+)$, $Y \in \Gamma(\mathcal{D}_-)$ then from (2.15) and Remark 1.2 we get that, for each $Z \in \Gamma(\mathcal{D}_+)$, $g(\nabla_X Y, Z) = -g(Y, \nabla_X Z) = 0$ and then we have (2.17). Analogously we prove (2.18). ■

Remark 2.5. It follows from (2.15)-(2.16) that \mathcal{D}_\pm define two orthogonal totally geodesic Legendrian foliations \mathcal{F}_\pm on M .

Example 2.1. Let \mathfrak{g} be a $(2n + s)$ -dimensional Lie algebra and let $\{X_1, \dots, X_n, Y_1, \dots, Y_n, \xi_1, \dots, \xi_s\}$ be a basis of \mathfrak{g} . The Lie bracket is defined as follows:

$$\begin{aligned} [X_\alpha, X_\beta] &= 0 \quad \text{for any } \alpha, \beta \in \{1, \dots, n\}, \\ [Y_\alpha, Y_\beta] &= 0 \quad \text{for any } \alpha \neq 2, \quad [Y_2, Y_\beta] = 2Y_\beta \quad \text{for any } \beta \neq 2 \\ [X_1, Y_1] &= 2\bar{\xi} - 2X_2, \quad [X_1, Y_\beta] = 0 \quad \text{for any } \beta \geq 2, \\ [X_h, Y_k] &= \delta_{hk} (2\bar{\xi} - 2X_2) \quad \text{for any } h, k \geq 3, \quad [X_2, Y_\beta] = 2X_\beta \quad \text{for any } \beta \neq 2, \\ [X_2, Y_2] &= 2\bar{\xi}, \quad [X_k, Y_1] = [X_k, Y_2] = 0 \quad \text{for any } k \geq 3, \\ [\xi_i, \xi_j] &= 0, \quad [\xi_i, X_\beta] = 0 \quad \text{and} \quad [\xi_i, Y_\beta] = 2X_\beta \quad \text{for any } \beta \in \{1, \dots, n\}, \end{aligned}$$

for all $i, j \in \{1, \dots, s\}$, where $\bar{\xi} = \xi_1 + \dots + \xi_s$. Let G be the Lie group whose Lie algebra is \mathfrak{g} . On G one can define an almost \mathcal{S} -structure by defining $\varphi(X_\alpha) = Y_\alpha$, $\varphi(Y_\alpha) = -X_\alpha$, $\varphi(\xi_1) = \dots = \varphi(\xi_s) = 0$, for all $\alpha \in \{1, \dots, n\}$, considering the left invariant Riemannian metric g such that $\{X_1, \dots, X_n, Y_1, \dots, Y_n, \xi_1, \dots, \xi_s\}$ is an orthonormal frame and, finally, defining each 1-form η_i as the dual 1-form of the vector field ξ_i with respect to the metric g . Taking into account the previous relations, we have, for all $h, k \in \{3, \dots, n\}$ and $i, j \in \{1, \dots, s\}$,

$$\begin{aligned} \nabla_{X_1} Y_2 &= 0, \quad \nabla_{X_1} Y_k = 0, \quad \nabla_{X_1} \xi_i = -2Y_1, \\ \nabla_{X_2} Y_1 &= 0, \quad \nabla_{X_2} Y_k = 0, \quad \nabla_{X_2} \xi_i = -2Y_2, \\ \nabla_{X_k} Y_1 &= 0, \quad \nabla_{X_k} Y_2 = 0, \quad \nabla_{X_k} Y_h = 2\delta_{hk}\bar{\xi}, \quad \nabla_{X_k} \xi_i = -2Y_k, \\ \nabla_{Y_1} Y_1 &= 2Y_2, \quad \nabla_{Y_1} \xi_i = 0, \\ \nabla_{Y_2} Y_1 &= 0, \quad \nabla_{Y_2} Y_2 = 0, \quad \nabla_{Y_2} \xi_i = 0, \\ \nabla_{Y_k} Y_1 &= 0, \quad \nabla_{Y_k} Y_2 = -2Y_k, \quad \nabla_{Y_k} Y_h = 2\delta_{hk}Y_2, \quad \nabla_{Y_k} \xi_i = 0, \end{aligned}$$

from which we get

$$\begin{aligned} R(X_1X_2)\xi_i &= R(X_1X_k)\xi_i = 0, \\ R(X_1Y_1)\xi_i &= R(X_1Y_2)\xi_i = R(X_1Y_k)\xi_i = 0, \\ R(X_2X_k)\xi_i &= R(X_2Y_1)\xi_i = R(X_2Y_2)\xi_i = R(X_2Y_k)\xi_i = 0, \\ R(X_kX_h)\xi_i &= R(X_kY_1)\xi_i = R(X_kY_2)\xi_i = R(X_kY_h)\xi_i = 0, \\ R(Y_1Y_2)\xi_i &= R(Y_1Y_k)\xi_i = R(Y_2Y_k)\xi_i = R(Y_hY_k)\xi_i = 0, \\ R(X_1\xi_j)\xi_i &= 4X_1, \quad R(X_2, \xi_j)\xi_i = 4X_2, \quad R(X_k\xi_j)\xi_i = 4X_k, \\ R(Y_1\xi_j)\xi_i &= -4Y_1, \quad R(Y_2\xi_j)\xi_i = -4Y_2, \quad R(Y_k\xi_j)\xi_i = -4Y_k, \\ R(\xi_i\xi_j)\xi_i &= 0. \end{aligned}$$

Moreover, for the tensor fields h_1, \dots, h_s we find that, for each $i \in \{1, \dots, s\}$, $h_i(X_\alpha) = X_\alpha$, $h_i(Y_\alpha) = -Y_\alpha$, for all $\alpha \in \{1, \dots, n\}$, and we conclude that G is an almost \mathcal{S} -manifold verifying the (κ, μ) -nullity condition with $\kappa = 0$ and $\mu = 4$. In this case, $\mathcal{D}_+ = \langle X_1, \dots, X_n \rangle$, $\mathcal{D}_- = \langle Y_1, \dots, Y_n \rangle$ and $\lambda = 1$. Note that each ξ_i is a foliate vector field with respect to the Legendrian foliation \mathcal{F}_+ defined by \mathcal{D}_+ , so that \mathcal{F}_+ is strongly flat (cf. [5]). Further note that a family of almost \mathcal{S} -manifolds satisfying the $(\kappa(a), \mu(a))$ -nullity condition for any $a > 0$ is obtained from $(G, \phi, \xi_i, \eta_j, g)$ by performing \mathcal{D} -homothetic transformations of constant a .

Lemma 2.5. *Let M be an almost \mathcal{S} -manifold verifying the (κ, μ) -nullity condition. Then for each $X, Y \in \Gamma(TM)$ we have*

$$\begin{aligned} (\nabla_X h)Y &= \left((1 - \kappa)g(X, \varphi Y) + g(X, h\varphi Y) \right) \bar{\xi} \\ &\quad + \bar{\eta}(Y)h(\varphi X + \varphi hX) - \mu\bar{\eta}(X)\varphi hY \end{aligned} \tag{2.19}$$

Proof. We fix $x \in M$ and a local φ -basis $\{e_1, \dots, e_n, \varphi e_1, \dots, \varphi e_n, \xi_1, \dots, \xi_s\}$ around x such that $\{e_1, \dots, e_n\}$ is a local basis of \mathcal{D}_+ . If $X \in \Gamma(\mathcal{D}_+)$, $Y \in \Gamma(\mathcal{D}_-)$ from (1.3), (2.17) and Remark 1.2 we have in x

$$(\nabla_X h)Y = -\lambda\nabla_X Y + \lambda \sum_{\alpha=1}^n g(\nabla_X Y, \varphi e_\alpha)\varphi e_\alpha = \lambda(1 + \lambda)g(X, \varphi Y)\bar{\xi}. \tag{2.20}$$

Moreover, from (1.3) and (2.18) we have $h(\nabla_Y X) = \lambda \sum_{\alpha=1}^n g(\nabla_Y X, e_\alpha)e_\alpha$. Then from (1.4) we have

$$(\nabla_Y h)X = \lambda(1 - \lambda)g(X, \varphi Y)\bar{\xi}. \tag{2.21}$$

Let $X, Y \in \Gamma(TM)$. We can write $X = X_+ + X_- + \sum_{i=1}^s \eta_i(X)\xi_i$, $Y = Y_+ + Y_- + \sum_{i=1}^s \eta_i(Y)\xi_i$ because of the decomposition $TM = \mathcal{D}_+ \oplus \mathcal{D}_- \oplus \ker(\varphi)$. On the other hand we have $\lambda(g(X_+, \varphi Y_-) - g(X_-, \varphi Y_+)) = g(hX, \varphi Y)$ and $\lambda^2(g(X_+, \varphi Y_-) + g(X_-, \varphi Y_+)) = g(hX, h\varphi Y)$. Then, from (2.7), (2.20), (2.21), (1.3) and (1.4) we get $(\nabla_X h)Y = \bar{\eta}(X)\mu h\varphi Y + \bar{\eta}(Y)h(\varphi X + \varphi hX) + \lambda^2(g(X_+, \varphi Y_-) + g(X_-, \varphi Y_+))\bar{\xi} + \lambda(g(X_+, \varphi Y_-) - g(X_-, \varphi Y_+))\bar{\xi}$. From the symmetry of h and (1.9) it follows (2.19). ■

Remark 2.6. Let M be an almost \mathcal{S} -manifold verifying the (κ, μ) -nullity condition. Then using (2.12), (2.19) and (1.8) we get, for all $X, Y \in \Gamma(TM)$

$$\begin{aligned} (\nabla_X \varphi h)Y &= \left[g(X, hY) - (1 - \kappa)g(X, \varphi^2 Y) \right] \bar{\xi} \\ &\quad + \bar{\eta}(Y) \left[hX - (1 - \kappa)\varphi^2 X \right] + \mu\bar{\eta}(X)hY. \end{aligned} \tag{2.22}$$

Lemma 2.6. *Let M be an almost \mathcal{S} -manifold verifying the (κ, μ) -nullity condition. Then for each $X, Y, Z \in \Gamma(\mathcal{D})$ we have*

$$R_{XY}hZ - hR_{XY}Z = s[\kappa(g(Y, \varphi Z)\varphi hX - g(X, \varphi Z)\varphi hY + g(Z, \varphi hY)\varphi X - g(Z, \varphi hX)\varphi Y) - 2\mu g(X, \varphi Y)\varphi hZ]. \tag{2.23}$$

Proof. Let $X, Y, Z \in \Gamma(TM)$. Then by a direct computation we get

$$\begin{aligned} (\nabla_X \nabla_Y h)Z &= (1 - \kappa) \left[(g(\nabla_X Y, \varphi Z) + g(Y, (\nabla_X \varphi)Z))\bar{\xi} - \bar{\eta}(Z) \left((\nabla_X \varphi)Y + \varphi(\nabla_X Y) \right) \right] \\ &+ \left[(1 - \kappa)g(Y, \varphi Z) + g(Y, h\varphi Z) \right] \nabla_X \bar{\xi} + \left[g(\nabla_X Y, h\varphi Z) + g(Y, (\nabla_X h\varphi)Z) \right] \bar{\xi} \\ &+ g(Z, \nabla_X \bar{\xi}) \left[h\varphi Y - (1 - \kappa)\varphi Y \right] + \bar{\eta}(Z) \left[(\nabla_X h\varphi)Y + h\varphi(\nabla_X Y) \right] \\ &- \mu \left[(\bar{\eta}(\nabla_X Y) + g(Y, \nabla_X \bar{\xi}))\varphi hZ - \bar{\eta}(Y)(\nabla_X \varphi h)Z \right] \end{aligned}$$

where we use (2.19), (1.8) and the antisymmetry of $\nabla_X \varphi$. Hence, using the Ricci identity $R_{XY}hZ - hR_{XY}Z = (\nabla_X \nabla_Y h)Z - (\nabla_Y \nabla_X h)Z - (\nabla_{[X,Y]}h)Z$, (2.19), the symmetry of $\nabla_X(h \circ \varphi)$ and (1.4), we obtain

$$\begin{aligned} R_{XY}hZ - hR_{XY}Z &= \mu \left[\bar{\eta}(Y)(\nabla_X \varphi h)Z - \bar{\eta}(X)(\nabla_Y \varphi h)Z - 2sg(X, \varphi Y)\varphi hZ \right] \\ &+ \left[g(Y, h\varphi Z) + (1 - \kappa)g(Y, \varphi Z) \right] \nabla_X \bar{\xi} - \left[g(X, h\varphi Z) + (1 - \kappa)g(X, \varphi Z) \right] \nabla_Y \bar{\xi} \\ &+ g(Z, \nabla_X \bar{\xi}) \left[h\varphi Y - (1 - \kappa)\varphi Y \right] - g(Z, \nabla_Y \bar{\xi}) \left[h\varphi X - (1 - \kappa)\varphi X \right] \\ &+ \left[(1 - \kappa)g((\nabla_Y \varphi)X - (\nabla_X \varphi)Y, Z) + g((\nabla_X h\varphi)Y - (\nabla_Y h\varphi)X, Z) \right] \bar{\xi} \\ &+ \bar{\eta}(Z) \left[(\nabla_X h\varphi)Y - (\nabla_Y h\varphi)X - (1 - \kappa)((\nabla_X \varphi)Y - (\nabla_Y \varphi)X) \right]. \end{aligned} \tag{2.24}$$

If we take $X, Y, Z \in \Gamma(\mathcal{D})$ then from (2.24), using identities (2.22), (2.12) and (1.4), we get (2.23). ■

Lemma 2.7. *Let M be an almost \mathcal{S} -manifold verifying the (κ, μ) -nullity condition. Then for each $X, Y, Z \in \Gamma(TM)$ we have*

$$\begin{aligned} R_{XY}\varphi Z - \varphi R_{XY}Z &= \left[\kappa(\bar{\eta}(Y)g(Z, \varphi X) - \bar{\eta}(X)g(Z, \varphi Y)) \right. \\ &+ \left. \mu(\bar{\eta}(Y)g(Z, \varphi hX) - \bar{\eta}(X)g(Z, \varphi hY)) \right] \bar{\xi} \\ &+ s[g(Z, \varphi X + \varphi hX)(hY - \varphi^2 Y) - g(Z, \varphi Y + \varphi hY)(hX - \varphi^2 X) \\ &- g(Z, hY - \varphi^2 Y)(\varphi X + \varphi hX) + g(Z, hX - \varphi^2 X)(\varphi Y + \varphi hY)] \\ &- \bar{\eta}(Z) [\kappa(\bar{\eta}(Y)\varphi X - \bar{\eta}(X)\varphi Y) + \mu(\bar{\eta}(Y)\varphi hX - \bar{\eta}(X)\varphi hY)]. \end{aligned}$$

Proof. We proceed fixing a point $x \in M$ and local vector fields X, Y, Z such that $\nabla X, \nabla Y$ and ∇Z vanish at x . Applying several times (2.12) and using (1.4) and the symmetry of $\nabla \varphi^2$, we get in x

$$\begin{aligned} \nabla_X((\nabla_Y \varphi)Z) - \nabla_Y((\nabla_X \varphi)Z) &= \left[g((\nabla_X h)Y - (\nabla_Y h)X, Z) \right. \\ &+ \left. \bar{\eta}(Y)g(Z, \varphi X + \varphi hX) - \bar{\eta}(X)g(Z, \varphi Y + \varphi hY) \right] \bar{\xi} \\ &+ s[g(Z, \varphi X + \varphi hX)(hY - \varphi^2 Y) - g(Z, \varphi Y + \varphi hY)(hX - \varphi^2 X) \\ &- g(Z, hY - \varphi^2 Y)(\varphi X + \varphi hX) + g(Z, hX - \varphi^2 X)(\varphi Y + \varphi hY)] \\ &- \bar{\eta}(Z) \left[(\nabla_X h)Y - (\nabla_Y h)X + \bar{\eta}(Y)(\varphi X + \varphi hX) - \bar{\eta}(X)(\varphi Y + \varphi hY) \right]. \end{aligned}$$

From the last identity, using $R_{XY}\varphi Z - \varphi R_{XY}Z = \nabla_X(\nabla_Y\varphi)Z - \nabla_Y(\nabla_X\varphi)Z$ and (2.13), we get the claimed identity. \blacksquare

Remark 2.7. In particular, from Lemma 2.7 it follows that for an \mathcal{S} -manifold $(M, \varphi, \xi_i, \eta_j, g)$ the following formula holds, for all $X, Y, Z \in \Gamma(TM)$,

$$R_{XY}\varphi Z - \varphi R_{XY}Z = \bar{\eta}(Y)g(Z, \varphi X) - \bar{\eta}(X)g(Z, \varphi Y) + s(g(Z, \varphi Y)\varphi^2 X - g(Z, \varphi X)\varphi^2 Y + g(Z, \varphi^2 Y)\varphi X - g(Z, \varphi^2 X)\varphi Y) - \bar{\eta}(Z)(\bar{\eta}(Y)\varphi X - \bar{\eta}(X)\varphi Y).$$

Theorem 2.2. Let M be an almost \mathcal{S} -manifold verifying the (κ, μ) -nullity condition with $\kappa < 1$. Then for each $X_+, Y_+, Z_+ \in \Gamma(\mathcal{D}_+)$, $X_-, Y_-, Z_- \in \Gamma(\mathcal{D}_-)$, we have

$$R_{X_-Y_-}Z_+ = s(\kappa - \mu)(g(\varphi Z_+, X_-)\varphi Y_- - g(\varphi Z_+, Y_-)\varphi X_-) \tag{2.25}$$

$$R_{X_+Y_+}Z_+ = s(2(1 + \lambda) - \mu)(g(Z_+, Y_+)X_+ - g(Z_+, X_+)Y_+) \tag{2.26}$$

$$R_{X_+Y_+}Z_- = s(\kappa - \mu)(g(\varphi Z_-, X_+)\varphi Y_+ - g(\varphi Z_-, Y_+)\varphi X_+) \tag{2.27}$$

$$R_{X_+Y_-}Z_- = s(\kappa g(\varphi X_+, Z_-)\varphi Y_- + \mu g(\varphi X_+, Y_-)\varphi Z_-) \tag{2.28}$$

$$R_{X_+Y_-}Z_+ = s(-\kappa g(\varphi Y_-, Z_+)\varphi X_+ - \mu g(\varphi Y_-, X_+)\varphi Z_+) \tag{2.29}$$

$$R_{X_-Y_-}Z_- = s(2(1 - \lambda) - \mu)(g(Y_-, Z_-)X_- - g(X_-, Z_-)Y_-) \tag{2.30}$$

Proof. Let $\{e_1, \dots, e_n, \varphi e_1, \dots, \varphi e_n, \xi_1, \dots, \xi_s\}$ be a local φ -basis such that $\{e_1, \dots, e_n\}$ is a basis of \mathcal{D}_+ . From Lemma 2.6 we get

$$\lambda R_{Z_+X_-}e_\alpha - hR_{Z_+X_-}e_\alpha = 2s\lambda[\kappa(g(X_-, \varphi e_\alpha)\varphi Z_+ + g(Z_+, \varphi e_\alpha)\varphi X_-) - \mu g(Z_+, \varphi X_-)\varphi e_\alpha]. \tag{2.31}$$

Taking the symmetry of h into account we have $g(\lambda R_{Z_+X_-}e_\alpha - hR_{Z_+X_-}e_\alpha, Y_-) = 2\lambda g(R_{Z_+X_-}e_\alpha, Y_-)$ and then, from (2.31) and Remark 1.2, $g(R_{Z_+X_-}e_\alpha, Y_-) = s(\kappa g(X_-, \varphi e_\alpha)g(\varphi Z_+, Y_-) - \mu g(Z_+, \varphi X_-)g(\varphi e_\alpha, Y_-))$. It follows that

$$g(R_{X_-Y_-}Z_+, e_\alpha) = s(\kappa - \mu)(g(X_-, \varphi e_\alpha)g(\varphi Z_+, Y_-) - g(Y_-, \varphi e_\alpha)g(\varphi Z_+, X_-)) \tag{2.32}$$

where we use $g(R_{X_-Y_-}Z_+, e_\alpha) = -g(R_{Z_+Y_-}e_\alpha, X_-) + g(R_{Z_+X_-}e_\alpha, Y_-)$. From (2.16) and Remark 1.2 we have $g(R_{X_-Y_-}Z_+, \varphi e_\alpha) = 0$; moreover (1.11) yields $g(R_{X_-Y_-}Z_+, \xi_i) = 0$. Then $R_{X_-Y_-}Z_+ = \sum_{\beta=1}^n g(R_{X_-Y_-}Z_+, e_\beta)e_\beta$. Using (2.32) we get (2.25). Identity (2.26) follows from (2.15), (1.11) Lemma 2.7, (1.2), (2.25) and $(1 + \lambda)^2 + \kappa = 2(1 + \lambda)$. The other identities follow in a similar way and then are omitted. \blacksquare

Theorem 2.3. Let M be an almost \mathcal{S} -manifold verifying the (κ, μ) -nullity condition with $\kappa < 1$. Then the sectional curvature K of M is determined by

$$K(X, \xi_i) = \kappa + \mu g(hX, X) = \begin{cases} k + \lambda\mu & \text{if } X \in \mathcal{D}_+ \\ k - \lambda\mu & \text{if } X \in \mathcal{D}_- \end{cases} \tag{2.33}$$

$$K(X, Y) = \begin{cases} s(2(1 + \lambda) - \mu) & \text{if } X, Y \in \mathcal{D}_+ \\ s(2(1 - \lambda) - \mu) & \text{if } X, Y \in \mathcal{D}_- \\ -s(\kappa + \mu)(g(X, \varphi Y))^2 & \text{if } X \in \mathcal{D}_+, Y \in \mathcal{D}_- \end{cases} \tag{2.34}$$

where X, Y are orthonormal and in the first two cases of (2.34) n has to be strictly greater than 1.

Proof. Identities (2.33) follow directly from (1.11), while identities (2.34) are a consequence of (2.26), (2.30) and (2.28) respectively. ■

Corollary 2.1. *Let M be an almost \mathcal{S} -manifold verifying the (κ, μ) -nullity condition with $\kappa < 1$. Then the Ricci operator verifies the following identities*

$$Q = s\left[\left(2(1 - n) + \mu n\right)\varphi^2 + \left(2(n - 1) + \mu\right)h\right] + 2n\kappa \bar{\eta} \otimes \bar{\xi}, \tag{2.35}$$

$$Q \circ \varphi - \varphi \circ Q = 2s\left(2(n - 1) + \mu\right)h \circ \varphi. \tag{2.36}$$

Proof. Let $\{e_1, \dots, e_n, \varphi e_1, \dots, \varphi e_n, \xi_1, \dots, \xi_s\}$ be a local φ -basis such that $\{e_1, \dots, e_s\}$ is a basis of \mathcal{D}_+ and let $X = X_+ + X_- \in \mathcal{D}_+ \oplus \mathcal{D}_-$. From (2.26), (2.28) and (1.11) we get

$$QX_+ = s\left[\left(2(1 + \lambda) - \mu\right)(n - 1)X_+ - (\kappa + \mu)X_+ + \kappa X_+ + \mu hX_+\right]. \tag{2.37}$$

On the other hand from (2.29) and (2.30) we obtain

$$QX_- = s\left[-(\kappa + \mu)X_- + \left(2(1 - \lambda) - \mu\right)(n - 1)X_- + \kappa X_- + \mu hX_-\right]. \tag{2.38}$$

Taking (2.37), (2.38) and (2.10) into account we obtain (2.35). Finally, identity (2.36) easily follows from (2.35). ■

Corollary 2.2. *Let M be an almost \mathcal{S} -manifold verifying the (κ, μ) -nullity condition. Then the scalar curvature S of (M, g) is constant and verifies the identity*

$$S = 2ns\left(2(n - 1) - \mu n + \kappa\right). \tag{2.39}$$

Proof. Let $\{e_1, \dots, e_n, \varphi e_1, \dots, \varphi e_n, \xi_1, \dots, \xi_s\}$ be a local φ -basis such that $\{e_1, \dots, e_s\}$ is a basis of \mathcal{D}_+ . Then from (2.26), (2.28) and (1.11) we have

$$\begin{aligned} g(Qe_\beta, e_\beta) &= s\left(2(1 + \lambda) - \mu\right)(n - 1) + s(-\kappa - \mu) + s(\kappa + \lambda\mu) \\ &= s\left[(n - 1)\left(2(1 + \lambda) - \mu\right) + (\lambda - 1)\mu\right]. \end{aligned} \tag{2.40}$$

Furthermore from (2.29), (2.30) and (1.11) we get

$$g(Q\varphi e_\beta, \varphi e_\beta) = s\left[(n - 1)\left(2(1 - \lambda) - \mu\right) - (1 + \lambda)\mu\right]; \tag{2.41}$$

then (2.40), (2.41) and (2.10) yield (2.39). ■

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Dipartimento di Matematica,
Università di Bari
Via E. Orabona 4, 70125 Bari, Italy
cappelletti@dm.uniba.it,terlizzi@dm.uniba.it