Third order TVD scheme for hyperbolic conservation laws

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Abstract

A new third order finite difference scheme for the solution of initial value problems for hyperbolic conservation laws is presented. The advantages of the scheme are its simplicity, third order accuracy and that it can be used for large time steps which saves more time. The scheme is proved stable for initial and initial boundary value problems for linear case. The technique of making the third order scheme oscillations free (TVD) is carried out. In this paper we extend TVD scheme to two dimension problems. The extension of the TVD scheme to nonlinear system of equations is illustrated by solving shallow water equations. Numerical results are presented and compared with exact solutions and other methods.

1 Introduction

In recent years there has been a substantial and productive effort to develop accurate computational techniques for partial differential equations, particularly the case of hyperbolic conservation laws. We are concerned with improved high order methods for solving hyperbolic conservation laws. In this paper we construct a finite difference scheme combining the explicit and implicit schemes to get an explicit scheme of third order accuracy which has a larger stability region compared with other known explicit schemes. The advantages of the scheme are its simplicity, third order accuracy and that it can be used for large time steps which reduces the number of steps and saves more time. The scheme is proved stable for initial and initial boundary value problems. To show the efficiency of the scheme, comparisons with the second order Lax-Wendroff (L-W) scheme and third order scheme presented in [5] are

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carried out. From the numerical results it is noticed that the new scheme gives accurate approximations for small time steps as well as for large steps and so reduces the computing time while the L-W scheme and the third order scheme [5] give good approximations only for small time steps and so they take more time, which means that the given scheme is economic and efficient. However, the high order second order (or higher) accurate are only suitable for linear systems or nonlinear systems with smooth solutions. It is well known that in solving nonlinear or linear systems with discontinuities when applying second order schemes or higher it is inevitable that oscillations will be observed with vicinity of the discontinuities which might trigger instabilities. In general, the difference schemes that are second order (or higher) accurate produce oscillations behind the waves and near discontinuities. It is well known that the monotone schemes always do not introduce oscillations and converge to the physical solutions. However, monotone schemes are only first order accurate schemes and then produce rather crude approximations wherever the solution varies strongly in space and time. When using a second (or higher) order accurate scheme, this difficulty can be overcome by adding a hefty amount of numerical dissipation (anti-diffusive term) to the scheme. This process is not successful and efficient in practice as in theory. This problem has frustrated people for many years until concept and theory of total variation diminishing (TVD) schemes was introduced by Harten [2]. The main property of TVD schemes is that it can be high order accurate and oscillations free. In this paper the TVD theory is applied to the new third order scheme using the techniques presented in [5] and [9]. This third order TVD scheme can avoid spurious oscillations and preserves high order in the smooth parts. This achieved by imposing a TVD constraints via the introducing of flux limiter functions. The extension of the scheme to the two dimensional problem is illustrated in this paper. The extension of the third order to nonlinear hyperbolic conservation laws is validated by solving a test problem for shallow water equations. The new scheme has an additional advantage that it gives very accurate results for large time steps as for small ones while the accuracy of most explicit schemes decreases as the time step increases. The paper is organized as follows. In section 2, we construct the third order scheme. Stability properties are analyzed for initial and initial boundary value problems. In section 3, we reformulate the new scheme to make it TVD (oscillations free). In section 4, we extend the scheme to constant coefficients linear hyperbolic conservation laws. Section 5 discusses nonlinear systems typically the shallow water equations. In section 6 we extend the scheme to two dimensions case. Numerical tests on the linear hyperbolic equations, in one and two dimensions space, with different initial conditions are performed in section 7. In this section we apply the scheme to shallow water equations. Numerical results are presented and compared with exact solutions and other methods. Conclusions are drawn in section 8.

2 Construction of the scheme

The initial boundary value problem for a first order hyperbolic conservation law is considered, namely

$$u_t + f(u)_x = 0, \quad b \le x \le d, \quad t \ge 0,\ f'(b) > 0,\ f'(d) > 0 \tag{2.1a}$$

$$u(x,0) = u_o(x), \quad u(b,t) = g(t) \quad t > 0 \tag{2.1b}$$

Here u is the unknown function and $f(u)$ is the physical flux. First we consider the linear case $f(u) = au$, so that a is a constant wave propagation speed. A uniform $x-t$ grid, with mesh size h in the x-direction and k in the t-direction, is introduced with a fixed ratio $\lambda = k/h$. Let u_j^n denote the numerical solution obtained at $(x,t) = (jh, nk)$ for $j = 0,1,2,...,N$ and $n = 0,1,2,....$

To compute the approximate solution u_j^n, we suggest the following two step method :

$$\tilde{u}_j^{n+1} = u_j^n - \frac{c}{2}(u_{j+1}^n - u_{j-1}^n) + \frac{c^2}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) \tag{2.2a}$$

$$u_j^{n+1} = u_j^n - \frac{c}{4}(\tilde{u}_{j+1}^{n+1} + u_{j+1}^n - \tilde{u}_{j-1}^{n+1} - u_{j-1}^n) \tag{2.2b}$$

where $c = \lambda$ a is the Courant number.

The first step of this method is the L-W difference scheme and the second is Crank-Nicolson implicit scheme. The scheme (2.2) can be written in one step method as :

$$u_j^{n+1} = u_j^n - \frac{c}{2}\left[u_{j+1}^n - u_{j-1}^n\right] + \frac{c^2}{8}\left[u_{j+2}^n - 2\,u_j^n + u_{j-2}^n\right]$$
$$- \frac{c^3}{8}\left[u_{j+2}^n - 2\,u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n\right] \qquad j = 2,3,...,N-2, n = 0,1,2,... \tag{2.3}$$

It can be shown that a scheme (2.3) is third order accurate in space and second order in time, i.e., of order $(h^3 + k^2)$. The scheme is stable if and only if

$$|c| \le \sqrt{2} \tag{2.4}$$

So the stability region for the scheme (2.3) is greater than for the L-W scheme ($|c| \le 1$).

The difference scheme (2.3) uses a five point lattice and so it cannot be used at the two left most points of the mesh. We consider the following boundary condition

$$\begin{cases} u_1^n = 2u_2^n - u_3^n \\ u_0^n \text{ is given} \end{cases} \quad \text{for } 0 \le c \le \sqrt{2} \tag{2.5a}$$

$$\begin{cases} u_1^n = 2u_2^n - u_3^n \\ u_0^n = 3u_2^n - 2u_3^n \end{cases} \quad \text{for } -\sqrt{2} \le c \le 0 \tag{2.5b}$$

These boundary conditions may be written in the form

$$u_1^n = 2u_2^n - u_3^n \quad , u_{N-1}^n = 2u_{N-2}^n - u_{N-3}^n \quad u_N^n = 3u_{N-2}^n - 2u_{N-3}^n \quad u_0^n \text{ is given.}$$

We now investigate the stability of the method defined by (2.3) and (2.5) in the sense of GKS [1]. We use the stability definition 3.3 of Gustaffson et al [1]. In [1] it is established (Theorem 5.4) that the stability of two related quarter-plane problem is equivalent to stability for the two boundary (see definition 3.3 in [1]). The quarter-plane problems are simply obtained by removing one or the other of the boundaries and extending the domain appropriately to $\pm\infty$. Firstly, we analyze the propagation of the initial data by taking a single Fourier mode

$$u(x,t) = e^{i(\omega t - \zeta x)} \tag{2.6}$$

where ω is the frequency and ζ is the wave number. Equation (2.6) can be written in more convenient form

$$u_j^n = \kappa^j z^n$$

where $z = e^{i\omega k,}$, $\kappa = e^{-i\zeta h}$. If ω or ζ is real, then $|z| = 1$ or $|\kappa| = 1$ respectively, but ω and ζ may be complex.

2.1 GKS Stability Theorem

Models (2.3) and (2.5) are stable if and only if they admit no eigensolutions (κ) with $|z| \geq 1$. Firstly, for the right quarter problem (2.3) $b \leq x \leq \infty$, c >0. The characteristic equation associated with scheme (2.3) is given by

$$P_1(\kappa) = (\frac{c^2}{8} - \frac{c^3}{8})\,\kappa^4 + (\frac{2c^2}{8} - \frac{c}{2})\,\kappa^3 + (1 - z - \frac{2c^2}{8})\,\kappa^2 + (\frac{c}{2} - \frac{2c^3}{8})\,\kappa + (\frac{c^3}{8} + \frac{c^2}{8}) = 0 \tag{2.7}$$

Lemma 2.1 *Let z be a complex number,*
a) For $z = 1$, the roots of $P_1(\kappa)$ are given by

$$q_1 = 1,\; q_2 = -1, \quad q_3 = \frac{(2-c^2) - \sqrt{4-3c^2}}{c - c^2},\; q_4 = \frac{(2-c^2) + \sqrt{4-3c^2}}{c - c^2}$$

b) for $|z| \geq 1$, $z \neq 1$, the roots of $P_1(\kappa)$ split into two groups (independent of z) :

$$M = (\kappa_1, \kappa_2),\; S = (\kappa_3, \kappa_4),\; \text{where} \quad |\kappa_1(z)| \leq 1,$$
$$|\kappa_2(z)| \leq 1, \quad |\kappa_3(z)| \geq 1 \text{ and } |\kappa_3(z)| \geq 1$$

and the inequalities can be taken strictly if $|z| > 1$.
c) The roots κ_1, κ_2 of part (b) are analytic continuations of the following roots q_i (at $z = 1$)

$$\begin{array}{ll} M = (q_1, q_3), & 0 < c \leq 1 \\ M = (q_1, q_2), & 1 < c \leq \sqrt{2} \\ M = (q_3, q_4), & -\sqrt{2} < c \leq -1 \\ M = (q_2, q_3), & -1 < c \leq 0 \end{array}$$

Proof.
Part (a) is trivial.
Part (b) follows from [1].
c) From (b) we know that for any $|z| \geq 1,\ z \neq 1$ there are exactly two roots for which $|\kappa| \leq 1$ and these groups are independent of z. We first consider $c > 0$. It is easy to verify that $\kappa = 1$ cannot be a double root at $z = 1$. If we consider a perturbation $z = 1+\delta$ and if $\kappa = 1+\varepsilon$, then we find from (2.7) that $\varepsilon = -\frac{\delta}{c+2\delta} < 0$. Hence the continuation of q_1 is inside the unit circle for $\delta > 0$ and q_1 is always in M. To examine the continuation of the root $q_2 = -1$ we take $z = 1+\delta\ ,\ \ \kappa = -(1+\varepsilon)$ then we have $\varepsilon = -\frac{1}{2}\frac{\delta}{\delta+\frac{1}{2}c(c^2-1)}$, *i.e.*, $\varepsilon < 0 \quad if\, c > 1$ and $\varepsilon > 0$ if $c < 1$. Therefore the continuation of q_2 is inside the unit circle for $c > 1$ and outside for $c < 1$. For $c = 1$, $\kappa = -1$ is a double root for $z = 1$ and as $z > 1$ it has two continuation, one moving inside the unit circle and the other outside. Therefore $M = (q_1, q_2), \quad in \quad 1 < c \leq \sqrt{2}$. For $0 < c < 1$ we have that $|q_3| < 1\ |q_3| < 1$ and so the perturbations of q_3 are in M. Since only two roots are in M we have $(q_1, q_3) \in M$. For $c < 0$, the above perturbation argument shows that if $z = 1+\delta\ ,\ \ \kappa = -(1+\varepsilon)$ then κ is inside the unit circle for $\delta > 0$ and for $-1 < c < 0$. it is easy to show that the continuation of q_1 is outside the unit circle. By continuity of the roots we have that the perturbation of q_3 is inside the unit circle and so $M = (q_2, q_3)$, in $-1 < c \leq 0$. For $-\sqrt{2} < c < -1$ we can see that $|q_3| < 1\ |q_3| < 1$ and $|q_4| < 1$ and then their continuations are in M for z close to one and hence for all z. Then $M = (q_4, q_3)$, in $-\sqrt{2} < c < -1$.

Theorem 2.1. *For fixed c, the scheme (2.3) with the boundary formula (2.5) is stable in the sense of GKS theory.*

Proof. For $c > 0$ the general solution of the difference equation (2.3) is $\tilde{u}_j = \sigma_1 \kappa_1^j + \sigma_2 \kappa_2^j$ where $\kappa_1 \neq \kappa_2$ are the roots of the characteristic equation $P_1(\kappa) = 0$ i.e., $\tilde{u}_1 = \sigma_1 \kappa_1 + \sigma_2 \kappa_2$ and $\tilde{u}_0 = \sigma_1 + \sigma_2$. The characteristic equation associated with the boundary condition (2.5) is given by

$$Q_1(\kappa) = \kappa(\kappa - 1)^2 \tag{2.8}$$

For condition (2.5a) we have the boundary equations (see [4] for details)

$$\begin{aligned} \cdots\cdots\cdots\cdots\sigma_1 + \cdots\cdots\cdots\cdots\sigma_2 = 0 \\ \kappa_1(\kappa_1 - 1)^2\,\sigma_1 + \kappa_2(\kappa_2 - 1)^2\sigma_2 = 0 \end{aligned} \tag{2.9}$$

The GKS theorem may be written in another form (see [1]) : "The homogeneous equations (2.9) have no nontrivial solutions for $|z| \geq 1$ " i.e.,

$$\det D(z) \neq 0 \tag{2.10}$$

The system (2.9) has a nontrivial solution only if $\kappa_1 = \kappa_2$ and $\kappa_1 = \kappa_2 = 1$ or if

$$\kappa_1^2 + (\kappa_2 - 2)\kappa_1 + (\kappa_2 - 2)^2 = 0$$

the solution of this equation has the property $\kappa_1 + \kappa_2 = 2$. Hence κ_1, κ_2 can be inside the unit circle only if $\kappa_1 = \kappa_2 = 1$, but we know that, from the last lemma,

$\kappa = 1$ is not a double root of $P_1(\kappa)$. Hence no simple root can satisfy (2.10). For a double root, $\tilde{u}_j = (\sigma_1 + j\sigma_2)\kappa^j$ we have the system

$$\cdots\cdots\cdots\cdots\sigma_1 = 0$$
$$\kappa(\kappa-1)^2\,\sigma_1 + (\kappa - 4\kappa^2 - 3\kappa^3)\sigma_2 = 0$$

This has nontrivial solutions if $\kappa = 0$ or $\kappa = 1$ or $\kappa = 1/3$. As before $\kappa = 1$ is not a double root of $P_1(\kappa)$ and so does not satisfy (2.10) while $\kappa = 0$ is the trivial solution. It remains only to verify that $\kappa = 1/3$ is not double root. If we put $P(\frac{1}{3}) = P'(\frac{1}{3})$, we get c has a complex value therefore $\kappa = 1/3$ is not double root. Hence the scheme (2.3) with the boundary (2.5) is stable for $0 \le c \le \sqrt{2}$.

For $c < 0$, the characteristic equation associated with the boundary condition (2.5b) is given by

$$Q_2(\kappa) = \kappa(\kappa-1)^2$$
$$Q_3(\kappa) = (\kappa-1)^2(\kappa + \frac{1}{2})$$

and the boundary equations

$$\kappa_1(\kappa_1-1)^2\,\sigma_1 + \kappa_2(\kappa_2-1)^2\sigma_2 = 0$$
$$(\kappa_1-1)^2\,(\kappa_1+\frac{1}{2})^2\;\sigma_1 + (\kappa_2-1)^2\,(\kappa_2+\frac{1}{2})^2\;\sigma_2 = 0$$

This system has a nontrivial solution only if $\kappa_1 = \kappa_2$ and $\kappa_1 = \kappa_2 = 1$. We have, from the last lemma, $\kappa_1 = 1$ is not in M for $c < 0$. Hence no simple root can satisfy (2.10). For a double root, we have the system

$$\kappa(\kappa-1)^2\,\sigma_1 + \kappa(\kappa-1)(3\kappa-1)\sigma_2 = 0$$
$$2(\kappa-1)^2\,(\kappa+\tfrac{1}{2})^2\;\sigma_1 + 6\kappa^2(\kappa-1)\,\sigma_2 = 0$$

This has nontrivial solutions if κ =0 or κ =1 As before κ =1 is not in M while κ =0 is the trivial solution. Hence the scheme (2.3) with the boundary (2.5b) is stable for κ. Hence the scheme (2.3) with the boundary conditions (2.5) is stable for $|c| \le \sqrt{2}$.

2.2 Numerical Experiments

In this section we compute approximate solutions to the problem (2.1) with a=1, with initial data defined by

$$u(x,0) = \sin 4\pi x, \quad x \in [0,1]$$

and the boundary data

$$u(0,t) = -\sin 4\pi t$$

This problem has exact solution $u(x,t) = \sin 4\pi(x-t)$.

We consider the scheme (2.3) and for comparison we include the L-W scheme of order $(h^2 + k^2)$. This scheme is stable if $c \le 1$. We denote this scheme by L-W. We have performed the computations with $N = 100$, i.e., $h = 0.01$ and $c = 0.25$ and the results are given in the table 1. We take

$$\|e^n\|_2 = h \sum_{j=0}^{100} \left|e_j^n\right|^2, \quad \|e^n\|_\infty = \max_j \left|e_j^n\right| \qquad e_j^n = u(x_j, nk) - u_j^n$$

is the truncation error at the grid points. We note from table (1) that the results obtained by (2.3) is more accurate than L-W results.

To show the benefit of the method the solution with $h = 1/150$, $c = 0.75$ for the same time is computed. The results are shown in table (2). It is clear from table (2) that the approximation solutions due to the scheme (2.3) is more accurate while the truncation errors due to L-W scheme are very big specially the maximum norm of the error which means that L-W scheme is not efficient for large time step $(0.5 \leq c \leq 1)$. If we compare the results obtained by the scheme (2.3) in both tables 1 and 2 we note that the approximation in the second case ($h = 1/150, c = 0.75$ and $k = 1/200$) is more accurate than the first case ($h = 1/100, c = 0.25$ and $k = 1/400$) although the number of time steps in the second case is half of the steps in the first case,which means that the scheme (2.3) is both efficient and economic scheme.

To show the efficiency of the scheme (2.3), comparison with the third order scheme, presented by Toro [5], is carried out. Here Toro third order scheme will be referred to as Tor3. Table 3 shows the results with $h = 0.01$. For each scheme, we select a value of c that satisfies the linear stability condition i.e., for Toro scheme $c \leq 1$ (we take $c = 0.9$) and for the scheme (2.3) $c \leq \sqrt{2}$ (we take $c = 1.25$). From the table we note that our scheme (2.3) is more accurate, specially for long time, and less expensive because it enjoys less restriction ($c = 0.9$ versus $c = 1.25$).

It is noticed that the third order finite difference scheme for solving initial boundary value problems gives good approximations when applied to linear equations in sense of truncation errors. But if we compare the exact solutions and numerical solutions, obtained by the difference scheme, graphically we notice that the numerical solution introduces spurious oscillations behind the waves while the exact solution is monotone wave. As already pointed out, the third order scheme will produces spurious oscillations behind the waves and near discontinuities. In the next section we address the problem of making the scheme (2.3) oscillation free.

Time	Method (2.3) ($\|e^n\|_2$)	Method L-W ($\|e^n\|_2$)	Method (2.3) ($\|e^n\|_\infty$)	Method L-W ($\|e^n\|_\infty$)
t=1	7.62056E-3	1.653622E-2	1.57776E-2	3.310253E-2
t=3	8.85979E-3	1.655050E-2	1.61285E-2	3.225204E-2
t=5	8.78500E-3	1.653950E-2	1.61285E-2	3.237981E-2
t=7	8.78123E-3	1.652717E-2	1.60802E-2	3.249586E-2
t=9	8.77885E-3	1.650998E-2	1.60514E-2	3.261712E-2

Table (1)

Time	Method (2.3) ($\|e^n\|_2$)	Method L-W ($\|e^n\|_2$)	Method (2.3) ($\|e^n\|_\infty$)	Method L-W ($\|e^n\|_\infty$)
t=1	4.24641E-3	7.131677E-2	9.75762E-3	8.719269E-1
t=3	4.20536E-3	7.193365E-2	9.81669E-3	8.794971E-1
t=5	4.20257E-3	7.254852E-2	9.72865E-3	8.870425E-1
t=7	4.19668E-3	7.316180E-2	9.73618E-3	8.945689E-1
t=9	4.19540E-3	7.377314E-2	9.75553E-3	9.020690E-1

Table 2

Time	Method (2.3)	Tor3
$t = 1$	9.28270E-3	5.6598E-3
$t = 3$	9.25572E-3	5.2789E-3
$t = 5$	9.27388E-3	4.7499E-3
$t = 7$	9.26566E-3	9.8499E-3
$t = 9$	9.25965E-3	9.6034E-3

Table 3

3 A Third Order TVD Scheme

In this section we construct and reformulate the conservative third order scheme presented in the last section. Sufficient conditions for this scheme to be TVD are derived for linear case.

Consider the model hyperbolic conservative law

$$u_t + f(u)_x = 0, \quad f(u) = au \tag{3.1}$$

where a constant is propagation speed. The scheme (2.3) can be written in the conservative form

$$u_j^{n+1} = u_j^n - \lambda \left[F^n_{j+\frac{1}{2}} - F^n_{j-\frac{1}{2}} \right] \tag{3.2}$$

where $F_{j+\frac{1}{2}}$ is the inter-cell numerical flux function which can be written in the form [5]

$$F_{j+\frac{1}{2}} = \frac{1}{2}(au_j + au_{j+1}) - \frac{1}{2}|a|\ \Delta_{j+\frac{1}{2}}u + |a|\ \left\{ A_0 \Delta_{j+\frac{1}{2}}u + A_1 \Delta_{j+L+\frac{1}{2}}u + A_2 \Delta_{j+M+\frac{1}{2}}u \right\} \tag{3.3}$$

where

$$A_0 = \frac{1}{2} - \frac{|c|}{4}, \quad A_1 = -\frac{|c|}{8} - \frac{c^2}{8}, \quad A_2 = -\frac{|c|}{8} + \frac{c^2}{8} \tag{3.4}$$

$L = -1, M = 1$ for $c > 0$ and $L = 1, M = -1$ for $c < 0$.

Where $\Delta_{j+\frac{1}{2}}u = u_{j+1} - u_j$.

3.1 TVD version of the method

The total variation $TV(u_j^n)$ of the mesh function un is defined as

$$TV(u^n) = \sum_{-\infty}^{\infty} \left| u^n_{j+1} - u^n_j \right| = \sum_{-\infty}^{\infty} \left| \Delta_{j+\frac{1}{2}} u^n \right| \tag{3.5}$$

The numerical scheme (3.2) is said to be TVD scheme if

$$TV(u^{n+1}) \leq TV(u^n) \tag{3.6}$$

which simply states that the total variations not increased as time evolves, so that TV(u^n) at any time n is bounded by TV(u^0) of the initial data. In fact, if the initial data of equation (3.1) is smooth, then the total variation of the solution remains constant and when the shock is formed the total variation decreases. To apply the TVD concept, we use Hartens theorem [2], which states that a scheme written as

$$u_j^{n+1} = u_j^n - B_{j-\frac{1}{2}}\, \Delta_{j-\frac{1}{2}}u \; + C_{j+\frac{1}{2}}\, \Delta_{j+\frac{1}{2}}u \tag{3.7}$$

is TVD provided that

$$B_{j+\frac{1}{2}} \geq 0, \qquad C_{j+\frac{1}{2}} \geq 0 \qquad B_{j+\frac{1}{2}} + C_{j+\frac{1}{2}} \leq 1 \tag{3.8}$$

where $B_{j+\frac{1}{2}}$ and $C_{j+\frac{1}{2}}$ are data dependent coefficients i.e., functions of the set $\{u_j^n\}$. Imposing a TVD constraint on (3.3) via a flux limiter function gives

$$F_{j+\frac{1}{2}} = \frac{1}{2}\,(au_j + au_{j+1}) - \frac{1}{2}\,|a|\;\Delta_{j+\frac{1}{2}}u + |a|\;\left\{A_0\Delta_{j+\frac{1}{2}}u + A_1\Delta_{j+L+\frac{1}{2}}u\right\}\phi_j + \\ |a| \quad A_2\Delta_{j+M+\frac{1}{2}}u\phi_{j+M} \tag{3.9}$$

where ϕ_j and ϕ_{j+M} are flux limiter functions. Scheme (3.2) and (3.9) is TVD for $|c| \leq 1$ if the limiter function is determined by

$$\phi_j \leq \frac{(1-|c|)r_j}{\eta(A_1 r_j + A_0 - A_2)} \tag{3.10a}$$

$$\phi_j \leq \frac{1-|c| + \eta A_2/r_j^*}{\eta(A_1 r_j + A_0)} \tag{3.10b}$$

$$\phi_j \geq \frac{A_2}{(A_1 r_j + A_0)r_j^*} \tag{3.10c}$$

$$\phi_j \geq 0 \tag{3.10d}$$

where r_j is called the local flow parameter and is defined by

$$r_j = \frac{\Delta_{j+L+\frac{1}{2}}u}{\Delta_{j+\frac{1}{2}}u} \tag{3.11a}$$

and r_j^* is called the upwind-downward flow parameter and is given by

$$r_j^* = \frac{\Delta_{j+L+\frac{1}{2}}u}{\Delta_{j+M+\frac{1}{2}}u} \tag{3.11b}$$

and η is defined by

$$\eta = \begin{cases} 1-|c| & \text{for} \quad 0 \leq |c| < \frac{1}{2} \\ |c| & \text{for } \frac{1}{2} \leq |c| \leq 1 \end{cases} \tag{3.12}$$

Proof. see[5] and [9].

By applying the last theorem to the scheme (3.2), (3.9), the flux limiter can be defined as

$$\phi_j = \begin{cases} \dfrac{(1-|c|)r_j}{\eta(A_1 r_j + A_0 - A_2)} & \text{for } \ 0 \le r_j \le r^L \\ 1 & \text{for } r^L \le r_j \le r^R \\ \dfrac{1-|c|+\eta A_2/r_j^*}{\eta(A_1 r_j + A_0)} & \text{for } r_j > r^R \\ 0 & \text{for } \ \ r_j < 0 \end{cases} \tag{3.13}$$

where

$$r^L = \frac{\eta\,(A_0 - A_2)}{1-|c|-\eta\,A_1}, \qquad r^R = \frac{1-|c|-\eta\,(A_0 - A_2/r_j^*)}{\eta\,A_1}$$

Therefore the scheme (3.2), (3.9) becomes TVD.

4 Application On Linear Hyperbolic Systems

In this section we extend the scalar schemes (3.2)-(3.9) to solve the initial value problem for linear hyperbolic systems with constant coefficients

$$U_t + AU_x = 0, \qquad U(x,0) = U_0(x) \tag{4.1}$$

where U is a column vector of m conserved variables and A is an mxm constant matrix. This is a system of conservation laws with flux function $F(U) = AU$ which is hyperbolic if A is diagonalizable with real eigenvalues, i.e., the matrix A can be written as

$$A = R\Omega R^{-1} \tag{4.2}$$

Where $\Omega = diag(\lambda^{(1)}, ..., \lambda^{(m)})$ $\Omega = diag(\lambda^{(1)}, ..., \lambda^{(m)})$ is the diagonal matrix of eigenvalues of A and $R = (r^{(1)}, ..., r^{(m)})$ is the matrix of right eigenvectors of A. Equation (4.2) means $AR = R\Omega$, i.e.,

$$Ar^{(p)} = \lambda^{(p)} r^{(p)}, \qquad p = 1, 2, \ldots, m \tag{4.3}$$

The natural way to extend the scalar scheme to linear systems is obtained by defining expressions for the flux differences $\Delta_{j+\frac{1}{2}}F = A\Delta_{j+\frac{1}{2}}U$.

This can be done by diagonalizing the system, solving local Riemann problems with left and right states U_j^n and U_{j+1}^n, i.e.,

$$U(x,0) = \begin{cases} U_j^n, & x < 0 \\ U_{j+1}^n & x > 0 \end{cases} \tag{4.4}$$

and letting

$$\alpha_{j+\frac{1}{2}} = R^{-1}_{j+\frac{1}{2}}\ \Delta_{j+\frac{1}{2}}U \tag{4.5}$$

where $R_{j+\frac{1}{2}}$ is the matrix of right eigenvectors at the interface$(j+1/2)$, which for the linear constant coefficient case is of course constant; $\alpha_{j+\frac{1}{2}}$ is called the wave strength vector with components $\alpha^{(p)}_{j+\frac{1}{2}}$, $(p = 1, 2, \cdots, m)$ across the p-the wave travelling at speed $\lambda^{(p)}_{j+\frac{1}{2}}$ in the $(j + 1/2)$ intercell. Then we have

$$\Delta_{j+\frac{1}{2}} U = \sum_{p=1}^{m} \alpha^{(p)}_{j+\frac{1}{2}} \; r^{(p)}_{j+\frac{1}{2}} \tag{4.6}$$

Since $F(U) = AU$, this leads to

$$\Delta_{j+\frac{1}{2}} F = A\Delta_{j+\frac{1}{2}} U = \sum_{p=1}^{m} \alpha^{(p)}_{j+\frac{1}{2}} A \; r^{(p)}_{j+\frac{1}{2}} = \sum_{p=1}^{m} \alpha^{(p)}_{j+\frac{1}{2}} \; \lambda^{(p)}_{j+\frac{1}{2}} \; r^{(p)}_{j+\frac{1}{2}} \tag{4.7}$$

Note that the single jump $\Delta_{j+q+\frac{1}{2}} F = \left| a_{j+q+\frac{1}{2}} \right| \Delta_{j+q+\frac{1}{2}} U$ in the scalar scheme (3.2) with the appropriate interpretation for $\left| a_{j+\frac{1}{2}} \right|$ is now substituted by a summation of jump (4.7), which gives a natural extension to linear systems with constant coefficients.

5 Nonlinear Hyperbolic Systems

Let the nonlinear system of equations

$$U_t + F(U)_x = 0 \tag{5.1}$$

where $F(U)$ is a vector flux such that $A(U) = \frac{\partial F}{\partial U}$ is the Jacobian matrix. A possible strategy for solving systems of nonlinear equations is to linearize the nonlinear system of equations (5.1) locally at each cell interface by an approximate the Jacobian matrix $A(U)$ and then implement the method of the last section using the linearized system

$$U_t + \bar{A} U_x = 0 \tag{5.2}$$

where $\bar{A}$ is a linearized constant matrix depending only on the local data U^n_j and U^n_{j+1}, i.e., $\bar{A} = \bar{A}\ (\ U^n_j,\ U^n_{j+1})$. Popular example of this approach is Roe's approximation [3]. Roe's matrix $\bar{A}\ (\ U^n_j,\ U^n_{j+1})$ is assumed to satisfy the following properties :

(i) $\bar{A}\Delta_{j+\frac{1}{2}} U = \Delta_{j+\frac{1}{2}} F$;
(ii) $\bar{A}$ is diagonalizable with real eigenvalues ;
(iii) $A \to F'(\bar{U})$ smoothly as $U^n_j,\ U^n_{j+1} \to \bar{U}$.

Denoting the Roe eigenvalues, eigenvectors and wave strengths as $\bar{\lambda}^{(p)}_{j+\frac{1}{2}}$, $\bar{r}^{(p)}_{j+\frac{1}{2}}$, $\bar{\alpha}^{(p)}_{j+\frac{1}{2}}$ $(p = 1, 2, \cdots, m)$ respectively, then applying the third order scheme of the last section, we solve the original nonlinear systems in a straightforward manner.

5.1 Shallow water equations

The one- dimensional shallow water equations as a typical nonlinear system of conservation laws, represents the motion of a free surface flow in a channel, take the form [8]

$$U_t + F(U)_x = 0 \tag{5.3a}$$

where

$$U = (S, Su)^T, F(U) = \left(Su, Su^2 + S^2\right)^T \tag{5.3b}$$

where S is the cross section of the flow, u velocity. With initial conditions

$$U(x, t_0) = \begin{cases} U_L & x < x_0 \\ U_R & x > x_0 \end{cases} \tag{5.3c}$$

The equations (5.3) can be written in the form

$$U_t + A(U)U_x = 0, A(U) = \frac{\partial F}{\partial U} \tag{5.4}$$

where $A(U)$ is the Jacobian matrix such that

$$A = \begin{pmatrix} 0 & 1 \\ 2S - u^2 & 2u \end{pmatrix} \tag{5.5}$$

The system (5.3)-(5.5) is hyperbolic with eigen values

$$\lambda^{(1)} = u - C, \quad \lambda^{(2)} = u + C \tag{5.6a}$$

where C= $\sqrt{2S}$ denotes the sound speed. The corresponding right eigenvectors of the Jacobian A are found to be

$$r^{(1)} = (1, u - C)^T, \quad r^{(2)} = (1, u + C)^T \, r^{(1)} = (1, u - C)^T, \quad r^{(2)} = (1, u + C)^T \tag{5.6b}$$

5.2 Linearization of shallow water equations

The nonlinear system of equations (5.3) can be linearized as

$$U_t + \bar{A}U_x = 0 \tag{5.7}$$

where $\bar{A}$ is an approximate Jacobian matrix of A with eigenvalues $\bar{\lambda}^{(p)}$ and eigenvectors $\bar{r}^{(p)}$ such that

$$\bar{A}\,\Delta U = \Delta F \tag{5.8}$$

The approximate matrix Jacobian $\bar{A}$ satisfying (5.8) can be written as

$$\bar{A}(U_j, U_{j+1}) = \begin{pmatrix} 0 & 1 \\ \bar{C}^2 - \bar{u}^2 & 2\bar{u} \end{pmatrix} \tag{5.9a}$$

where $\bar{C}$ and $\bar{u}$ are given by [3]

$$\bar{C} = \sqrt{S_j + S_{j+1}} \tag{5.9b}$$

$$\bar{u} = \frac{u_{j+1}\sqrt{S_{j+1}} + u_j\sqrt{S_j}}{\sqrt{S_{j+1}} + \sqrt{S_j}} \tag{5.9c}$$

The eigenvalues and eigenvectors of the linearized matrix $\bar{A}$ are

$$\bar{\lambda}^{(1)} = \bar{u} - \bar{C}, \quad \bar{\lambda}^{(2)} = \bar{u} + \bar{C}, \bar{r}^{(1)} = \left(1, \bar{u} - \bar{C}\right)^T, \quad \bar{r}^{(2)} = \left(1, \bar{u} + \bar{C}\right)^T \tag{5.10}$$

The wave strengths are

$$\bar{\alpha}^1 = 0.5\,\Delta S + \frac{1}{2C}(\bar{u}\,\Delta S - \Delta Su), \quad \bar{\alpha}^2 = 0.5\,\Delta S - \frac{1}{2C}(\bar{u}\,\Delta S - \Delta Su) \tag{5.11}$$

where $\Delta pq = \bar{p}\Delta q + \bar{q}\Delta p$. Here $\bar{S} = \sqrt{S_j S_{j+1}}$.

6 Extension to two dimensional conservation laws

We now consider the scalar two dimensional linear equation

$$u_t + f(u)_x + g(u)_y = 0, \tag{6.1}$$

where $f(u) = a, g(u) = b$. Here a and b are constant wave speeds in the x and y directions respectively. We consider only regular, Cartesian grids with spacing Δx *and* Δy. We can define Courant numbers in the x and y directions as $C_x = a\,(\frac{\Delta t}{\Delta x})$ and $C_x = b\,(\frac{\Delta t}{\Delta y})$.

We are interested in un-split explicit finite difference scheme in the form

$$u^{n+1}_{j,i} = u^n_{j,i} - \frac{\Delta t}{\Delta x}\left[f^n_{j+\frac{1}{2},i} - f^n_{j-\frac{1}{2},i}\right] - \frac{\Delta t}{\Delta y}\left[g^n_{j,i+\frac{1}{2}} - g^n_{j,i-\frac{1}{2}}\right] \tag{6.2}$$

where

$$f_{j+\frac{1}{2},i} = \frac{1}{2}\left(au_{j,i} + au_{j+1,i}\right) - \frac{1}{2}|a|\ \Delta_{j+\frac{1}{2},i}u + |a|\ \left\{A_0\Delta_{j+\frac{1}{2},i}u + A_1\Delta_{j+L+\frac{1}{2},i}u\right\}\ \varphi_{j,i} + |a|\ \ A_2\Delta_{j+M+\frac{1}{2},i}u\ \phi_{j+M,i}$$

$$g_{j,i+\frac{1}{2}} = \frac{1}{2}\left(bu_{j,i} + au_{j,i+1}\right) - \frac{1}{2}|b|\ \Delta_{j,i+\frac{1}{2}}u + |b|\ \left\{A_0\Delta_{j,i+\frac{1}{2}}u + A_1\Delta_{j,i+L+\frac{1}{2}}u\right\}\ \psi_{j,i} + |b|\ A_2\Delta_{j,i+M+\frac{1}{2}}u\ \psi_{j,i+M} \tag{6.3}$$

here $\Delta_{j+\frac{1}{2},i}u = u_{j+1,i} - u_{j,i}$ and $\Delta_{j,i+\frac{1}{2}}u = u_{j,i+1} - u_{j,i}$

The limiter function (3.14) takes the form

$$\phi_{j,i} = \begin{cases} \dfrac{(1-|c_x|)\theta_{j,i}}{\eta(A_1\theta_{j,i} + A_0 - A_2)} & \text{for } 0 \le \theta_{j,i} \le \theta^L \\ 1 & \text{for } \theta^L \le \theta_{j,i} \le \theta^R \\ \dfrac{1-|c_x| + \eta A_2\phi_{j+M,i}/\theta^*_j}{\eta(A_1\theta_{j,i} + A_0)} & \text{for } \theta_{j,i} > \theta^R \\ 0 & \text{for } \quad \theta_{j,i} < 0 \end{cases} \tag{6.4a}$$

$$\phi_{j+M,i} = \begin{cases} \eta\theta_{j+M,i} & \text{for } 0 \le \theta_{j+M,i} < 0.5 \\ 1 & \text{for } \theta_{j+M,i} > 0.5 \\ 0 & \text{for } \theta_{j,i} = 0 \end{cases} \tag{6.4b}$$

where $\theta^L = \dfrac{\eta\ (A_0 - A_2)}{1 - |c_x| - \eta\ A_1}$, $\qquad \theta^R = \dfrac{1 - |c_x| - \eta\ (A_0 - A_2\phi_{j+M,i}/\theta^*_{j,i})}{\eta\ A_1}$

where $\theta_{j,i}$ is called the local flow parameter and is defined by

$$\theta_{j,i} = \frac{\Delta_{j+L+\frac{1}{2},i}u}{\Delta_{j+\frac{1}{2},i}u} \tag{6.4c}$$

and $\theta^*_{j,i}$ is called the upwind-downward flow parameter and is given by

$$\theta^*_j = \frac{\Delta_{j+L+\frac{1}{2},i}u}{\Delta_{j+M+\frac{1}{2},i}u} \tag{6.4d}$$

similarly we define ψ_j as

$$\psi_{j,i} = \begin{cases} \dfrac{(1-|c_y|)r_{j,i}}{\eta(A_1 r_{j,i} + A_0 - A_2)} & \text{for } 0 \le r_{j,i} \le r^L \\ 1 & \text{for } r^L \le r_{j,i} \le r^R \\ \dfrac{1-|c_y| + \eta A_2 \phi_{j,i+M}/r^*_{j,i}}{\eta(A_1 r_{j,i} + A_0)} & \text{for } r_{j,i} > r^R \\ 0 & \text{for } \quad r_{j,i} < 0 \end{cases} \tag{6.5a}$$

$$\psi_{j,i+M} = \begin{cases} \eta r_{j,i+M} & \text{for } 0 \le r_{j,i+M} < 0.5 \\ 1 & \text{for } r_{j,i+M} > 0.5 \\ 0 & \text{for } r_{j,i} = 0 \end{cases} \tag{6.5b}$$

$$r^L = \frac{\eta\,(A_0 - A_2)}{1-|c_y| - \eta\,A_1}, \qquad r^R = \frac{1-|c_y| - \eta\,(A_0 - A_2\phi_{j,i+M}/r^*_{j,i})}{\eta\,A_1}$$

where r_j is called the local flow parameter and is defined by

$$r_{j,i} = \frac{\Delta_{j,i+L+\frac{1}{2}}u}{\Delta_{j,i+\frac{1}{2}}u} \tag{6.5c}$$

and $r^*_{j,i}$ is called the upwind-downward flow parameter and is given by

$$r^*_j = \frac{\Delta_{j,i+L+\frac{1}{2}}u}{\Delta_{j,i+M+\frac{1}{2}}u} \tag{6.5d}$$

7 Numerical Experiments

In this section we give numerical results on the computation of the solutions of some test problems.

7.1 Scalar problems

Consider the scalar equation

$$u_t + u_x = 0, \qquad -\infty < x < \infty, \qquad t \ge 0 \tag{7.1a}$$

$$u(x,0) = g(x) \tag{7.1b}$$

We will approximate equation (7.1) with different initial conditions :

I. wave problem Consider equation (7.1) where

$$g(x) = \begin{cases} 0, & 0 \le x \le 1 \\ \sin[8\pi(x-1)] & 1 \le x \le 2 \\ 0 & 2 \le x \le \infty \end{cases} \tag{7.2}$$

The exact solution of this problem is $u(x,t) = g(x-t)$, i.e., the wave propagates to the right with speed $c = 1$. Using $h = 0.01$ and for large Courant number $c = 0.9$, the numerical solution is displayed at very long time $t = 8$. Figures (1a) and (1b) show the results obtained by the third order Toro TVD scheme and the third order TVD scheme (3.2) with (3.9) respectively. The results from figure 1a shows that Toro scheme is not very satisfactory for long time while the scheme (3.2) gives very good approximations for long time. In these figures the numerical solutions are shown in symbols and the exact solution in full lines.

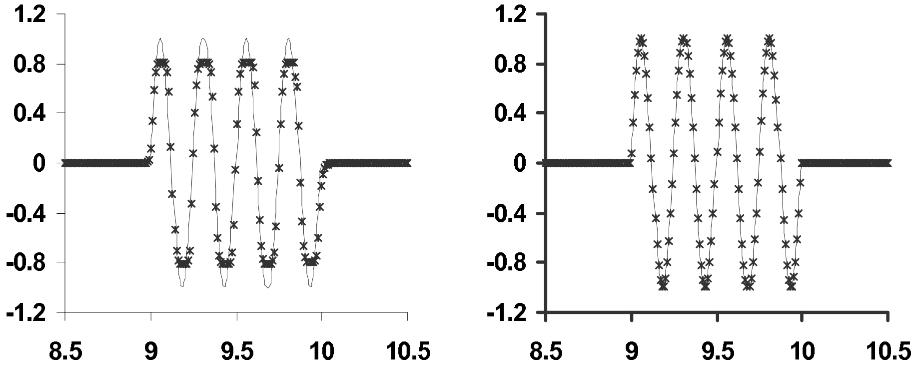

Figure 1. Solution of the equation (7.1) with (7.2) (a) Toro scheme (b) scheme (3.2) with (3.9)

II. Riemann problem Here we solve the equation (7.1) with the initial condition

$$g(x) = \begin{cases} 1 & \frac{1}{3} \leq x \leq \frac{2}{3} \\ 0 & \text{elsewhere} \end{cases}. \tag{7.3}$$

For this problem we have moving discontinuities. The numerical solution is computed at $t = 0.2$, with $h = 0.01$ and $c = 0.9$. Figures 2a,b show the results obtained by the third order Toro TVD scheme and the third order TVD scheme (3.2) with (3.9) respectively. We notice that, from figure (2a), the discontinuities are smeared with four interior points while the third order scheme (3.2) with (3.9) reproduce the exact solutions.

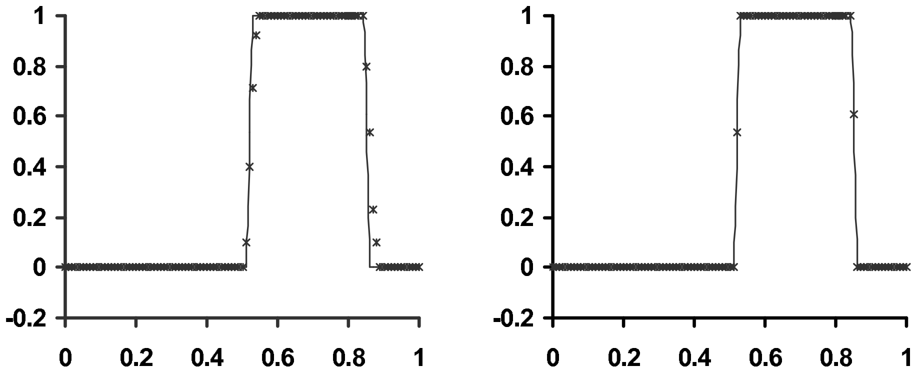

Figure 2. Solution of the equation (7.1) with (7.3) (a) Toro scheme (b) scheme (3.2) with (3.9)

7.2 Nonlinear Shallow Water Equations

Here we discuss numerical test results of the solution of the RP (5.3) with initial data [8]

$$U_L = (0.597,\ 0)^T, \qquad U_R = (0.04166,\ 0)^T \tag{7.4}$$

The numerical experiments were performed using the linearized system (5.8)-(5.11) and the scheme (3.2),(3.9) with limiter (3.31). Figure 3 show the exact solution in full lines for the cross section $S(x,t)$ and the velocity $u(x,t)$ together with the numerical solution, shown in symbols. We take $\Delta x = 0.01$ and the Courant number used is 0.9. Note that the results show good approximation in smooth parts and the discontinuities are absolutely sharp and their positions are exact.

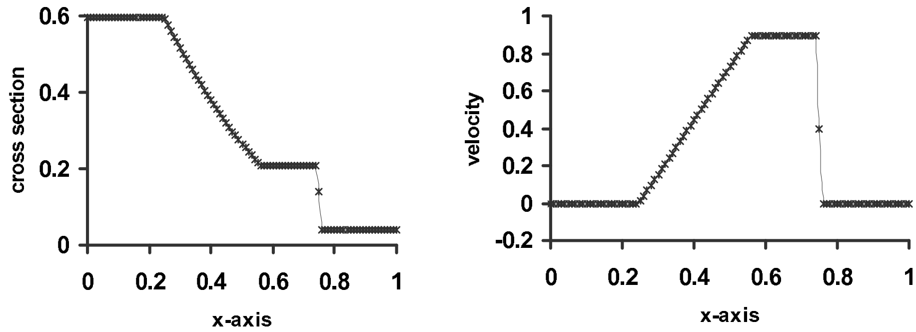

Figure 3. Solution of shallow water equations using the scheme (3.2) with (3.9) using the linearization method

7.3 Two dimension problems

We solve the two dimension linear equation (6.1) with $a = b = 1$. To demonstrate the efficiency of the method, we consider the linear rotation of square patch on $[0,1] \times [0,1]$, with initial condition

$$u(x,y,0) = \begin{cases} 1, & |x-0.5| \le 0.5, \quad |y-0.5| \le 0.5 \\ 0, & \text{otherwise} \end{cases} \tag{7.5}$$

we show the solution after rotation of $\frac{\pi}{4}\ and \frac{\pi}{2}$ at $t = 0.5$ and $t = 1$ in figure 4 with a regular mesh 40×40 cells and Courant number $c = 0.9$. The numerical solutions are free of oscillations.

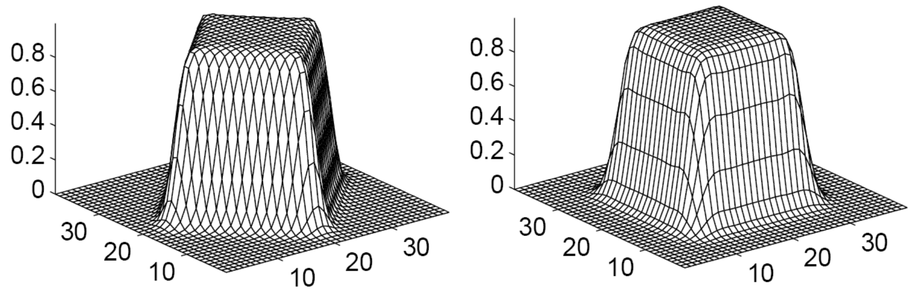

Figure 4. Solution of the equation (6.1) with (7.5) at $T = 0.5$ (left) and $T = 1.0$ (right)

8 Conclusions

A third order finite difference scheme for computing solutions to hyperbolic conservation laws has been established. It has two important features: simplicity and robustness. An oscillations-free version of the scheme is constructed by use of flux limiters. Applications of the scheme to scalar equations and nonlinear systems in one dimension give results that compare very well with those obtained by existing high resolution methods. Application to two dimensions is illustrated by treatment of a simple model problem.

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