Relaxing stratification

Thomas Forster Olivier Esser

Abstract

A number of ways of relaxing the stratification constraint for the axioms of Quine's NF are reviewed. It is shown how most of them result in inconsistency

Introduction

NF is a odd system, but it is not odd in the way people think. The oddness lies not in the apparently purely syntactic nature of the insight that underlies it, for that insight is less perverse and less syntactic than one might suppose. NF is not so much oddly *conceived* as oddly *fragile*. It is very striking that every known weakening of it results in a system with a relatively simple consistency proof, and on the other hand almost any weakening of the syntactic chains that Keep Chaos At Bay is swiftly punished with inconsistency. (This is not to say that there are no natural strengthenings of NF that appear to be consistent: there are. The point is that these strengthenings do not arise from any relaxation of syntactic constraints in the comprehension scheme.) If stratification works at all—and it might—then it is very finely balanced on a knife edge.

Mathematicians on their first exposure to NF often spontaneously wonder if it might be safe to take some risks with the syntactic constraint they see exploited in NF, perhaps because—it being less natural at first blush than it becomes on mature aquaintance—one is initially more inclined to question its value as a source of axioms than one is to question the value of the cumulative hierarchy as such a source.

The authors of this little survey thought that it might be useful to collect in one place the various liberties that people have from time thought about taking with the syntactic restraints that seem to keep *NF* safe. Most of them have been shown to fail, and although most of these failures can be made to happen fairly quickly if one

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knows how to overload the machinery, beginners—even mature beginners—cannot be expected to find these counterexamples swiftly. Accordingly there is merit to be gained—and time and effort to be saved—by collecting them all into one place.

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Definitions and Summary

We discuss the obvious relaxations of the rules, and discuss in some detail a well-motivated and attractive but ultimately doomed idea of the second author that there might somehow be a kind of inhomogeneous equality relation.

One *aperçu* we will be making repeated use of is the fact that no set theory can survive having the collection of all *genuine* (seen-from-outside) wellorderings as a set. In ZF style theories all such "universal" collections are easily shown to be proper classes anyway, so there is no particular significance to the class of genuine wellorderings above and beyond the class of common-or-garden wellorderings. However in *NF* the class of common-or-garden wellorderings *is* a set; indeed in NF the extension of any stratified predicate is a set, so any relaxation of the syntactic constraints that would enable us to give a stratified definition of genuine wellordering will be fatal. It is striking how many of these proofs turn on this one feature. Perhaps this hides a moral. If it does, then it is a moral that was first pointed in 1940 by Rosser, who noticed that if one relaxes the device of stratification in NF-with-classes to allow bound class variables to appear in stratified formulæ in set existence axioms then the collection of genuine wellorderings is a set.

Before we embark on the details a word is in order on the proof sketches to be found in the body of the paper. We are considering a programme of incorporating stratification into logics which extend ordinary first-order logic and claiming that this gives rise to inconsistency. Clearly proofs of some sort have to be provided: one cannot just say "this enables one to define genuine wellorderings and thus prove the Burali-Forti paradox". However, not all of these logics have satisfactory proof systems available in which one could provide rigorous proof objects. The compromise we have reached, in order not to try the readers' or our own patience unduly, is to provide proof sketches which should blossom into proper proofs when placed in the context of a proper proof system. In some cases we have felt obliged to supply more than one proof to allay suspicions.

The usual definition of 'stratified' is developed as follows.

Let ϕ be a formula on the language $\mathcal{L} : (\in, =)$ of set theory. A stratification of ϕ is a map s from the set of variables of ϕ to the integers such that if x and y are two variables of ϕ and $x = y$ is a subformula of ϕ, then $s(x) = s(y)$ and if $x \in y$ is a subformula of ϕ, then $s(x) + 1 = s(y)$. A formula ϕ is stratified if there is a stratification of ϕ. If ϕ is a formula and if there is a map s such as before but defined only on the bound variables of ϕ, the formula is said to be weakly stratified. We still call such an s a stratification but we precise that it is defined on the bound variables of ϕ.

An axiom of comprehension is an axiom saying that for all tuples $\vec{x}$ and any weakly stratified ϕ the collection $\{y : \phi(y, \vec{x})\}$ is a set.

1 An apercu of Holmes

Tn is the natural number $|\{\{y\} : y \in x\}|$,where $x \in n$, with the effect that if σ is a stratification then $\sigma(`Tn') = \sigma(`n') + 1$. The result is that '$n = Tn$' is unstratified and the assertion '$(\forall n \in \mathbb{N})(n = Tn)$' cannot be proved by induction and would have to be added if we use to exploit it. This is Rosser's Axiom of Counting. NF + the Axiom of Counting is NFC

Randall Holmes has remarked to us that although one usually thinks of the Axiom of Counting as a flagrantly unstratified—albeit natural—assertion, one can in fact relax the definition of 'stratified' in such a way that it becomes stratified, with the effect that if one expresses NF as before—extensionality plus weakly stratified comprehension—one obtains precisely NFC. The idea is as follows. In any formula ϕ a **natural number variable** is a bound variable, 'x' say, whose binding quantifier is restricted to the Natural Numbers: $\forall x \in \mathbb{N}$ or $\exists x \in \mathbb{N}$. If ϕ can be turned into a stratified formula by prefixing 'T's to some occurrences of some of the number variables in ϕ then a stratification of ϕ in the new relaxed sense is a stratification defined on the bound variables of ϕ other than the natural number variables.

Thus although the axiom of counting does not arise from a relaxation of the stratification discipline it can be (mis)represented as arising in that way. Holmes makes the point that the same move can be used to assert that any other definable set is strongly cantorian.

2 The Burali-Forti Paradox in the First Edition of Quine's ML

In [4] Quine thought to extend NF the way ZF is extended to obtain NGB, namely by addition of proper classes. In the ZF case this is a useful manœuvre, since it enables one to replace an infinite axiom scheme (replacement) by finitely many axioms of class existence and the single axiom "The image of a set in a function is a class".

In NF there is no infinite scheme waiting to be finitised in this way (the set existence scheme of NF is finitisable even without the use of classes) but there is no harm in adding classes: none, that is, if we add them properly. In the first edition of [4] the set existence scheme of NF is modified to allow bound class variables into instances of the comprehension scheme, but the stratification constraint remains. This means that as well as being able to define common-or-garden wellorders (as usual) as total orders all of whose subsets have least members, one is also able to define wellorderings-seen-from-outside as total orders all of whose sub*classes* have least members. With hindsight, one is surprised at how long it took for people to realise what had gone wrong. It is at least in part by reflection on this little episode that the authors were led to the conclusion that the parallel dangers in other relaxations of the stratification discipline might be as initially mysterious as that one was, and that by spelling them out we might be doing a service.

3 Cumulative stratification

Suppose we were to require of a stratification only that if there is a subformula '$x \in y$' then the type of 'y' must be *greater* than that of 'x', and doesn't have to be greater by precisely *one*? Then

$$(\exists y_2)(\forall z_0)((z_0 \in x_1 \longleftrightarrow z_0 \in y_2) \wedge x_1 \notin y_2)$$

is stratified in this new weak sense, but says $x \notin x$. This would give us Russell's paradox.

4 Typing mod n

In the theory of Types Amb^n is the scheme that says that types repeat themselves every n applications of power set. The ambiguity scheme is just Amb^1. We know that when $m|n$ then $Amb^n \subseteq Amb^m$. Now the usual apparatus with type shifting automorphisms and general model-theoretic nonsense will accept a model of Amb^n and return a "circular" model of type theory. One in which, for every natural number k, type $k+n$ is just the same as type k. Now what holds in such models, if there are any? AC fails, as we know that Amb^n refutes choice, but there is no reason to suppose that this theory is inconsistent—if we are careful.

One might think that this is a model for a kind of typed set theory where the types are integers mod n. That is to say every variable has a type subscript that is an integer mod n and whenever '$x_i \in y_j$' occurs in a formula then $j = i+1$ (mod n). The axioms are are extensionality and comprehension for well-typed formulæ.

However, if this were the case, one would be able to form, at any type k, the set $A_k = \{y_k : \neg(\exists x_{k+1}, x_{k+2} \ldots x_{k-1})(y_k \in x_{k+1} \in \ldots x_{k-1} \in y_k)\}$, namely the set of things at level k that do not belong to an n-cycle. This of course gives us a version of the standard (if obscure) n-ary version of Russell's paradox. The feature peculiar to this typed setting is that A_k has to exist at each type k. Then we reason as follows: A_k cannot belong to an n-cycle, for if it did, one of its members would belong to an n-cycle, which they don't. So, for all k, $A_k \in A_{k+1}$. So the A_k form an n-cycle. Contradiction.

This is not to say that there is no consistent typed set theory of this kind. What it means is that the notion of typing it uses is more restrictive than the one we have just considered, and is the same as the notion of typing in negative type theory.

If we think about the case $k = 1$ then it becomes obvious: *every* set-theoretic formula is well-typed if our types are allowed to be integers mod 1!

5 Infinitary languages

5.1 L_{ω_1,ω_1}

There is a stratified L_{ω_1,ω_1} formula

$$(\forall x_0 \ldots \forall x_n \ldots)(\neg(\bigwedge_{n\in\mathbb{N}} x_{n+1} \in x_n))$$

that says that x_0 is wellfounded. If the collection of wellfounded sets is a set then we have Mirimanoff's paradox.

5.1.1 Formulæ of L_{ω_1,ω_1} which use only finitely many types

In this language we can still define genuine wellorderings.

$$(\forall x_0 \ldots \forall x_n \ldots)(\neg(\bigwedge_{n \in \mathbb{N}} x_{n+1} < x_n))$$

5.2 $L_{\omega_1 \omega}$

Consider $\{T^n\Omega : n \in \mathbb{N}\}$. This collection has a stratified definition in $L_{\omega_1\omega}$. But then it is a set of ordinals with no least member.

5.2.1 Formulæ of $L_{\omega_1,\omega}$ which use only finitely many types

The first thing to notice is that the previous section's example, $\{T^n\Omega : n \in \mathbb{N}\}$, cannot be used in this case, since it uses infinitely many types. At this stage we know of no proof that this liberalisation results in an inconsistency. However it does give us a strong system.

The alert reader might expect that in $L_{\omega_1,\omega}$ one might be able to exploit the infinite family of approximants of [1] to branching quantifier formulæ and obtain thereby any contradiction obtaining by exploiting the branching-quantifier language. However, the contradictions obtained thereby all rely on our being able to define genuine wellorderings. Since there doesn't seem to be any way of exploiting the set-theoretic machinery here available, one returns to the fact that wellordering cannot be defined in $L_{\omega_1,\omega}$ and that therefore one should not expect this relaxation to fail—at least not on those grounds alone.

We will prove that the models of NF + comprehension for stratified formulæ of the language $L_{\omega_1,\omega}$ using only finitely many types are exactly the models for which the set $\mathbb{N}$ (the internal natural numbers) and $\mathcal{P}(\mathbb{N})$ (the powerset of the natural numbers) are the real external ones. We assume the following comprehension scheme: For any formula φ of the language $L_{\omega_1,\omega}$, using only finitely many types, we say that:

$$(\forall a_1 \ldots \forall a_n)(\exists u)(\forall t)\ (t \in u \longleftrightarrow \varphi)$$

where the free variables of φ are among $a_1, \ldots, a_n$. Notice that we consider only a finite number of $a_1, \ldots, a_n$; without this requirement, it would be easy to show that $\{\Omega, T\Omega \ldots T^n\Omega \ldots\}$ can be defined with two types and infinitely many parameters. It is easy to prove that the condition is necessary; we will prove that the condition is sufficient. The idea will be to give a truth definition for formulæ of fixed types and to use this truth definition to express arbitrary formulæ of the language $L_{\omega_1,\omega}$ as finite usual formulæ. We will need our hypothesis in order to represent formulæ of $L_{\omega_1,\omega}$ that use only finitely many types inside the theory.

Let n and $t_1, \ldots, t_{n+1}$ be fixed concrete natural numbers. In our proof, we will use the notion of an n-ary-function $f(x_1, \ldots, x_n)$, for which the formula $f(x_1, \ldots x_n) = y$ is stratified by a function s such that $s(x_1) = t_1, \ldots, s(x_n) = t_n$ and $s(y) = t_{n+1}$. It is clear that such functions can be defined in NF.

Using the fact that the set of natural numbers and the powerset of the set of natural numbers are the real ones, one can represent a formula φ by an element $\ulcorner\varphi\urcorner$

of our model. Let $n, t_1, \ldots, t_n$ be fixed natural numbers, We will consider an $(n+1)$-ary-function $\mathsf{T}_{t_1,\ldots,t_n}$, for which the formula $\mathsf{T}_{t_1,\ldots,t_n}(p, a_1, \ldots, a_n) = y$ is stratified by a function s for which $s(p) = 0$, $s(a_1) = t_1$, $\ldots$, $s(a_n) = t_n$ and having the following property. For a formula $\varphi(a_1, \ldots, a_n)$, whose free variables are among $a_1, \ldots, a_n$ and stratified by a function s such that $s(a_1) = t_1, \ \ldots, \ s(a_n) = t_n$, the following holds:

$$\mathsf{T}_{t_1,\ldots,t_n}(\ulcorner\varphi\urcorner, a_1, \ldots, a_n) = 1 \ \text{ iff } \ \varphi(a_1, \ldots, a_n)$$

Notice that the formula φ may have fewer than n free variables since it is always possible to add dummy variables. The function $\mathsf{T}_{t_1,\ldots,t_n}$ is defined by induction in the usual way. Clearly $\mathsf{T}_{t_1,\ldots,t_n}$ is defined by a finite stratified formula. The only thing that we have to check is that this definition is stratified: this is why we have had to fix $t_1, \ldots, t_n$.

With these functions $T_{t_1,\ldots,t_n}$, it is possible to show that our model satisfies the comprehension scheme for formulæ of $L_{\omega_1,\omega}$, by replacing each formula by a finite formula. Indeed, consider a formula φ whose free variables are among $a_1, \ldots, a_n$ and stratified by a function s with $s(a_1) = t_1, \ \ldots, \ s(a_n) = t_n$; we have

$$(\forall a_1 \ldots \forall a_n)(\exists u)(\forall t) \ (t \in u \longleftrightarrow \varphi)$$

$$\text{iff}$$

$$(\forall a_1 \ldots \forall a_n)(\exists u)(\forall t) \ (t \in u \longleftrightarrow \mathsf{T}_{t_1,\ldots,t_n}(\ulcorner\varphi\urcorner, a_1, \ldots, a_n) = 1)$$

6 Branching quantifiers

Allowing the incorporation of branching quantifiers into stratified formulæ will lead to Burali-Forti. The proof is hard. In fact—given the cute and easily established fact which we are about to show—it is surprisingly hard.

It is standard that the following formula says that A and B are the same size

$$\begin{pmatrix} \forall x \in A & \exists y \in B \\ \forall y' \in B & \exists x' \in A \end{pmatrix} (y = y' \longleftrightarrow x = x') \tag{1}$$

Let us write this as $A \sim B$. Its significance for us is that we have immediately that $A \sim \iota``A$ (the set of singletons of A). Surely, one thinks, a proof that every set is cantorian should be just round the corner, and with it a proof of Cantor's paradox.

Sadly it seems not. The idea is good, but one has to try this machinery with ordinals not cardinals. After all, if we are in a countable model, then all infinite sets of the model are the same size seen from outside. But not all wellorderings are the same length!

If $\langle A, \leq_A \rangle$ is a totally ordered set then

$$\begin{pmatrix} \forall n \in \mathbb{N} & \exists x \in A \\ \forall m \in \mathbb{N} & \exists y \in A \end{pmatrix} (n < m \longleftrightarrow y <_A x) \tag{2}$$

is a stratified formula which says that $<_A$ has a descending ω-sequence (seen from outside). Indeed, we easily see that, in formula 2, we have $y = x$ if we choose

$n = m$ (for if $n = m$, $x \not< y$ and $y \not< x$). From this we infer that there is an external function $x \mapsto x_n$ from $\mathbb{N}$ to A such that $x_m < x_n$ whenever $n < m$.

Henceforth let $\langle A, \leq_A \rangle$ be an internal wellordering. We will show that it is a genuine (external) wellordering as well.

$$\begin{pmatrix} \forall n \in \mathbb{N} & \exists x \in A \\ \forall m \in \mathbb{N} & \exists y \in A \\ \forall n' \in \mathbb{N} & \exists x' \in A \\ \forall m' \in \mathbb{N} & \exists y' \in A \end{pmatrix} \bigwedge \begin{cases} (n < m \longleftrightarrow y <_A x) \\ (n' < m' \longleftrightarrow y' <_A x') \\ (m + 1 = m' \to y' = y) \end{cases} \tag{3}$$

Expression (3) has a quantifier prefix of four rows. Rows three and four say that there is a descending ω-chain, and that it is the tail of the descending ω-chain given by rows one and two. Clearly (3) follows from (2)

Now let us assume that $\langle A, \leq_A \rangle$ was an internal wellordering but not an external wellordering, and let a be the $<_A$-minimal element of A such that

$$\begin{pmatrix} \forall n \in \mathbb{N} & \exists x \in A|a \\ \forall m \in \mathbb{N} & \exists y \in A|a \end{pmatrix} (n < m \longleftrightarrow y <_A x) \tag{4}$$

where $A|a$ is $\{u \in A : u \leq a\}$. Such an a exists because (4) is stratified. But then we have

$$\begin{pmatrix} \forall n \in \mathbb{N} & \exists x \in A|a \\ \forall m \in \mathbb{N} & \exists y \in A|a \\ \forall n' \in \mathbb{N} & \exists x' \in A|a \\ \forall m' \in \mathbb{N} & \exists y' \in A|a \end{pmatrix} \bigwedge \begin{cases} (n < m \longleftrightarrow y <_A x) \\ (n' < m' \longleftrightarrow y' <_A x') \\ (m + 1 = m' \to y' = y) \end{cases} \tag{5}$$

If we now instantiate 'n' to 1 in (5) we see that x must be a, by minimality of a. But then rows three and four of the quantifier prefix tell us there is a descending ω-chain consisting entirely of things below a, contradicting minimality of a.

6.1 Another proof

We will first prove that if $\langle A, \leq_A \rangle$ and $\langle B, \leq_B \rangle$ are two (internal) well-orderings then there can be at most one external isomorphism between them. Our point of departure is formula 6 which says there is an external isomorphism between $\langle A, \leq_A \rangle$ and $\langle B, \leq_B \rangle$.

$$\begin{pmatrix} \forall x \in A & \exists y \in B \\ \forall y' \in B & \exists x' \in A \end{pmatrix} (x \leq_A x' \longleftrightarrow y \leq_B y') \tag{6}$$

This gives rise to

$$\begin{pmatrix} \forall x_1 \in A.\ \exists y_1 \in B \\ \forall y_1' \in B.\ \exists x_1' \in A \\ \forall x_2 \in A.\ \exists y_2 \in B \\ \forall y_2' \in B.\ \exists x_2' \in A \end{pmatrix} \bigwedge \begin{cases} (x_1 \leq_A x_1' \leftrightarrow y_1 \leq_B y_1') \\ (x_2 \leq_A x_2' \leftrightarrow y_2 \leq_B y_2') \\ (x_1 = x_2 = u \to y_1 \neq y_2) \end{cases} \tag{7}$$

(7) is a stratified formula, with 'u' free, that says that there are two external isomorphisms between $\langle A, \leq_A \rangle$ and $\langle B, \leq_B \rangle$ that differ on u. Since it is stratified, and $\langle A, \leq_A \rangle$ is a wellordering, the set of u in A satisfying (7) will have a least

member. Notice that without loss of generality we could have replaced the last line by by '$x_1 = x_2 = u \to (y_1 <_B y_2)$'.

The next formula needs some explanation! Like (7) it is a stratified formula, with 'u' free, and it says that there are two external isomorphisms between $\langle A, \leq_A\rangle$ and $\langle B, \leq_B\rangle$ that differ on u, *but agree below* u. The reader will notice that lines four to eight of the quantifier prefix are duplicates of lines one-to-four of the prefix (and lines three-and-four of the matrix analogously duplicate lines one-and-two). The duplicated lines are copied bodily from formula (7), and the duplicated variables are upper-case copies of the lower-case originals. This duplication is necessary because the branching-quantifier language affords us no variables (as it might be 'F' and 'G') to range over the second order objects captured by the branching quantifiers which would enable us to say '$F = G$'. We have to say—*twice*—that there are two external isomorphisms, and by writing down equations between the old variables and their duplicates we say that the two isomorphisms announced *second* are the same as the two isomorphisms announced *first* (this is what lines five and six are doing). Finally we use the first announcements to say that the two isomorphisms differ at u and use the second to say that they agree below u.

$$\left(\begin{array}{l} \forall x_1 \in A.\ \exists y_1 \in B \\ \forall y_1' \in B.\ \exists x_1' \in A \\ \forall x_2 \in A.\ \exists y_2 \in B \\ \forall y_2' \in B.\ \exists x_2' \in A \\ \forall X_1 \in A.\ \exists Y_1 \in B \\ \forall Y_1' \in B.\ \exists X_1' \in A \\ \forall X_2 \in A.\ \exists Y_2 \in B \\ \forall Y_2' \in B.\ \exists X_2' \in A \end{array}\right) \bigwedge \left\{\begin{array}{l} (x_1 \leq_A x_1' \leftrightarrow y_1 \leq_B y_1') \\ (x_2 \leq_A x_2' \leftrightarrow y_2 \leq_B y_2') \\ (X_1 \leq_A X_1' \leftrightarrow Y_1 \leq_B Y_1') \\ (X_2 \leq_A X_2' \leftrightarrow Y_2 \leq_B Y_2') \\ (x_1 = X_1 \leftrightarrow y_1 = Y_1) \\ (x_2 = X_2 \leftrightarrow y_2 = Y_2) \\ (x_1 = x_2 = u \to (y_1 <_B y_2) \\ (X_1 = X_2 <_A u \to Y_1 = Y_2) \end{array}\right. \tag{8}$$

Let us instantiate 'u' for the lower-case variables in (8) ranging over members of A. This gives

$$\left(\begin{array}{l} \quad \exists y_1 \in B \\ \forall y_1' \in B.\ \exists x_1' \in A \\ \quad \exists y_2 \in B \\ \forall y_2' \in B.\ \exists x_2' \in A \\ \forall X_1 \in A.\ \exists Y_1 \in B \\ \forall Y_1' \in B.\ \exists X_1' \in A \\ \forall X_2 \in A.\ \exists Y_2 \in B \\ \forall Y_2' \in B.\ \exists X_2' \in A \end{array}\right) \bigwedge \left\{\begin{array}{l} (u \leq_A x_1' \leftrightarrow y_1 \leq_B y_1') \\ (u \leq_A x_2' \leftrightarrow y_2 \leq_B y_2') \\ (X_1 \leq_A X_1' \leftrightarrow Y_1 \leq_B Y_1') \\ (X_2 \leq_A X_2' \leftrightarrow Y_2 \leq_B Y_2') \\ (u = X_1 \leftrightarrow y_1 = Y_1) \\ (u = X_2 \leftrightarrow y_2 = Y_2) \\ (u = u = u \to (y_1 <_B y_2) \\ (X_1 = X_2 <_A u \to Y_1 = Y_2) \end{array}\right. \tag{9}$$

This simplifies to

$$\left(\begin{array}{l} \quad \exists y_1 \in B \\ \forall y_1' \in B.\ \exists x_1' \in A \\ \quad \exists y_2 \in B \\ \forall y_2' \in B.\ \exists x_2' \in A \\ \forall X_1 \in A.\ \exists Y_1 \in B \\ \forall Y_1' \in B.\ \exists X_1' \in A \\ \forall X_2 \in A.\ \exists Y_2 \in B \\ \forall Y_2' \in B.\ \exists X_2' \in A \end{array}\right) \bigwedge \left\{\begin{array}{l} (u \leq_A x_1' \leftrightarrow y_1 \leq_B y_1') \\ (u \leq_A x_2' \leftrightarrow y_2 \leq_B y_2') \\ (X_1 \leq_A X_1' \leftrightarrow Y_1 \leq_B Y_1') \\ (X_2 \leq_A X_2' \leftrightarrow Y_2 \leq_B Y_2') \\ (u = X_1 \leftrightarrow y_1 = Y_1) \\ (u = X_2 \leftrightarrow y_2 = Y_2) \\ (y_1 <_B y_2) \\ (X_1 = X_2 <_A u \to Y_1 = Y_2) \end{array}\right. \tag{10}$$

We notice that in the first four lines of this last formula the $\leq_A$ and $\leq_B$ can be replaced by $=$. Using this we deduce:

$$\begin{pmatrix} \exists y_1 \in B \\ \forall y_1' \in B.\ \exists x_1' \in A \\ \exists y_2 \in B \\ \forall y_2' \in B.\ \exists x_2' \in A \\ \forall X_1 \in A.\ \exists Y_1 \in B \\ \forall Y_1' \in B.\ \exists X_1' \in A \\ \forall X_2 \in A.\ \exists Y_2 \in B \\ \forall Y_2' \in B.\ \exists X_2' \in A \end{pmatrix} \bigwedge \begin{cases} (u \leq_A x_1' \leftrightarrow y_1 \leq_B y_1') \\ (u \leq_A x_2' \leftrightarrow y_2 \leq_B y_2') \\ (X_1 \leq_A X_1' \leftrightarrow Y_1 \leq_B Y_1') \\ (X_2 \leq_A X_2' \leftrightarrow Y_2 \leq_B Y_2') \\ (u = X_1 \leftrightarrow y_1 = Y_1) \\ (u = X_2 \leftrightarrow y_2 = Y_2) \\ (y_1 <_B y_2) \\ (X_1 = X_2 <_A u \to Y_1 = Y_2) \\ (Y_2' = y_2 \leftrightarrow X_2' = u) \end{cases} \tag{11}$$

Let us fix an $X_2 \in A$ and take $X_1 = X_2$. We see that if $X_2 \leq_A u$ then $Y_2 \neq y_1$. If $X_2 \geq_A u$ then $Y_2 \geq_B y_2 >_B y_1$. This shows that formula (11) implies that $Y_2 \neq y_1$, contradicting the fact that we have isomorphisms.

We have now established that if there is an external isomorphism between $\langle A, \leq_A \rangle$ and $\langle B, \leq_B \rangle$ then there is precisely one. Therefore the following formula, which says that F is the union of all graphs of external isomorphisms between $\langle A, \leq_A \rangle$ and $\langle B, \leq_B \rangle$, says that F is the graph of the unique external isomorphism between $\langle A, \leq_A \rangle$ and $\langle B, \leq_B \rangle$. Since the formula is stratified, F actually exists.

$$F = \left\{ \langle u, v \rangle \;\middle|\; \begin{pmatrix} \forall x \in A\ \exists y \in B \\ \forall y' \in B.\ \exists x' \in A \end{pmatrix} (x \leq_A x' \leftrightarrow y \leq_B y') \wedge (u = x \to y = v) \right\} \tag{12}$$

So we can conclude that if there is an external isomorphism between $\langle A, \leq_A \rangle$ and $\langle B, \leq_B \rangle$ then its graph is a set. But the ordinals are externally isomorphic to the ordinals below Ω, and cannot be internally isomorphic.

7 Least-fixed point Logics[1]

Fixed-point logics can be seen historically as arising from an attempt to prove a completeness theorem. In an ideal world the following ought to be true: We are thinking about finite structures only (otherwise the concepts of polynomial and exponential time make no sense). It ought to be the case that first-order = polytime and second-order = exptime. However, this doesn't work, at least not straightforwardly. One can tell in polynomial time whether or not a graph is connected, but there is no first-order expression ϕ such that a finite graph is connected iff it is a model of ϕ. There is a polytime algorithm for testing whether or not a group is simple, even though being a simple group is emphatically not a first-order property. The counterexamples to the identification of polytime with first-order all seem to be of the one flavour: polytime things that are not first-order rather than the other way round. This suggests that in order to get the missing completeness theorem we need a more expressive first-order syntax. One way of getting more formulæ is to go to **fixed point logic**.

[1] see Dawar-Gurevich [2]

In this logic one can obtain the effect of a predicate modifier of *transitive closure* of a binary relation in a manner that is in a clear sense first-order, so the logic is genuinely richer than ordinary predicate calculus. However if we add $\in$ to the language (which we will have to if we are to do NF) we can also characterise the unary predicate $WF(x)$ (meaning that x is well-founded) by saying it is the least fixed point for the function taking the predicate $W(x)$ to the predicate $(\forall y \in x)(W(y))$. That is to say one defines $WF(z)$ to be $\text{lfp}_{W,x}[\forall y \in x.W(y)](z)$

The "lfp" syntax is as follows. 'W' is a second-order variable, ranging over predicates. If $W(y)$ is a predicate, so too is $\forall y \in x.W(y)$. $\text{lfp}_{W,x}[\forall y \in x.W(y)](z)$ is the least fixed point for the operation taking the first predicate to the second. This is obviously inconsistent, since the collection of wellfounded sets cannot be a set.

This particular application of lfp-logic is inhomogeneous, in that the free variables in the two formulæ '$W(u)$' and '$(\forall y \in x)(W(x))$' are at different types, and that is evidently the root of the trouble.

If we restrict use of this syntax to cases where the free variables are of the same type we find that the least fixed points can be shown to exist in NF anyway. This is because in those circumstances we are considering a monotone homogeneous operation F taking sets to sets and this must have a fixed point by the Tarski-Knaster theorem, provable in NF.

The fixed point logic we have just considered has a weaker version. Add to first-order logic a predicate modifier of **transitive closure**. If we add to a language containing a stratified but inhomogeneous predicate (such as $\in$) we have to decide how to assign types to formulæ containing the transitive closure. How are we to type '$x \in^* y$'? The obvious solution is to allow it to be stratified as long as x is given a lower type than y. But if we do this, it becomes possible to say in a stratified way that x is a transitive set. This implies that the collection of transitive sets is a set, contradicting an old result of Forster [3]. A predicate modifier of transitive closure of *homogeneous* predicates is definable in NF anyway.

8 Inhomogeneous equality

In the course of preparing a paper on permutations methods in NF (still in preparation) the authors toyed at one point with the idea of adding to the language of NF countably many binary relations: $=_n$, one for each integer number n (where n can be positive or negative), which would behave like equality relations and for which '$x =_n y$' is stratified with 'y' n types higher than 'x' for $n \geq 0$ or 'x' $-n$ types higher than 'y' for $n \leq 0$. The motivation can be seen from the chapter on permutation methods in [3].

The motivation comes from topologies on permutation models. Let us fix a model V of NF. It is natural to say that for any formula (maybe stratified) $\phi(a_1, \ldots, a_n)$ and for any parameters $(a_1, \ldots, a_n)$, the following set is open

$$\{\sigma \ : \ V^\sigma \models \phi(a_1, \ldots, a_n)\}$$

If we consider only stratified ϕ, we have just, by a lemma of Henson (see [3]:

$$\{\sigma \ : \ V^\sigma \models \phi(a_1, \ldots, a_n)\} = \{\sigma \ : \ V \models \phi(H_{k_1}(\sigma, a_1), \ldots, H_{k_n}(\sigma, a_{k_n}))\}$$

where the k_i ($i = 1, \ldots n$) are the levels of stratification of $a_1, \ldots, a_n$ in ϕ. Here (and in our forthcoming paper on permutation methods in NF) we write $H_k(\sigma, x)$ where in [3] (and Henson's original paper) we would have written $\sigma_k(x)$.

In order to study this topology in more depth, it would be interesting to know whether or not, given $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_n)$ and $k_1, \ldots, k_n$, there is a permutation σ with:

$$H_{k_1}(\sigma, a_1) = b_1 \wedge \cdots \wedge H_{k_n}(\sigma, a_n) = b_n$$

If $k_1 = \cdots = k_n$ (which corresponds to the case where all a_i have the same level of stratification in ϕ) this is easy: there will be such a permutation iff the tuples $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ satisfy the same equalities, in the sense that $a_i = a_j$ iff $b_i = b_j$.

Moreover this last fact can be expressed by a stratified formula. In the general case where the k_i are not all equal, the answer would be true if $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ satisfies the same "equality relations" but where "$a_{k_i} = a_{k_j}$" is stratified where a_{k_i} is of level k_i and a_{k_j} is of level k_j. At this point, we thought that all will be all right just by adding some supplementary "inhomogeneous equality relations" to NF and to put suitable axioms on these.

Let us describe more precisely these "equality relations":

The language is $(\in, =_n)$, (one symbol $=_n$ for each concrete integer number n). We will work here in the first order theory without equality. Consider the following sheme:

1. "Extensionality": $(\forall xy)(x =_n y \leftrightarrow (\forall z \in x)(\exists w \in y)(z =_n w) \wedge (\forall z \in y)(\exists w \in x)(w =_n z))$

2. "Substitution scheme": From $x =_n y$ and $\phi(y, z_1 \ldots z_k)$ infer $(\exists w_1 \ldots w_k)(w_1 =_n z_1 \ldots w_k =_n z_k \wedge \phi(x, w_1 \ldots w_n))$

We add the comprehension scheme for stratified formulas:

3. Comprehension scheme: If $\phi(x, y_1, \ldots, y_n)$ is stratified then

$$(\forall y_1, \ldots, \forall y_n)\ \exists u\ \forall t\ t \in u \iff \phi(t, y_1, \ldots, y_n)$$

The idea is that $=_0$ should play the rôle of the "real" equality. Notice however that the axioms we have considered for $=_0$ are much weaker than the usual axioms for the equality relation: for example, *prima facie* there appears to be no way to infer $x =_0 x$.

It would have been natural to add some more axioms for these equality relations in order to ensure that they look more like genuine equality. Here we have retained only the minimal axioms needed to derive the paradox.

Consider the formula $\phi(y)$:

$$(\exists y')(\ y' =_1 y \wedge y' \notin y)$$

and consider the formula $\psi(x)$ with one free variable x which is

$$\forall y.\ y \in x \leftrightarrow \phi(y)$$

the formula $\exists y'\ y' =_1 y \wedge y' \notin y$ being stratified the comprehension scheme say that there is at least one x with $\psi(x)$. Pick such an x and call it R. We have:

$$\exists R'\ R' =_1 R \tag{13}$$

To prove (13) take a formula with two free variables $\phi(x, y)$ such that $\phi(x, y)$ is true for any value of x and y (for example take $x =_0 y \vee x \neq_0 y$). Take a set z satisfying $\forall x.\ x \in z \to \perp$ and call this set $\emptyset$. We have $\phi(\emptyset, R)$. By the extensionality axiom for $=_1$, we have $\emptyset =_1 \emptyset$ and the substitution scheme tells us now that $\exists R'.\ R' =_1 R$ such that $\phi(\emptyset, R')$ which shows (13).

Now we derive Russell's paradox: suppose that $R \in R$, from the definition of R we infer $\exists R'\ R' =_1 R \wedge R' \notin R$. This implies $\neg\phi(R')$ (by the definition of R). By the substitution scheme we have $\neg\phi(R)$ and this means $R \notin R$ (by the definition of R).

Conversely, suppose that we have $R \notin R$. So $(\forall R'.\ R' =_1 R \to R' \in R)$. Take R' with $R' =_1 R$ which exists by (13). As $R' \in R$, we have $\phi(R')$ and also $\phi(R)$ by the substitution scheme and thus $R \in R$.

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Thomas Forster
DPMMS, CMS
Wilberforce road
CB3OWB United Kingdom
e-mail: T.Forster@dpmms.cam.ac.uk

Olivier Esser
Université Libre de Bruxelles
Service de logique C.P.211
Boulevard du Triomphe
B-1050 Bruxelles
Belgium
e-mail: oesser@ulb.ac.be