Weyl's theorem for Algebraically class A Operators

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Abstract

Let A be a bounded linear operator acting on a Hilbert space H. In [32], A. Uchiyama proved that Weyl's theorem holds for class A operators with the additional condition that $\ker A|_{[TH]} = 0$ and he showed that every class A operator whose Weyl spectrum equals to zero is compact and normal. In this paper we show that Weyl's theorem holds for algebraically class A operator without the additional condition $\ker A|_{[TH]} = 0$. This leads as to show that a class A operator whose Weyl spectrum equals to zero is always compact and normal.

1 Introduction

Let $B(H)$ and $K(H)$ denote, respectively, the algebra of bounded linear operators and the ideal of compact operators acting on infinite dimensionel separable Hilbert space H. If $A \in B(H)$ we shall write $N(A)$ and $R(T)$ for the null space and the range of A, respectively. Also, let $\alpha(A) := dimN(A)$, $\beta(A) := dimN(A^*)$, and let $\sigma(A)$, $\sigma_a(A)$ and $\pi_0(A)$ denote the spectrum, approximate point spectrum and point spectrum of A, respectively. An operator $A \in B(H)$ is called Fredholm if it has closed range, finite dimensional null space, and its range has finite co-dimension. The index of a Fredholm operator is given by

$$I(A) := \alpha(A) - \beta(A).$$

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A is called Weyl if it is of index zero, and Browder if it is Fredholm of finite ascent and descent, equivalently ([19], Theorem 7.9.3) if A is Fredholm and $A-\lambda$ is invertible for sufficiently small $|\lambda| > 0$, $\lambda \in \mathbb{C}$. The essential spectrum $\sigma_e(A)$, the Weyl spectrum $\sigma_w(A)$ and the Browder spectrum $\sigma_b(A)$ of A are defined by [18, 19]

$$\sigma_e(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is not Fredholm}\},$$

$$\sigma_w(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is not Weyl}\},$$

$$\sigma_b(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is not Browder}\},$$

respectively. Evidently

$$\sigma_e(A) \subseteq \sigma_w(A) \subseteq \sigma_b(A) = \sigma_e(A) \cup acc\sigma(A),$$

where we write $accK$ for the accumulation points of $K \subseteq \mathbb{C}$. If we write $isoK = K \setminus accK$, then we let

$$\pi_{00}(A) := \{\lambda \in iso\sigma A : 0 < \alpha(A-\lambda) < \infty\}.$$

We say that Weyl's theorem holds for A if

$$\sigma(A) \setminus \sigma_w(A) = \pi_{00}(A).$$

For any operator A in $B(H)$ set, as usual, $|A| = (A^*A)^{\frac{1}{2}}$ and $[A^*, A] = A^*A - AA^* = |\ A\ |^2 - |\ A^*\ |^2$ (the self commutator of A), and consider the following standard definitions: A is normal if $A^*A = AA^*$, hyponormal if $A^*A - AA^* \geq 0$, p-hyponormal if $(|A|^{2p} - |A^*|^{2p}) \geq 0$.

A is said to be log-hyponormal if A is invertible and satisfies the following equality

$$log(A^*A) \geq log(AA^*).$$

It is known that invertible p-hyponormal operators are log-hyponormal operators but the converse is not true [30]. However it is very interesting that we may regards log-hyponormal operators as 0-hyponormal operators [30, 29]. The idea of log-hyponormal operator is due to Ando [2] and the first paper in which log-hyponormality appeared is [15]. See [1, 30, 29, 31] for properties of log-hyponormal operators. We say that an operator $A \in B(H)$ belongs to the class A if $|A^2| \geq |A|^2$. Class A was first introduced by Furuta-Ito-Yamazaki [16] as a subclass of paranormal operators which includes the classes of p-hyponormal and log-hyponormal operators. The following Theorem is one of the results associated with class A operator.

Theorem 1.1. *[16] Every log-hyponormal operator is a class A operator.*

In [33], H. Weyl proved that Weyl's theorem holds for hermitian operators. Weyl's theorem has been extended from hermitian operators to hyponormal and Toeplitz operators [7], and to several classes of operators including semi-normal operators ([4, 5]). Recently W.Y.Lee [23] showed that Weyl's theorem holds for algebraically hyponormal operators. In [11] the authors showed that Weyl's theorem holds for algebraically p-hyponormal operators. A.Uchiyama [32] extended this result to a class A operator with the additional condition $\ker A|_{[TH]} = 0$. In this paper we show that Uchiyama's results remains holds without additional condition. Stamplfli [27] proved that if A is hyponormal and $\sigma_w(A) = 0$, Then A is compact and normal. In this paper we extend Stampfli's result to a class A operator.

2 Main results

Let $r(A)$ and $W(A)$ denote the spectral radius and the numerical range of A, respectively. It is well known that $r(A) \leq ||A||$ and that $W(A)$ is convex with convex hull $conv\sigma(A) \subseteq \overline{W(A)}$. A is said convexoid if $conv\sigma(A) = \overline{W(A)}$.

Lemma 2.1. *Let A be a class A operator and $\lambda \in \mathbb{C}$. If $\sigma(A) = \{\lambda\}$, then $A = \lambda$.*

Proof. We consider two cases:

Case 1 ($\lambda = 0$). Since A is class A operator, A is normaloid [1]. Therefore $A = 0$.

Case 2 ($\lambda \neq 0$). Here A is invertible, and since A is a class A operator, A^{-1} is also a class A operator [31]. Therefore A^{-1} is normaloid. On the other hand, $\sigma(A^{-1}) = \{\frac{1}{\lambda}\}$. Hence $||A||||A^{-1}|| = |\lambda||\frac{1}{\lambda}| = 1$. It follows from ([24], Lemma 3) that A is convexoid. Hence $W(A) = \{\lambda\}$ and $A = \lambda$. ■

The following lemma is well known.

Lemma 2.2. *[32] Let $A \in B(H)$ be class A operator. If $\lambda \in \sigma_p(A) - \{0\}$, then $\lambda \in \sigma_p(A^*)$).*

We say that A is algebraically class A operator if there exists a nonconstant complex polynomial p such that $p(A)$ is a class A operator.

Lemma 2.3. *Let A be a quasinilpotent algebraically class A operator. Then A is nilpotent.*

Proof. Assume that $p(A)$ is class A operator for some nonconstant polynomial p. Since $\sigma(p(A)) = p(\sigma(A))$, the operator $p(A) - p(0)$ is quasinilpotent. Thus Lemma 2.1 would imply that

$$cA^m(A - \lambda_1)...(A - \lambda_n) \equiv p(A) - p(0) = 0,$$

where $m \geq 1$. Since $A - \lambda_i$ is invertible for every $\lambda \neq 0$, we must have $A^m = 0$. ■

Lemma 2.4. *Let A be algebraically class A operator. Then A is isoloid.*

Proof. Let $\lambda \in iso\sigma(A)$ and let

$$P := \frac{1}{2\pi i} \int_{\partial D} (\mu - T)^{-1} d\mu$$

be the associated Riesz idempotent, where D is a closed disk centered at λ which contains no other points of $\sigma(A)$. We can then represent A as the direct sum

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \text{ where } \sigma(\mathrm{A}_1) = \{\lambda\} \text{ and } \sigma(\mathrm{A}_2) = \sigma(\mathrm{A}) \setminus \{\lambda\}.$$

Since A is algebraically class A operator, $p(A)$ is a class A operator for some nonconstant polynomial p. Since $\sigma(A_1) = \lambda$, we must have

$$\sigma(p(A_1)) = p(\sigma(A_1)) = \{p(\lambda)\}.$$

Therefore $p(A_1) - p(\lambda)$ is quasinilpotent. Since $p(A_1)$ is a class A operator, it follows from lemma 2.2 that $p(A_1) - p(\lambda) = 0$. Put $q(z) := p(z) - p(\lambda)$. Then $q(A_1) = 0$, so A_1 is algebraically class A operator. Since $A_1 - \lambda$ is quasinilpotent and algebraically class A operator, it follows from Lemma 2.3 that $A_1 - \lambda$ is nilpotent. Therefore $\lambda \in \pi_0(A_1)$, and hence $\lambda \in \pi_0(A)$. This shows that A is isoloid. ■

Lemma 2.5. *Let $A \in B(H)$ be class A operator. Then A has SVEP.*

Proof. If A is class A operator, then $|A|^2 \leq |A^2|$. By the Schwartz inequality,

$$||Ax||^2 = (|A|^2x;x) \leq (|A^2|;x) \leq |||A^2|x||x|| = ||A^2x||||x||$$

for every $x \in H$. Thus

$$||Ax||^2 \leq ||A^2x||$$

for each unit vector $x \in H$. If $x \in N(A^2)$, then

$$||Ax||^2 \leq ||A^2x|| = 0$$

and $x \in N(A)$. Since the non-zero eigenvalues of a class A operator are normal eigenvalues of A by Lemma 2.2, if $0 \neq \sigma_p(A)$ and $(A-\lambda)^2x = 0$, then

$$(A-\lambda)(A-\lambda)x = 0 = (A-\lambda)^*(A-\lambda)x$$

and

$$||(A-\lambda)x||^2 = ((A-\lambda)^*(A-\lambda)x, x) = 0.$$

Hence, if A is a class A operator, then $asc(A-\lambda) = 2$. Since operators with finite ascent have SVEP [21], A has SVEP. This completes the proof. ∎

Theorem 2.1. *Let A be an algebraically class A operator. Then Weyl's theorem holds for A*

Proof. Assume that $\lambda \in \sigma(A) \setminus \sigma_w(A)$. Then $A-\lambda$ is Weyl and not invertible. We claim that $\lambda \in \partial\sigma(A)$. Assume to the contrary that λ is an interior point of $\sigma(A)$. Then there exists a neigborhood U of λ such that $dim(A-\mu) > 0$ for all $\mu \in U$. It follows from ([12], Theorem 10) that A does not have SVEP. On the other hand, since $p(A)$ is class A operator for nonconstant polynomial p, it follows from Lemma 2.5 that $p(A)$ has SVEP. Hence by ([22], Theorem 3.3.9), A has SVEP, a contradiction. Therefore $\lambda \in \partial\sigma(A)$. Conversely, assume that $\lambda \in \pi_{00}(A)$, with associated Riesz idempotent

$$P := \frac{1}{2\pi i}\int_{\partial D}(\mu - T)^{-1}d\mu,$$

where D is a closed disk centered at λ which contains no other points of $\sigma(A)$. We can then represent A as the direct sum

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \text{where}\, \sigma(\mathrm{A}_1) = \{\lambda\}\, \text{and}\, \sigma(\mathrm{A}_2) = \sigma(\mathrm{A}) \setminus \{\lambda\}.$$

We consider two cases:

Case 1 ($\lambda = 0$). Here A_1 is algebraically class A and quasinilpotent. Hence it follows from Lemma 2.3 that A_1 is nilpotent. We claim that $dimR(P) < \infty$, where $R(P)$ is the range of P. For, if $N(A_1)$ were infinite dimensional, then $0 \notin \pi_{00}(A)$, a contradiction. Therefore A_1 is a finite dimensional operator, therefore Weyl. But since A_2 is invertible, we can conclude that A is Weyl. Thus $0 \in \sigma(A) \setminus \sigma_w(A)$.

Case 2 ($\lambda \neq 0$): As in the proof of Lemma 2.3, $A_1 - \lambda$ is nilpotent. Since $\lambda \in \pi_{00}(A)$, $A_1 - \lambda$ is finite demensionel operator. Therefore $A_1 - \lambda$ is Weyl. Since $A_2 - \lambda$ is invertible, $A - \lambda$ is Weyl and Weyl's theorem holds for A. ∎

As a consequence of the above theorem, we obtain

(1) Every Algebraically hyponormal operator satisfies Weyl's theorem. In particular Weyl's theorem holds for hyponormal operators.

(2) Every Algebraically *log*-hyponormal operator satisfies Weyl's theorem. In particular Weyl's theorem holds for *log*-hyponormal operators.

(3) Every Algebraically p-hyponormal operator satisfies Weyl's theorem. In particular Weyl's theorem holds for p-hyponormal operators.

Theorem 2.2. *Let $A \in B(H)$ be algebraically class A operator. Then Weyl's theorem holds for $f(A)$ for every function f analytic on a neighborhood of $\sigma(A)$.*

Proof. We prove that $f(\sigma_w(A)) = \sigma_w(f(A))$ for every function f analytic on a neighborhood of $\sigma(A)$. Let f be an analytic function on a neighborhood of $\sigma(A)$. Since $\sigma_w(f(A)) \subseteq f(\sigma_w(A))$ with no restrection on A, it is sufficiant to prove that $f(\sigma_w(A)) \subseteq \sigma_w(f(A))$. Assume that $\lambda \notin \sigma_w(f(A))$. Then $f(A) - \lambda$ is Weyl and

$$f(A) - \lambda = c(A - \alpha_1)(A - \alpha_2)...(A - \alpha_n)g(A), \tag{2.1}$$

where $c, \alpha_1, \alpha_2, ...\alpha_n \in \mathbb{C}$ and $g(A)$ is invertible. Since the operators on the right-hand side of (2.1) commute, every $A - \alpha_i$ is Fredholm. Since A is algebraically class A operator, A has SVEP by Lemma 2.5. It follows from ([3], Theorem 2.6) that $i(A - \alpha_i) \leq 0$ for each $i = 1, 2, ..., n$. Hence $\lambda \notin f(\sigma_w(A))$, and so $f(\sigma_w(A)) = \sigma_w(f(A))$.

It is known [23], that if A is isoloid then

$$f(\sigma(A)) \setminus \pi_{00}(A) = \sigma(f(A)) \setminus \pi_{00}(A)$$

for every analytic function on a neighborhood of $\sigma(A)$. Since A is isoloid by Lemma 2.3 and Weyl's theorem holds for $f(A)$,

$$\sigma(f(A)) \setminus \pi_{00}(A) = f(\sigma(A)) \setminus \pi_{00}(A) = f(\sigma_w(A)) = \sigma_w(f(A)).$$

This completes the proof. ∎

Theorem 2.3. *Let $A \in B(H)$ be a class A operator and let $\sigma_w(A) = 0$. Then A is compact and normal.*

Proof. Since Weyl's theorem holds for A by the above theorem and $\sigma_w(A) = 0$ and since a class A operator is normaloid, every non zero spectrum of A is an isolated normal eigenvalue with finite dimensional eigenspace, which reduces A. Hence $\sigma(A) \setminus \sigma_w(A)$ is a finite set or a countable infinity set whose accumulation point is only zero. Let $\sigma(A) \setminus \sigma_w(A) = \{\lambda_n\}$ with $|\lambda_1| \geq |\lambda_2| \geq ... \geq 0$ and let E_n be the orthogonal projection onto $ker(A - \lambda_n)$. Then $AE_n = E_nA = \lambda_n E_n$ and $E_nE_m = 0$ if $n \neq m$. Put $E = \oplus_n E_n$. Then

$$A = \oplus_n \lambda_n E_n \oplus A|_{(1-E)H}$$

and $\sigma(A|_{(1-E)H}) = \{0\}$. Since $A|_{(1-E)H}$ also a class A operator because EH is a reducing subspace of A, $A|_{(1-E)H} = 0$ by Lemma 2.1. This implies that $A = \oplus_n \lambda_n E_n$ is normal. The compactness of A follows from the finiteness or the countability of $\{\lambda_n\}_n$ satisfying $|\lambda_n| \downarrow 0$ and each E_n is a finite rank projection. ∎

As a consequence of the above theorem, we obtain

Corollary 2.1. *Let $A \in B(H)$. Then*
(1) Every class A operator with $\sigma_w(A) = 0$ is compact and normal.
(2) Every log-hyponormal operator with $\sigma_w(A) = 0$ is compact and normal.
(3) Every p-hyponormal operator with $\sigma_w(A) = 0$ is compact and normal.

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