On the number of orderings of n items

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Abstract

Suppose that consumers have to classify n items or baskets of goods according to their individual preferences or utility and such that ties are allowed. In this paper we study the number of possible classifications or outcomes $f(n)$. We obtain different representations for $f(n)$ and use singularity analysis to determine the asymptotic behaviour of $f(n)$. We also give a probabilistic interpretation of $f(n)$ and use a renewal argument to study $f(n)$ as $n \to \infty$. Assuming that each of the $f(n)$ outcomes has equal probability to occur, we study the random variable N_n where N_n equals the number of most preferred items, i.e. the number of items on the top of the list.

1 Introduction

Suppose that we ask consumers to express their preferences with respect to 3 items A, B and C. Allowing for ties we can have the following tastes:

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1	A	B	C
2	A	C	B
3	A	BC	
4	B	A	C
5	B	C	A
6	B	AC	
7	C	A	B
8	C	B	A
9	C	AB	
10	AB	C	
11	AC	B	
12	BC	A	
13	ABC		

We find a total of $f(3) = 13$ possible arrangements. In case 1 we prefer A to B and B to C. In case 12 we are indifferent between B and C and prefer B (and C) to A. In case 13 we are totally indifferent. In the example there are 9 cases with a single item in the first place, we have 3 cases with 2 items in the first place and we have 1 case with all 3 items in the same place. Assuming that each arrangment has equal probability (1/13), we can simply determine the p.d. of $N_3 =$ the number of items on the first place. We find $P(N_3 = 1) = 9/13$, $P(N_3 = 2) = 3/13$ and $P(N_3 = 3) = 1/13$. In the case where there are 2 items A and B, we find $f(2) = 3$ possible arrangements and now we have $P(N_2 = 1) = 2/3$ and $P(N_2 = 2) = 1/3$.

In the paper we plan to study this more generally. In section 2 we discuss the number of arrangements $f(n)$. In section 3 we use singularity analysis to obtain the asymptotic behaviour of $f(n)$. In section 4 we briefly discuss the relation between renewal theory and $f(n)$ and give a moment representation of $f(n)$. We also discuss the asymptotic behaviour of $P(N_n = k)$. We finish the paper with some concluding remarks.

2 The number of arrangements

When we have n items, the number of classifications will be denoted by $f(n)$. The number of items on the first place will be denoted by N_n. In our first result we show how to obtain $f(n)$ recursively.

Proposition 1. *(i) With $f(0) = 1$, we find*

$$f(n) = \sum_{k=1}^{n} \binom{n}{k} f(n-k),\ n \geq 1$$

(ii) For $1 \leq k \leq n$ we have

$$P(N_n = k) = \binom{n}{k} \frac{f(n-k)}{f(n)}$$

Proof. (i) When we classify n things, we have the following possibilities:

- we prefer 1 thing above the other things: there are $\binom{n}{1}$ ways to choose 1 thing. The remaining $n-1$ things can be arranged in $f(n-1)$ ways;

- we put k things ($1 \le k \le n-1$) on the first place: there are $\binom{n}{k}$ ways to choose k things. The remaining $n-k$ things can be arranged in $f(n-k)$ ways.

- we can put all n things on the first place. This can be done in $1 = f(0)$ way.

Summing up the different possibilities, we have (i). Assuming that each of the $f(n)$ arrangements has equal probability, result (ii) follows from (i). ∎

Using this result we find the following first few terms of the sequence $f(n)$:

$$\begin{pmatrix} n & f(n) \\ 1 & 1 \\ 2 & 3 \\ 3 & 13 \\ 4 & 75 \\ 5 & 541 \\ 6 & 4683 \\ 7 & 47293 \\ 8 & 545835 \\ 9 & 7087261 \\ 10 & 102247563 \end{pmatrix}$$

This sequence is included as sequence $A000670/M2952$ and $A034172$ in Sloane's on-line encyclopedia of integer sequences. The sequence also appears in a problem related to locks, cf. Velleman and Call (1995). We can also see $f(n)$ as the number of possible outcomes in a race with n people, where ties are allowed. Another interpretation is related to the Stirling numbers of the second kind $S(n,m)$, cf. Weisstein (2003, p. 2865). The number $S(n,m)$ counts the number of ways of partitioning a set of n elements into m nonempty sets. Clearly $S(n,1) = S(n,n) = S(0,0) = 1$, $S(n,0) = S(0,n) = 0$ and $S(n,m) = S(n-1,m-1) + mS(n-1,m)$. If we partition n into m subsets, we can consider these m subsets as sets of items with equal utility. It follows that each of the $S(n,m)$ partitionings leads to $m!$ orderings. We conclude that

$$f(n) = \sum_{m=0}^{n} m!S(n,m)$$

The table above shows that the sequence $f(n)$ grows exponentially fast. To obtain the precise asymptotic behaviour of $f(n)$ (cf. section 3 below) it is convenient to use generating functions. In the next result we determine the generating function $B(z)$ of the related sequence $\{b(n)\}$ defined by $b(n) = f(n)/n!$.

Proposition 2. *$B(z)$ is given by*

$$B(z) = \frac{1}{2 - \exp(z)}$$

Proof. Using Proposition 1 we have $b(0) = 1$ and

$$b(n) = \sum_{k=1}^{n} \frac{1}{k!} b(n-k),\ n \geq 1$$

It follows that

$$\sum_{n=1}^{\infty} b(n) z^n = \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{z^k}{k!} b(n-k) z^{n-k}$$

Changing the order of summation, we find that

$$B(z) - 1 = \sum_{k=1}^{\infty} \frac{z^k}{k!} \sum_{n=k}^{\infty} b(n-k) z^{n-k}$$

and consequently also that

$$B(z) - 1 = B(z) \sum_{k=1}^{\infty} \frac{z^k}{k!} = B(z)(\exp z - 1)$$

Rearranging the terms, we find the desired result. ∎

Remark. Using Proposition 2, a Taylor expansion shows that

$$B(z) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{2^k} e^{kz} = \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{2^k} \sum_{n=0}^{\infty} \frac{1}{n!} (kz)^n$$

It follows that an explicit expression for $f(n)$ is given by

$$f(n) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{k^n}{2^k},\ n \geq 0$$

3 Singularity analysis

Returning to $B(z)$, note that $B(z)$ has simple singularities given by $s(k) = \log(2) + i2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$. The dominant singularity is given by $s(0) = \log(2)$. Rescaling the function $B(z)$, it is more convenient to use $W(z)$ defined by $W(z) = B(z \log(2))$, i.e.

$$W(z) = \frac{1}{2 - 2^z}$$

Clearly $W(z)$ is the generating function of the sequence $w(n) = (\log 2)^n f(n)/n!$. The dominant singularity of $W(z)$ is given by 1. Using $\mu = 2\log(2)$ the following relations are easily established: as $z \to 1$ we have

$$W(z)(1 - z) \to \frac{1}{\mu} \tag{1}$$

$$W(z) - \frac{1}{\mu(1 - z)} \to \frac{1}{4} \tag{2}$$

$$\left(W(z) - \frac{1}{\mu(1-z)} - \frac{1}{4}\right)/(1-z) \to \frac{\log(2)}{24} \tag{3}$$

In order to extract from here information about the asymptotic behaviour of $w(n)$ the classical approach is to use Tauberian theory and regular variation as in Bingham et al. (1987). Although these methods are powerfull and have a huge amount of applications in analysis and probability theory, Tauberian theory is not applicable here. A modern approach is based on singularity analysis, a method introduced and stimulated by Flajolet, see Flajolet and Odlyzko (1990), Flajolet and Sedgewick (2005). See also Wilf (1990, Chapter 5). Chapter 6 of the forthcoming book of Flajolet and Sedgewick (2005) contains the state of the art concerning singularity analysis. Using (1), (2) and the transfer theorem of Flajolet and Sedgewick (2005, Corollary VI.1) we find the following result.

Theorem 3. *(i) The coefficients $w(n)$ satisfy*

$$w(n) \to \frac{1}{\mu}$$

and consequently we have

$$f(n) \sim \frac{1}{\mu}\frac{n!}{(\log(2))^n}$$

(ii) Moreover we have

$$\sum_{n=0}^{\infty}(w(n) - \frac{1}{\mu}) = \frac{1}{4}$$

Remark. Using (3) we also see that

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(w(k) - \frac{1}{\mu}) - \frac{1}{4}\right) = -\sum_{k=1}^{\infty} k(w(k) - \frac{1}{\mu}) = \frac{\log(2)}{24}$$

Using more terms in the asymptotic expansion of $W(z)$ we can obtain more information about the rate of convergence in Theorem 3(i).

4 Relation with probability theory

As an alternative to analyse $B(z)$, we introduce random variables X and Y where X has a Poisson distribution with parameter $\lambda = \log(2)$ and where Y has the probability distribution (p.d.) given by

$$P(Y = k) = \frac{(\log(2))^k}{k!},\ k \geq 1$$

Note that Y has the same distribution as the truncated variable $X \mid X \geq 1$. The generating function of Y is given by $\phi(z) = E(z^Y) = 2^z - 1$. Clearly all moments of Y are finite. The first few moments of Y are given by $\mu = E(Y) = 2\log(2)$ and $\sigma^2 = Var(Y) = 2\log(2)(1 - \log(2))$. The generating function of X is given by $\psi(z) = E(z^X) = \frac{1}{2}2^z$. It follows that $\phi(z) = 2\psi(z) - 1$. Using $W(z) = B(z\log(2))$ as before, we find the followings relation between $W(z)$, $\phi(z)$ and $\psi(z)$.

Proposition 4. *We have*

$$W(z) = \frac{1}{1-\phi(z)} = \frac{1}{2}\frac{1}{1-\psi(z)}$$

Proposition 4 shows that $W(z)$ is the generating function of the renewal sequence generated by Y. Using this renewal interpretation, we can use results from discrete renewal theory to obtain the asymptotic behaviour of the sequences $w(n)$ and $f(n)$. One such result is the following, see e.g. Feller (1970).

Proposition 5. *As $n \to \infty$ we have*

$$w(n) = \frac{(\log(2))^n f(n)}{n!} \to \frac{1}{\mu}$$

and

$$\sum_{n=0}^{\infty}(w(n) - \frac{1}{\mu}) = \frac{\sigma^2 - \mu + \mu^2}{2\mu^2} = \frac{1}{4}$$

Evidently, we find back the results of Theorem 3.

Proposition 4 also shows that $2W(z)$ is the generating function of the renewal sequence generated by X. From this it follows that $w(0) = 1$ and

$$2w(n) = \sum_{k=1}^{\infty} P(S_k = n),\ n \geq 1$$

where S_k is the sum of i.i.d. copies of X. Since S_k has a Poisson distribution with parameter $k\log(2)$, it follows that

$$2w(n) = \sum_{k=1}^{\infty} \frac{(k\log(2))^n}{n!2^k}$$

Returning to the sequence $f(n)$ we find back that

$$f(n) = \frac{1}{2}\sum_{k=1}^{\infty}\frac{k^n}{2^k},\ n \geq 1 \tag{4}$$

Moreover, expression (4) has the following probabilistic interpretation.

Proposition 6. *For $n \geq 1$ we have $f(n) = \frac{1}{2}E(U^n)$ where U is a r.v. with $P(U = k) = 2^{-k}$, $k \geq 1$.*

Combining Propositions 5 and 6, as a Corollary, we find the asymptotic behaviour of $E(U^n)$: we have

$$E(U^n) \sim \frac{n!}{(\log(2))^{n+1}}$$

5 The number of items in the first place

For the r.v. N_n we find the following result (we use the notation of section 4).

Theorem 7. *As $n \to \infty$, we have $P(N_n = k) \to P(Y = k)$, $k \geq 1$.*

Proof. First observe that

$$P(N_n = k) = \frac{(\log(2))^k}{k!} \frac{w(n-k)}{w(n)} = P(Y = k) \frac{w(n-k)}{w(n)}$$

Since $w(n) \to 1/\mu$ we find that $w(n-k)/w(n) \to 1$. This proves the result. ■

For the moments of N_n we have the following result

Proposition 8. *(i) For $r \geq 1$ we have $E\left(\begin{array}{c} N_n \\ r \end{array} \right) = 2P(N_n = r)$*

(ii) For $r \geq 1$ we have $E\left(\begin{array}{c} N_n \\ r \end{array} \right) \to 2P(Y = r) = E\left(\begin{array}{c} Y \\ r \end{array} \right)$

Proof. (i) We have

$$E\left(\begin{array}{c} N_n \\ r \end{array} \right) = \sum_{k=r}^{n} \binom{k}{r} \binom{n}{k} \frac{f(n-k)}{f(n)}$$

Using

$$\binom{k}{r}\binom{n}{k} = \binom{n-r}{k-r}\binom{n}{r}$$

it follows that

$$\binom{k}{r}\binom{n}{k} \frac{f(n-k)}{f(n)} = \binom{n-r}{k-r} \frac{f(n-r-(k-r))}{f(n-r)} P(N_n = r)$$

For $k = r$ this is

$$\binom{k}{r}\binom{n}{r} \frac{f(n-k)}{f(n)} = P(N_n = r)$$

while for $r + 1 \leq k \leq n$ this is

$$\binom{k}{r}\binom{n}{k} \frac{f(n-k)}{f(n)} = P(N_{n-r} = k - r) P(N_n = r)$$

Summing up we find that

$$E\left(\begin{array}{c} N_n \\ r \end{array} \right) = 2P(N_n = r)$$

and the result.

(ii) The second result follows from the first result. ■

6 Concluding remarks

1) In the consumer context Theorem 7 shows that when consumers express their preferences completely randomly, then there is a high probability (approximately $P(Y = 1) = \log(2)$) to have one winner. Moreover, in this case each of the n items has equal probability to be this winner.

2) Let $N_{2,n}$ denote the number of items that were put in the second place and as before, let $N_n = N_{1,n}$ denote the number of items ranked in the first place. We clearly have $P(N_{1,n} = n) = P(N_{2,n} = 0) = 1/f(n)$. For $1 \leq k \leq n-1$ and $1 \leq j \leq n-k$ we have

$$P(N_{1,n} = k, N_{2,n} = j) = P(N_{1,n} = k)P(N_{1,n-k} = j)$$

As n tends to infinity, for fixed k, j we obtain that $P(N_{1,n} = k, N_{2,n} = j) \to P(Y = k)P(Y = j)$.

3) In the paper we obtained the asymptotic behaviour of $E(U^n)$ where $P(U = k) = 2^{-k}$, $k \geq 1$. Using a similar approach we can also consider $E(V^n)$ where $P(V = k) = pq^{k-1}$, $k \geq 1$ and where $0 < p = 1 - q < 1$. Now we find that

$$F(z) = E(e^{zV}) = \sum_{k=0}^{\infty} \frac{E(V^k)}{k!} z^k = \frac{pq\exp(z)}{q(1 - q\exp(z))}$$

Clearly $F(z)$ has a dominant singularity given by $s = -\log(q)$. Now we obtain that

$$E(V^n) \sim \frac{pn!}{q(-\log(q))^{n+1}}$$

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